Cumulative-Damage Reliability for Random-Independent (Normal- or Weibull-Distributed) Fatigue Stress, Random-Fixed Strength, and Deterministic Usage

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ABSTRACT

To determine helicopter component retirement intervals, it is common for fatigue engineers to utilize Miner's rule for cumulative damage. Time consuming Monte Carlo simulations are considered the state of the art method for determining reliability in the presence of load variations. However, a new set of methods employed on a recent AHS fatigue reliability round robin problem provide an indication that near term advances are on the horizon. This paper presents the development of a set of analytical formulas for the calculation of cumulative damage per Miner's rule for the case with randomly varying oscillatory fatigue loads applied to a given component selected randomly from a population of various components. Although further development is required to cover all material characterizations, the method shows great promise in providing efficient fatigue reliability calculations during iterations of the design process. This paper provides a detailed derivation of the method, useful charts and tables, and guidance regarding implementation of the formula in various computer applications.

NOTATION

$a(\bullet,\bullet)$	"a function" derived herein
$b(\bullet,\bullet,\bullet)$	"b function" derived herein
COV_E	endurance limit coefficient of variation
COV_S	load amplitude coefficient of variation
D	accumulated fatigue damage
E	endurance limit
$erfc(\bullet)$	complementary error function
$\exp(\bullet)$	exponential function
$f(\bullet)$	normal probability density function
$g(\bullet)$	Weibull probability density function
i	load cycle index
j	component index
k	material constant (factor)
l	binomial expansion term index
m	material constant (exponent)
N	number of cycles to failure
n	number of applied load cycles
p	load condition index
$p_\bullet(\bullet)$	primary polynomials used to evaluate the a function
$q_\bullet(\bullet)$	secondary polynomials used to evaluate the a function
R	component reliability
S	oscillatory load amplitude
w	"standardized" Weibull load amplitude
w_0	"standardized" Weibull damaging load amplitude
x	generalized oscillatory load amplitude
y	generalized endurance limit
z	standardized normal load amplitude
z_0	standardized normal damaging load amplitude
α	material constant (endurance limit adjustment)
β	Weibull slope (for load amplitudes)
β_S	load amplitude Weibull slope
$\Gamma(\bullet)$	Gamma function
$\Gamma(\bullet,\bullet)$	upper incomplete Gamma function
$\gamma(\bullet,\bullet)$	"standardized" Weibull probability density function
δ	Weibull minimum expected value (for load amplitudes)
δm	small increase in m with $0 < \delta m < 1$
η	Weibull characteristic value (for load amplitudes)
μ	mean (of load amplitudes)
μ_E	endurance limit mean
μ_S	load amplitude mean
ξ	load condition usage spectrum weighting factor
σ	standard deviation (of load amplitudes)
$\phi(\bullet)$	standardized normal probability density function

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INTRODUCTION

Fatigue reliability of helicopter components is an example of the time-dependent stress-strength models discussed by Kapur and Lamberson [1]. The three categories of variables discussed in chapter 8 of [1] are Deterministic, Random-Fixed, and Random-Independent. For a given helicopter component selected randomly from a pool of components and experiencing various load conditions throughout its useful life, one may categorize the stress as random independent and the strength as random fixed. Kapur and Lamberson recommend numerical simulation to solve the resulting time-dependent stress-strength model. Although the work in [1] is dated, the reader may recognize application of Monte Carlo techniques as state of the art for this type of simulation.

Arden [2] has recently posed a Round Robin Problem to determine component reliability in the context of usage monitoring (referenced herein as the "Arden Round Robin problem"). By presenting representative but hypothetical helicopter-component fatigue data, the Arden Round Robin problem has allowed for necessary public discussion (and the potential for industry-wide collaboration) regarding the important class of problems under consideration without requiring inappropriate disclosure of proprietary or sensitive data. In addition, the Arden Round Robin problem is an example of the subject time-dependent stress-strength model which has proven useful in sparking discussion of various novel solution methods [3].

This paper presents novel analytical methods which are useful in solving cumulative-damage fatigue problems with Normal- or Weibull-Distributed Random-Independent Stress and Random-Fixed Strength. A new set of analytical functions are derived. For a special class of material characterizations, the functions may be evaluated using classical analytical functions. Advantages of the resulting methods include high speed and numerical precision. The methods are demonstrated on realistic example problems derived from the Arden Round Robin problem in [2] and possible future extensions to the present work are discussed.

BACKGROUND

To determine helicopter component retirement intervals, it is common for fatigue engineers to utilize the Palmgren-Miner cycle-ratio summation rule for cumulative damage based on strength characterized by some form of the equation

$$\frac{1}{N} = \frac{1}{k}\left(\frac{S}{E} - \alpha\right)^m \tag{1}$$

where N is the number of cycles to failure, S is a given oscillatory or vibratory load amplitude corresponding to a component substantiating parameter, E is the endurance limit for that parameter, and the given material constants k, m, and α, as well as the endurance limit Coefficient Of Variation (COV) are derived from coupon data. Typically, this curve is fit to failure data (cycles to failure N recorded for a corresponding constant-amplitude test load S) for a limited number of full-scale fatigue test specimens to determine the mean value of E for the tested specimens. This paper considers the special case where the material constant m is restricted to be an integer.

THEORY

Consider the j-th member of a set of components with randomly distributed endurance limit used in a given fleet with application of the i-th cycle of a randomly distributed oscillatory load. Substituting S_i and E_i into Eqn. (1) results in the (Palmgren-Miner) damage fraction for the i-th cycle applied to the j-th component described in Eqn. (2).

$$\frac{1}{N_{i,j}} = \begin{cases} \dfrac{1}{k}\left(\dfrac{S_i}{E_j} - \alpha\right)^m & \mathit{if}\ S_i > \alpha\, E_j \\ 0 & \mathit{otherwise} \end{cases} \tag{2}$$

Defining $x_i = S_i$ and $y_j = \alpha E_j$, one may express the damage fraction as in Eqn. (3).

$$\frac{1}{N_{i,j}} = \begin{cases} \left[\dfrac{1}{k}\left(\dfrac{\alpha}{y_j}\right)^m\right]\left(x_i - y_j\right)^m & \mathit{if}\ x_i > y_j \\ 0 & \mathit{otherwise} \end{cases} \tag{3}$$

The damage accumulated in the j-th component according to the Palmgren-Miner cycle-ratio summation rule would be as described in Eqn. (4).

$$D_j = \sum_{i=1}^{n} \frac{1}{N_{i,j}} = \left[\frac{1}{k}\left(\frac{\alpha}{y_j}\right)^m\right] \sum_{i=1}^{n} \begin{cases} \left(x_i - y_j\right)^m & \mathit{if}\ x_i > y_j \\ 0 & \mathit{otherwise} \end{cases} \tag{4}$$

If the loads x_i were distributed with normal density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right),$$

then the damage accumulated in n cycles would be

$$D_j = n\left[\frac{1}{k}\left(\frac{\alpha}{y_j}\right)^m\right] \int_{y_j}^{\infty} \left(x - y_j\right)^m f(x)dx\,. \tag{5}$$

Defining $z = \dfrac{x-\mu}{\sigma}$ results in

$$D_j = n\left[\frac{1}{k}\left(\frac{\alpha\sigma}{y_j}\right)^m\right] \int_{z_{0,j}}^{\infty} \left(z - z_{0,j}\right)^m \phi(z)dz\,,$$

where

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$$

and

$$z_{0,j} = \frac{y_j - \mu}{\sigma}.$$

Defining the integral expression

$$a(z_0, m) = \int_{z_0}^{\infty} (z - z_0)^m \phi(z) dz , \quad (6)$$

results in

$$D_j = n \left[\frac{1}{k} \left(\frac{\alpha}{z_{0,j} + \frac{\mu}{\sigma}} \right)^m \right] a(z_{0,j}, m), \quad (7)$$

where determination of the cumulative damage in the j-th component has been reduced to integration and tabulation of the various "a functions" in Eqn. (6).

Similarly, if the loads x_i were distributed with Weibull density function

$$g(x) = \begin{cases} \frac{\beta}{\eta - \delta} \left(\frac{x - \delta}{\eta - \delta} \right)^{\beta - 1} \exp\left(-\left(\frac{x - \delta}{\eta - \delta} \right)^{\beta} \right) & \text{if } x \geq \delta \\ 0 & \text{otherwise} \end{cases},$$

then the damage accumulated in n cycles would be

$$D_j = n \left[\frac{1}{k} \left(\frac{\alpha}{y_j} \right)^m \right] \int_{y_j}^{\infty} (x - y_j)^m g(x) dx .$$

Because the Weibull density function is zero below $x = \delta$, the lower limit of integration, y_j, in the previous expression may be replaced by $\max(y_j, \delta)$, resulting in

$$D_j = n \left[\frac{1}{k} \left(\frac{\alpha}{y_j} \right)^m \right] \int_{\max(y_j, \delta)}^{\infty} (x - y_j)^m g(x) dx . \quad (8)$$

In other words, the lower limit of integration in Eqn. (8) represents the lowest damaging load for the j-th component. Because no load smaller than $x = y_j$ causes damage and $x = \delta$ is the lowest load in the distribution, the lower limit of integration in Eqn. (8) is simply the larger of y_j and δ.

Defining $w = \frac{x - \delta}{\eta - \delta}$ allows Eqn. (8) to be restated as

$$D_j = n \left[\frac{1}{k} \left(\frac{\alpha(\eta - \delta)}{y_j} \right)^m \right] \int_{\max(w_{0,j}, 0)}^{\infty} (w - w_{0,j})^m \gamma(w, \beta) dw ,$$

where

$$\gamma(w, \beta) = \begin{cases} \beta\, w^{\beta - 1} \exp(-w^{\beta}) & \text{if } w \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$w_{0,j} = \frac{y_j - \delta}{\eta - \delta}.$$

Defining the integral expression

$$b(w_0, m, \beta) = \int_{\max(w_0, 0)}^{\infty} (w - w_0)^m \gamma(w, \beta) dw , \quad (9)$$

results in

$$D_j = n \left[\frac{1}{k} \left(\frac{\alpha}{w_{0,j} + \left(\frac{\delta}{\eta - \delta} \right)} \right)^m \right] b(w_{0,j}, m, \beta), \quad (10)$$

where determination of the cumulative damage in the j-th component has been reduced to integration and tabulation of the various "b functions" in Eqn. (9).

Delaying, for now, further development of the a function and b function tools required for use of Eqns. (6) and (9), one may wish to explore how the cumulative damage in the j-th component, D_j, from Eqns. (7) and (10) might be useful in predicting component reliability.

Component reliability depends on variations in component usage, loads, and strengths. Although Eqns. (7) and (10) currently only account for total damage due to variation in a given set of load cycles, these equations may be readily expanded as in the alternate Eqns. (11) and (12) to cover a fixed usage spectrum as a weighted sum of terms similar to those provided in Eqns. (7) and (10), where the weighting factor, ξ_p, is the ratio of the number of cycles, n_p, at each load condition to the total number of cycles, n.

$$D_j = n \sum_{p=1}^{p_{\max}} \xi_p \left[\frac{1}{k} \left(\frac{\alpha}{z_{0,j,p} + \frac{\mu_p}{\sigma_p}} \right)^m \right] a(z_{0,j,p}, m) \quad (11)$$

$$D_j = n \sum_{p=1}^{p_{\max}} \xi_p \left[\frac{1}{k} \left(\frac{\alpha}{w_{0,j,p} + \frac{\delta_p}{\eta_p - \delta_p}} \right)^m \right] b(w_{0,j,p}, m, \beta_p) \quad (12)$$

Eqns. (11) and (12) may be combined to cover hybrid cases where some load conditions are best represented by a normal distribution and others are best represented by a Weibull distribution.

In the present study, only the special case of deterministic usage is considered. Usage variation may be considered variation in weighting factor ξ_p for each load level with the constraint that the weighting factors sum to unity, but this topic is reserved for future explorations. Although further development is necessary to extend the method for investigation of the impact of usage variation on component reliability, the method presented herein is directly applicable to all cases of known usage, including the case of assessing component damage accumulated to date due to accurately monitored past usage.

Now consider a large population of identically used components with randomly distributed endurance limit. The reliability, R, of any component in the population is the probability that $D_j < 1$, where D_j is calculated from Eqns. (11) and/or (12). Examination of Eqn. (2) for the case of $m > 0$ reveals that each term accumulated in the damage fraction increases with decreasing component endurance limit, E_j. If one could determine the critical endurance limit which corresponds to $D_j = 1$ in Eqns. (11) and/or (12) for a given number of applied cycles, then one could evaluate R based on the cumulative endurance limit distribution as the probability that E_j is greater than the critical endurance limit. It is noted that this method is not limited to any specific type of endurance limit distribution, such as normal, Weibull, or other.

Derivation of a function and b function tools

Having established the potential usefulness of a function and b function tools described respectively in Eqns. (6) and (9) for use in cumulative-damage reliability studies, this article now explores methods of computing the integrals in Eqns. (6) and (9) for the special case where the material constant m is restricted to be an integer. Eqns. (6) and (9) defining the a function and b function, respectively, are repeated here for the convenience of the reader.

$$a(z_0, m) = \int_{z_0}^{\infty} (z - z_0)^m \phi(z) dz \tag{6}$$

$$b(w_0, m, \beta) = \int_{\max(w_0, 0)}^{\infty} (w - w_0)^m \gamma(w, \beta) dw \tag{9}$$

According to the binomial theorem [4],

$$(z - z_0)^m = \sum_{l=0}^{m} \left(\frac{m!}{l!(m-l)!} \right) z^l (-z_0)^{m-l},$$

which may be used to rearrange Eqns. (6) and (9) as

$$a(z_0, m) = \sum_{l=0}^{m} \left(\frac{m!}{l!(m-l)!} \right) (-z_0)^{m-l} \int_{z_0}^{\infty} z^l \phi(z) dz \tag{13}$$

and

$$b(w_0, m, \beta) = \sum_{l=0}^{m} \left(\frac{m!}{l!(m-l)!} \right) (-w_0)^{m-l} \int_{\max(w_0,0)}^{\infty} w^l \gamma(w, \beta) dw, \tag{14}$$

respectively.

Turning now to the integral expression in Eqns. (13) and (14), it may be shown via integration that

$$\int_{z_0}^{\infty} z^l \phi(z) dz = \frac{2^{\left(\frac{l}{2}-1\right)}}{\sqrt{\pi}} \Gamma\left(\frac{l+1}{2}, \frac{z_0^2}{2} \right), \quad if\ z_0 \ge 0, \tag{15}$$

and

$$\int_{\max(w_0,0)}^{\infty} w^l \gamma(w, \beta) dw = \Gamma\left(\frac{l+\beta}{\beta}, w_0^{\beta} \right), \quad if\ w_0 \ge 0, \tag{16}$$

where

$$\Gamma(a, x) = \int_x^{\infty} t^{a-1} \exp(-t) dt$$

is the upper incomplete gamma function [5], which may be calculated easily using standard mathematical software packages (see Appendix A for selected example commands). In the particular case of $w_0 \le 0$, the integral in Eqn. (16) may be evaluated as

$$\int_0^{\infty} w^l \gamma(w, \beta) dw = \Gamma\left(\frac{l+\beta}{\beta} \right)$$

where

$$\Gamma(a) = \int_0^{\infty} t^{a-1} \exp(-t) dt$$

is the gamma function [5].

The expression in Eqn. (15) solves the integration in Eqn. (13) for the case of $z_0 \ge 0$. It is also necessary to solve the integration in Eqn. (13) for the case of $z_0 < 0$. which represents an important case with mean load above the endurance limit for the j-th component. Integration by parts provides Eqn. (17), which may be applied recursively.

$$\int_{z_0}^{\infty} z^l \phi(z) dz = z_0^{l-1} \phi(z_0) + (l-1) \int_{z_0}^{\infty} z^{l-2} \phi(z) dz \tag{17}$$

For even and odd values of l, recursive application of Eqn. (17) results in the need for the primary and secondary basis functions described in Eqns. (18) and (19), respectively, and in Figure 1.

$$\int_{z_0}^{\infty} \phi(z) dz = \frac{1}{2} erfc\left(\frac{z_0}{\sqrt{2}} \right) \tag{18}$$

$$\int_{z_0}^{\infty} z \phi(z) dz = \phi(z_0) \tag{19}$$

Recursive application of Eqn. (17) leads to the results summarized in Table 1, which allow evaluation in the case of $z_0 < 0$. Figure 2 illustrates that the results in Table 1 match the function given in Eqn. (15) for the case of $z_0 \ge 0$. Use of the results in Table 1 for Eqns. (17) through (19) in Eqn. (13) leads to

$$a(z_0, m) = p_m(z_0) \left[\frac{1}{2} erfc\left(\frac{z_0}{\sqrt{2}} \right) \right] + q_m(z_0) \phi(z_0), \tag{20}$$

where $p_m(z_0)$ and $q_m(z_0)$ are polynomials defined in Table 2, which may be applied directly to Eqn. (11). Although Table 2 is limited to $m \le 5$, one may easily apply Eqn. (17) to extend these results to larger values, as necessary.

Table 1. Evaluation of $\int_{z_0}^{\infty} z^l \phi(z)dz$ for Various l Values

l	$\int_{z_0}^{\infty} z^l \phi(z)dz$
0	$\frac{1}{2}erfc\left(\frac{z_0}{\sqrt{2}}\right)$
1	$\phi(z_0)$
2	$z_0\phi(z_0)+\frac{1}{2}erfc\left(\frac{z_0}{\sqrt{2}}\right)$
3	$\left(z_0^2+2\right)\phi(z_0)$
4	$\left(z_0^3+3z_0\right)\phi(z_0)+\frac{3}{2}erfc\left(\frac{z_0}{\sqrt{2}}\right)$
5	$\left(z_0^4+4z_0^2+8\right)\phi(z_0)$

Table 2: Polynomials Used in Evaluation of $a(z_0,m)$ in Equation (20) for Various Values of m

m	$p_m(z_0)$	$q_m(z_0)$
0	1	0
1	$-z_0$	1
2	z_0^2+1	$-z_0$
3	$-z_0^3-3z_0$	z_0^2+2
4	$z_0^4+6z_0^2+3$	$-z_0^3-5z_0$
5	$-z_0^5-10z_0^3-15z_0$	$z_0^4+9z_0^2+8$

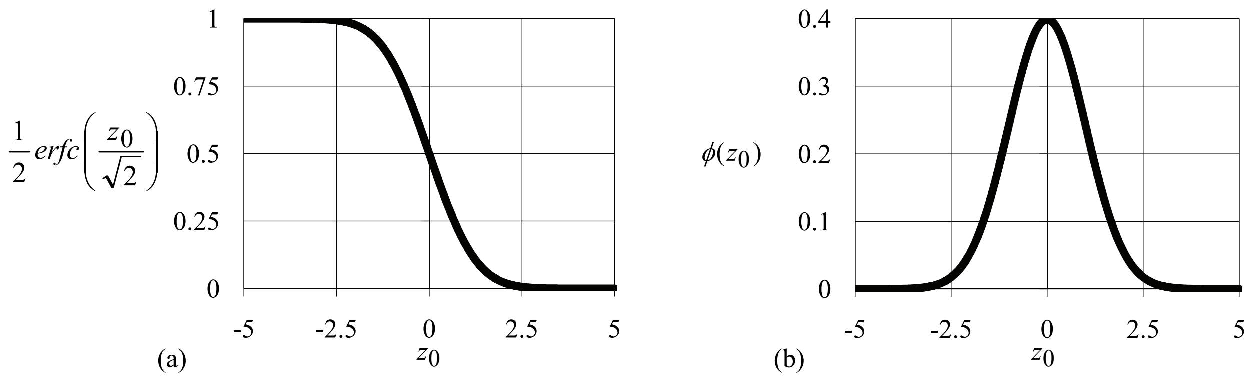

Figure 1: Basis function components of the a function, $a(z_0,m)$, used herein to solve for cumulative damage in a particular component with normally distributed loads, including (a) Primary Basis Function and (b) Secondary Basis Function

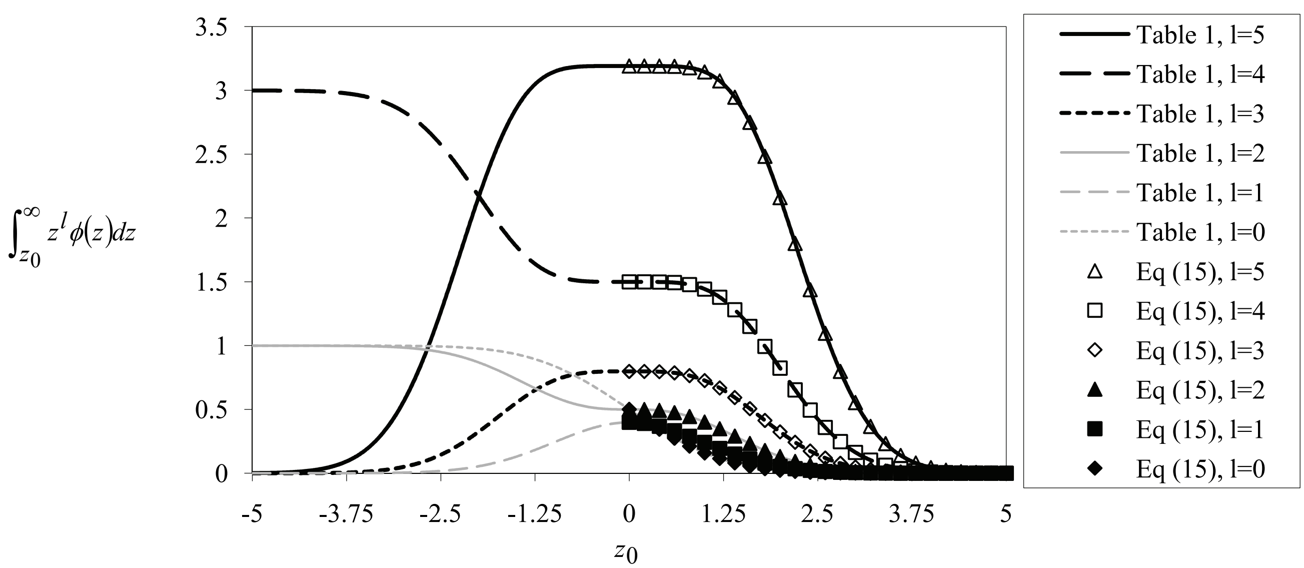

Figure 2: Comparison of Results of Table 1 and Equation (15)

Using Eqn. (20), one may rewrite Eqn. (11) as

$$D_j = n \sum_{p=1}^{p_{\max}} \left[\xi_p \frac{1}{k} \left(\frac{\alpha}{z_{0,j,p} + \dfrac{\mu_p}{\sigma_p}} \right)^m \left\{ p_m\left(z_{0,j,p}\right) \left[\frac{1}{2} erfc\left(\frac{z_{0,j,p}}{\sqrt{2}} \right) \right] + q_m\left(z_{0,j,p}\right) \phi\left(z_{0,j,p}\right) \right\} \right].$$

Similarly, using Eqns. (14) and (16), one may rewrite Eqn. (12) as

$$D_j = n \sum_{p=1}^{p_{\max}} \left[\xi_p \frac{1}{k} \left(\frac{\alpha}{w_{0,j,p} + \dfrac{\delta_p}{\eta_p - \delta_p}} \right)^m \begin{cases} \displaystyle\sum_{l=0}^{m} \left[\left(\frac{m!}{l!(m-l)!} \right) \left(-w_{0,j,p}\right)^{m-l} \Gamma\left(\frac{l+\beta_p}{\beta_p}, \left(w_{0,j,p}\right)^{\beta_p} \right) \right] & if \; w_{0,j,p} > 0 \\ \displaystyle\sum_{l=0}^{m} \left[\left(\frac{m!}{l!(m-l)!} \right) \left(-w_{0,j,p}\right)^{m-l} \Gamma\left(\frac{l+\beta_p}{\beta_p} \right) \right] & otherwise \end{cases} \right].$$

Although the a function and b function tools derived in this study may be implemented directly in *MS Excel*® using complex expressions combining standard functions, the author has implemented them as *Visual Basic*® functions for use with *MS Excel*® (see Appendix B for details).

For various values of m, Figures 3 and 4 show a and b functions, respectively. In Figure 4, the value $\beta = 1.69$ allows comparison between a and b functions. Figure 5 shows b functions for various values of β.

ILLUSTRATIVE EXAMPLE

Two examples based on the Arden Round Robin problem [2] allow demonstration of a and b function methods without publishing proprietary or sensitive data. Figure 6 displays the fatigue curve from [2]. Conversion of the material data in [2] to the format of Eqn. (1) reveals an integer m value.

First, a functions are used to solve the Normal Stress-Normal Strength case for a single load condition with a 10% Coefficient of Variation (COV) for stress and strength. Figure 7 shows probability of fatigue failure curves for various ratios of mean stress to mean strength, where about 220 values of n are found using repeated applications of Eqn. (6) with $D = 1$ for a set of critical $z_{0,j}$ values corresponding to critical endurance limits. The probability of failure is the likelihood that a critical endurance limit exceeds that of a given component. Similarly, Figure 8 shows the ratio of mean stress to mean strength as a function of cycles for various levels of reliability. From Figure 8, it is apparent that for the case of a 10% COV for stress and strength, multiplying strength by a factor of 0.5 (or $\approx 1 - 5 \times COV_E$) results in approximately 6-nines of reliability (0.9_6), while a factor of 0.6 (or $\approx 1 - 4 \times COV_E$) results in approximately 4-nines (0.9_6).

Ratios of mean stress to mean strength for 6-nines of reliability are shown in Figures 9 and 10 in order to illustrate increases in stress COV from 10% to 50% and increases in strength COV from 10% to 14%, respectively. Revealing the nature of problems combining random-independent stress with random-fixed strength, these COV ranges achieve similar reductions in stress to strength ratio at 10^8 cycles and demonstrate that the reliability for this class of problems depends more strongly on variation in strength than stress. According to Figure 9, the effects of increased variation in stress are greater at higher cycles. This is due to a greater number of cycles above the endurance limit, even though the mean stress is well below the endurance limit.

Having illustrated use of a functions to solve the Normal Stress-Normal Strength case using the material strength curve from [2] for a single load condition, it is important to note that the Arden Round Robin problem in [2] was originally posed as an example of Weibull Stress-Normal Strength case with multiple load conditions for each of mild, moderate, and severe usage. Solutions using b functions are depicted in Figure 11 as probabilities of failure versus cycles for each usage. Also shown in Figure 11 are curves representing three cases of Failure Rate Allocation (FRA) to achieve a system failure rate of less than one fatigue failure in 10 million flight hours for systems comprised of 10, 100, and 1000 Critical Safety Items (CSIs). Details of the calculation are illustrated in Table 3 for two cases, including a reliability adjusted to correspond with a life of 1056.2 hours (1056.2 hours was selected based on results presented in [3]) and an FRA for 100 CSIs. The reported negative value of the summand term in the last column (-2.67655E-24 for a peak load of 260 ksi in the 100 CSI case) is due to numerical errors related to precision of the calculation. Although this error does not impact the significant digits of the result in this case, numerical issues causing this error should be addressed.

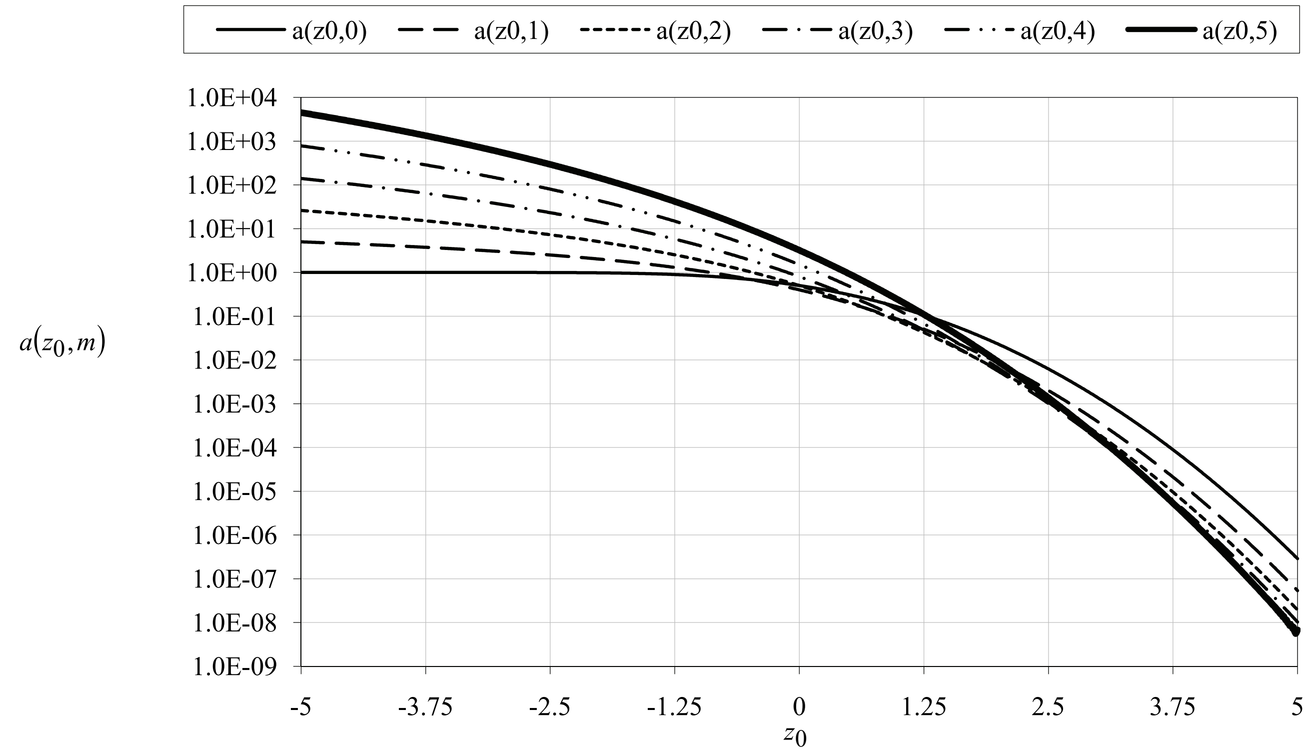

Figure 3: a **functions for various values of** m

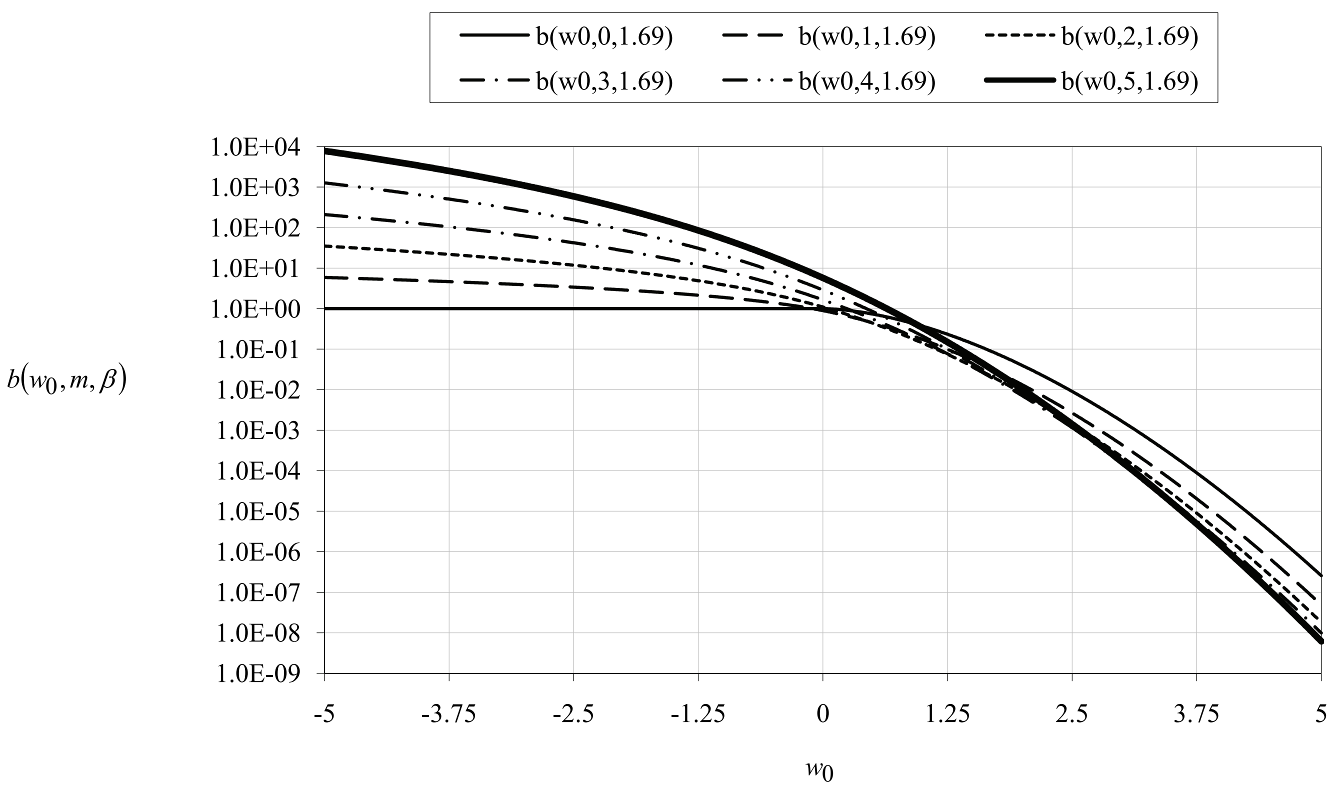

Figure 4: b **functions for** $\beta = 1.69$ **and various values of** m

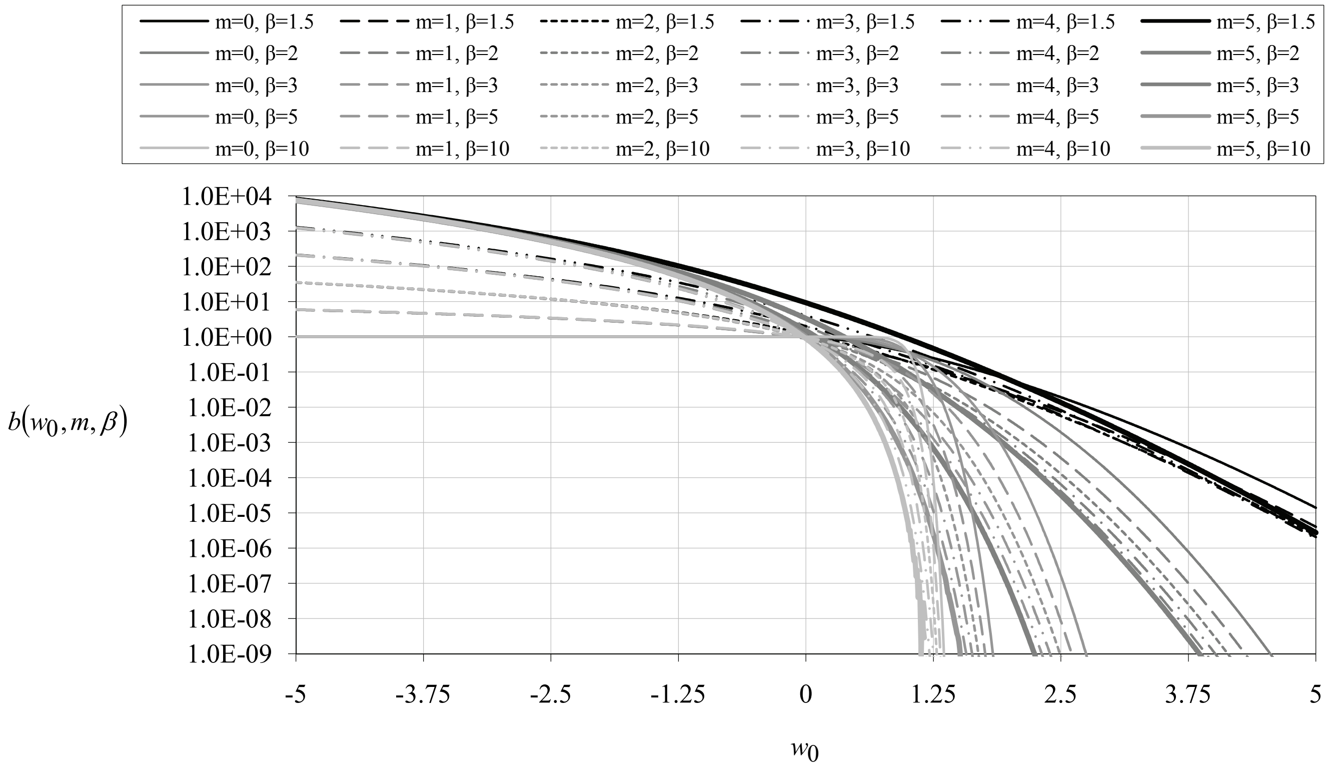

Figure 5: b functions for various values of m and β

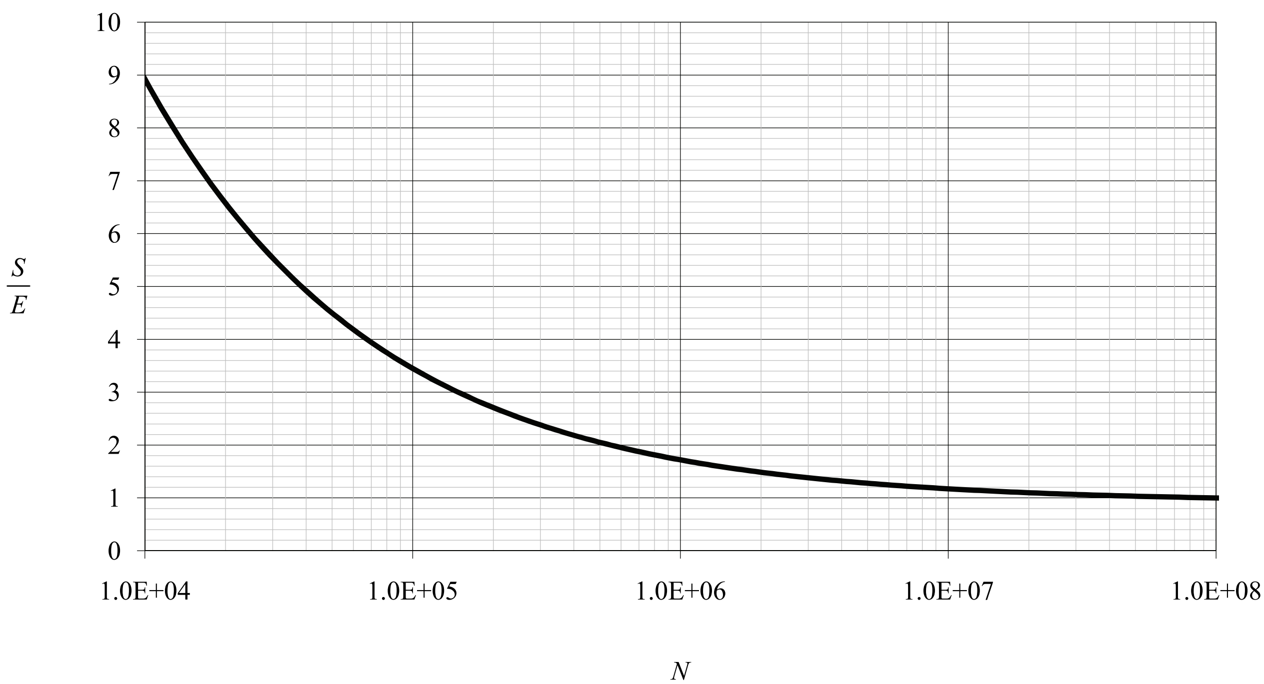

Figure 6: Fatigue Curve per Equation (1) with $k = 640000$, $m = 2$, and $\alpha = 0.92$

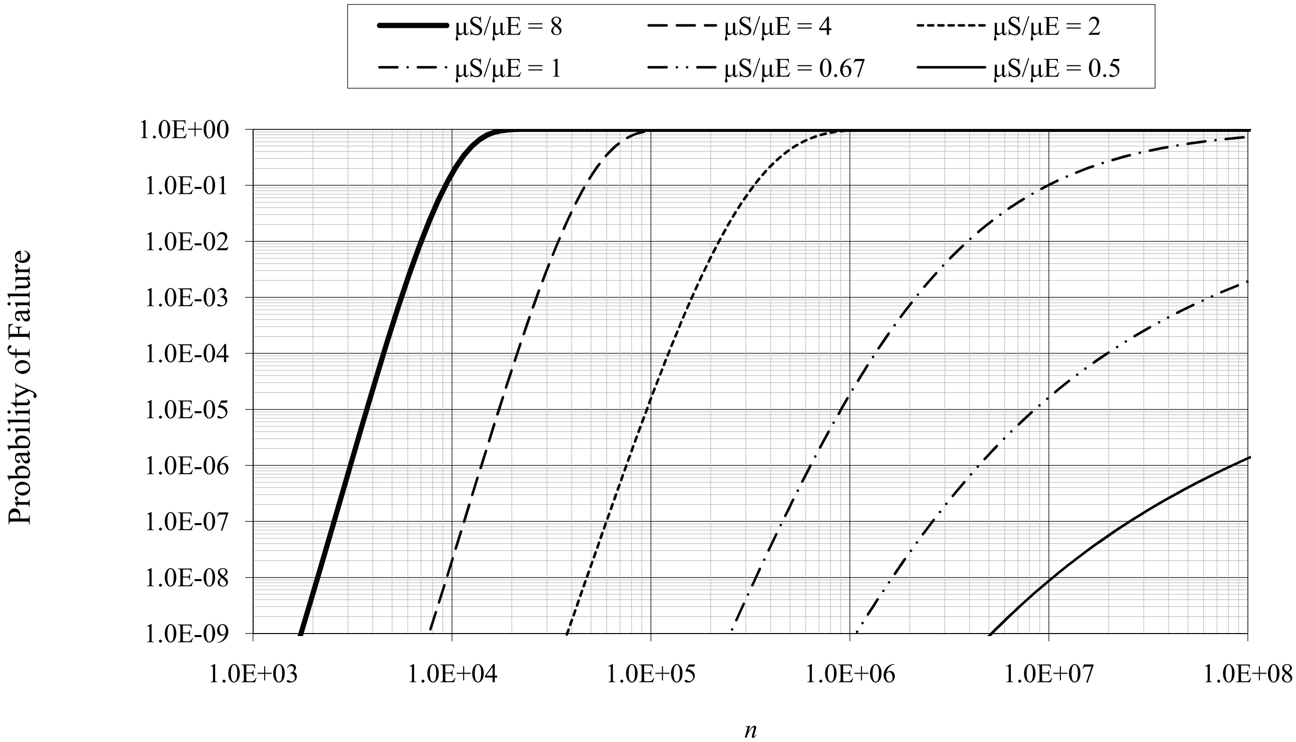

Figure 7: Probability of Failure versus Cycles with $k = 640000$, $m = 2$, $\alpha = 0.92$, $COV_S = 0.10$, and $COV_E = 0.10$ (Normal Stress-Normal Strength case)

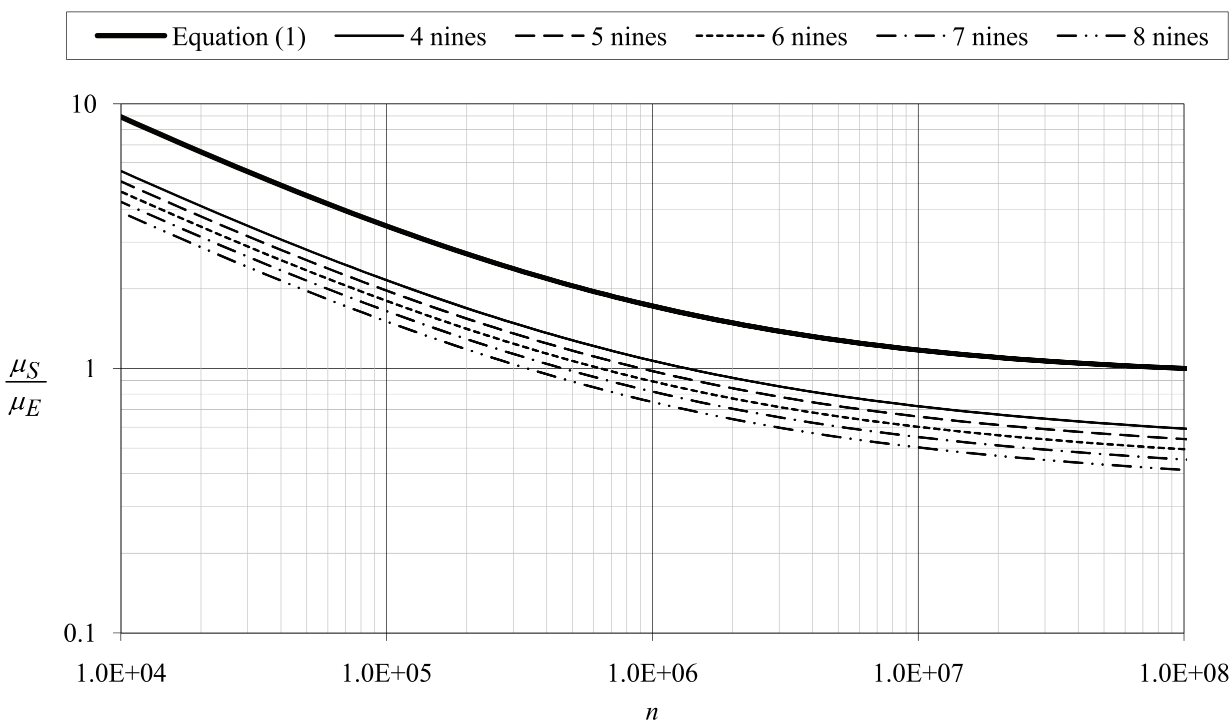

Figure 8: Ratio of Mean Stress to Mean Strength as a Function of Cycles with $k = 640000$, $m = 2$, $\alpha = 0.92$, $COV_S = 0.10$, and $COV_E = 0.10$ for $R = 0.9_4$, 0.9_5, 0.9_6, 0.9_7, and 0.9_8 (Normal Stress-Normal Strength case)

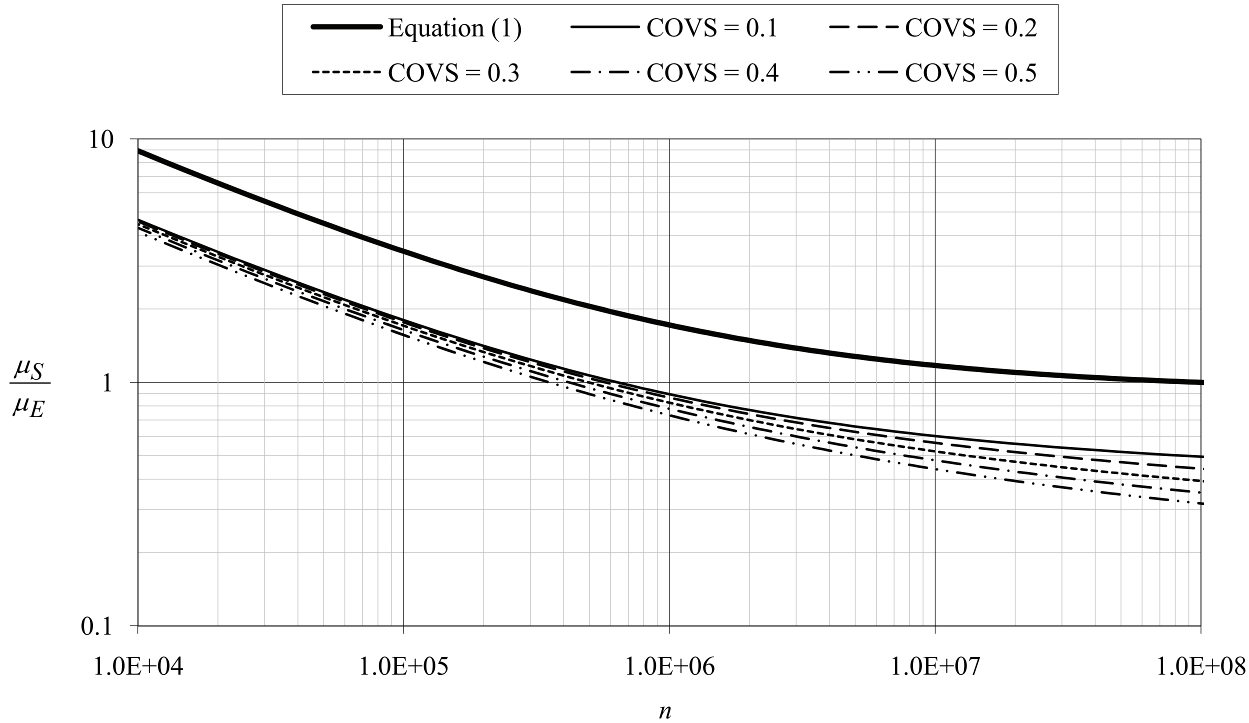

Figure 9: Sensitivity of Ratio of Mean Stress to Mean Strength to changes in COV_S with $R = 0.9_6$, $k = 640000$, $m = 2$, $\alpha = 0.92$, and $COV_E = 0.10$ (Normal Stress-Normal Strength case)

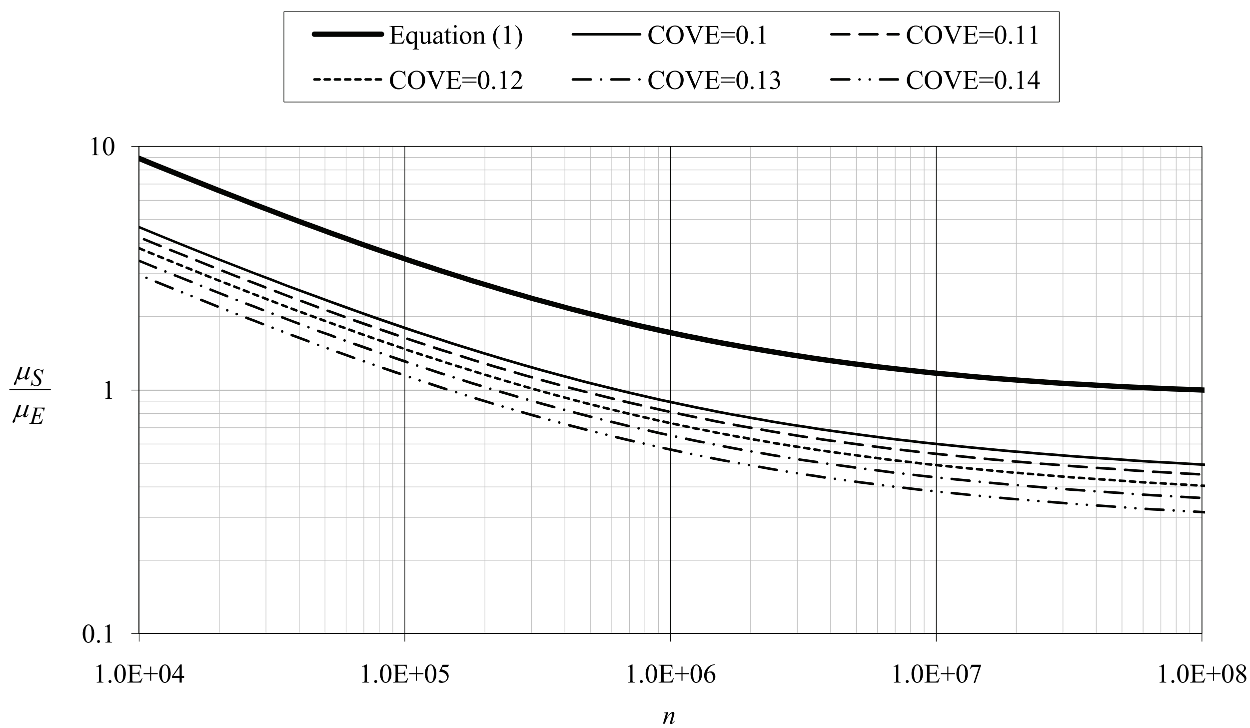

Figure 10: Sensitivity of Ratio of Mean Stress to Mean Strength to changes in COV_E with $R = 0.9_6$, $k = 640000$, $m = 2$, $\alpha = 0.92$, and $COV_S = 0.10$ (Normal Stress-Normal Strength case)

FUTURE EXTENSIONS

Readers attempting to implement the methods proposed in this paper to other realistic problems may immediately confront a desire for several future extensions. For example, the current method provides results for the special case where the material constant m is restricted to be an integer, but it has not been universal practice to restrict all material characterizations to that special case. Furthermore, it is not reasonable to expect industry to change current practice or legacy (and often proprietary) material characterizations for the sake of the current method because such restrictions would likely increase each corresponding strength COV and standard working-curve reduction. For the case of a non-integer material constant, m, several options are available. Primarily, one might consider numerical integration of the relevant a and b functions. For $m \geq 2$, the author has found that an *MS Excel*® implementation of the classic trapezoid rule with as few as 500 integration steps can provide adequate results for such cases.

For the case of a functions, Figure 12 shows results of a 500 step trapezoid integration of Eqn. (6) for non-integer values of m. Although the author remains interested in finding analytical solutions for a and b functions with non-integer m values, inspection of the semi-log graph in Figure 12 leads the author to propose that an alternative to numerical integration would be semi-log interpolation between evaluations of a and b functions at integer values of m such as in Eqn. (21), where $0 < \delta m < 1$.

$$\log a(z_0, m + \delta m) \approx \log a(z_0, m) + \delta m[\log a(z_0, m+1) - \log a(z_0, m)] \tag{21}$$

An estimate of the upper bound of the a function is desired to provide a lower bound on the number of cycles allowed per Eqn. (7), and hence a lower bound on the predicted life for a given reliability or FRA. The author asserts that Eqn. (21) provides such an upper bound of the a function. The curves shown in Figure 13 demonstrate that the value of $a(z0, m + \delta m)$ predicted by Eqn. (21) exceeds that calculated using numerical integration at every value of m under consideration due to the positive curvature of the a function with respect to m; note that the curvature is positive in Figure 12. As one might expect from inspection of the curve shapes in Figure 13, quadratic semi-log interpolation schemes further reduce the difference from the numerical integration results. However, the author does not assert that quadratic or higher-order polynomial interpolation schemes should be viewed as an upper limit of the a function.

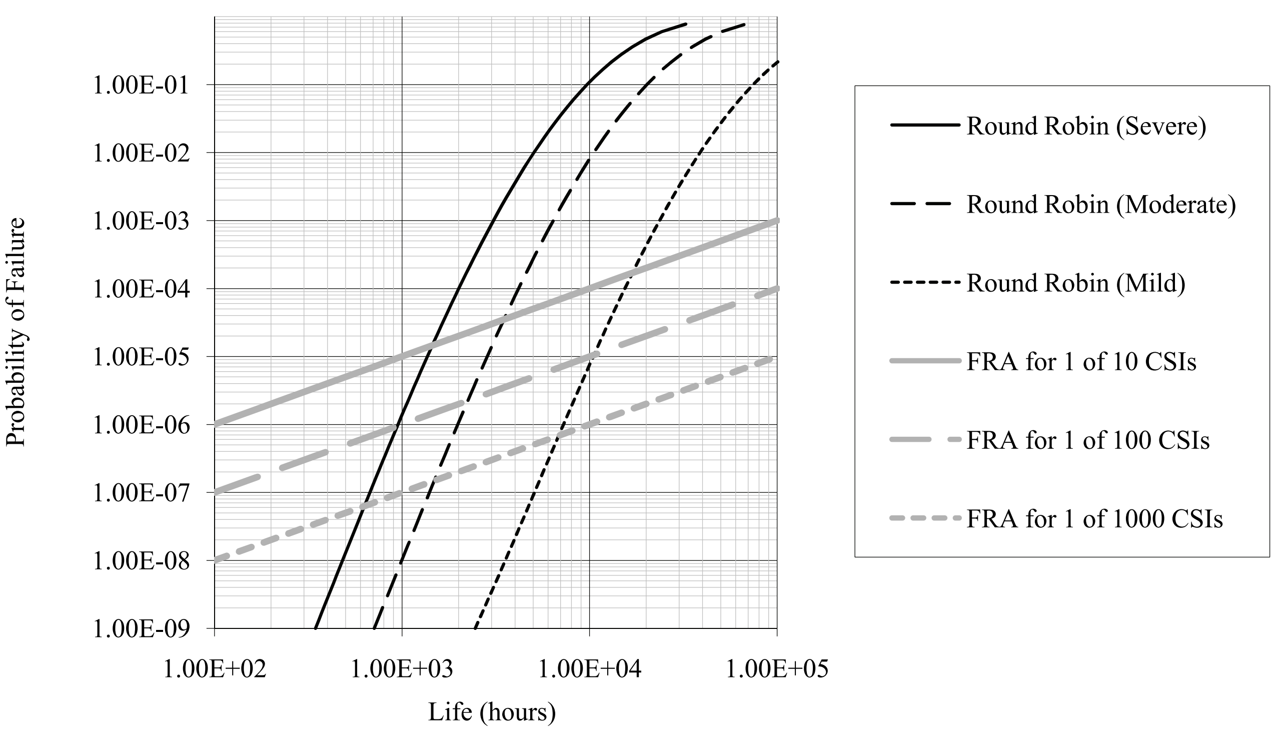

Figure 11: Solution to Arden Round Robin problem from [2] enhanced with a Failure Rate Allocation (FRA) for each case of 10, 100, and 1000 Critical Safety Items (CSIs)

(Weibull Stress-Normal Strength case with $k = 640000$, $m = 2$, $\alpha = 0.92$, $\mu_E = 1000$ psi, $COV_E = 0.10$, and $\beta_S = 4$, various cycle-counted load conditions, and usage classified as severe, moderate, or mild)

Figure 13 also demonstrates that increasing the number of integration steps for low values of m tends to decrease the difference in results predicted by the two methods. Additional effort should also address precision issues related to numerical integration of a and b functions for non-integer values of m. For example, Figure 13 illustrates the potential benefits of specialized numerical integration methods for a and b functions. Candidate techniques include variable transformation, variable-step-size numerical methods, higher-order Newton-Cotes formulas, and Gauss quadrature [6]. Development of parametric methods of plotting and transforming the integrand in Eqns. (6) and (9) may enable optimization of numerical methods for evaluating $a(z_0, m)$ and $b(w_0, m, \beta)$, respectively. It is recommended that selected numerical methods should be compared to the methods of this paper for test cases with integer values of m.

Table 3: Detailed Solution of the Arden Round Robin Problem, includes Original Problem and Failure Rate Allocation (FRA) Enhancement for 100 Critical Safety Items (CSIs)

Round Robin Problem Data for Severe Usage Spectrum Case ($\beta_p = 4$, $\delta_p = 0$)			E_{crit} = 538.4492308 R = 0.999998039 (life of 1056.2 hours)	E_{crit} = 523.4949291 R = 0.999999056 (FRA 1 of 100 CSIs)
Peak Load (95%)	% Time, ξ_p	η_p, Weibull Parameter	$\xi_p\left[\frac{1}{k}\left(\frac{\alpha}{\bullet}\right)^m\right] b(\bullet, m, \beta_p)$	$\xi_p\left[\frac{1}{k}\left(\frac{\alpha}{\bullet}\right)^m\right] b(\bullet, m, \beta_p)$
2300	0.0005	1748.244161	4.05357E-09	4.38198E-09
1840	0.0015	1398.595329	6.35026E-09	6.91793E-09
1750	0.002	1330.185775	7.24786E-09	7.91307E-09
1400	0.006	1064.148620	1.02836E-08	1.1371E-08
1380	0.002	1048.946497	3.25376E-09	3.60151E-09
1300	0.004	988.1380042	5.2097E-09	5.79323E-09
1050	0.008	798.1114649	4.25611E-09	4.84123E-09
1040	0.012	790.5104034	6.10539E-09	6.95383E-09
920	0.0025	699.2976645	6.85982E-10	7.97E-10
900	0.007	684.0955414	1.70205E-09	1.98601E-09
780	0.016	592.8828025	1.60466E-09	1.93976E-09
720	0.021	547.2764331	1.16749E-09	1.45032E-09
700	0.01	532.0743100	4.42485E-10	5.55927E-10
600	0.011	456.0636942	1.08423E-10	1.48123E-10
540	0.028	410.4573248	7.03321E-11	1.04956E-10
520	0.02	395.2552017	2.80292E-11	4.35566E-11
500	0.0255	380.0530785	1.82632E-11	2.97868E-11
480	0.033	364.8509554	1.08357E-11	1.87341E-11
460	0.0035	349.6488323	4.60028E-13	8.5383E-13
400	0.0765	304.0424628	1.8602E-13	4.81449E-13
360	0.035	273.6382165	9.85589E-16	3.80627E-15
360	0.044	273.6382165	1.23903E-15	4.78503E-15
350	0.014	266.0371550	8.88978E-17	3.94136E-16
300	0.102	228.0318471	7.1562E-21	9.52183E-20
260	0.028	197.6276008	0	-2.67655E-24†
240	0.055	182.4254777	0	0
200	0.1275	152.0212314	0	0
180	0.049	136.8191083	0	0
120	0.077	91.21273885	0	0
100	0.1785	76.01061571	0	0
$\frac{1}{n} = \sum_{p=1}^{p_{\max}} \xi_p\left[\frac{1}{k}\left(\frac{\alpha}{\bullet}\right)^m\right] b(\bullet, m, \beta_p)$			5.25995E-08	5.88493E-08
n			1.901160E+07	1.699255E+07
Life (hours) $= n/(RPM \times 60)$			1056.199992	944.0307599

† Although not a significant error in this case, $b(2.4370, 2, 4)$ returns -3.9493E-16 due to numerical precision issues.

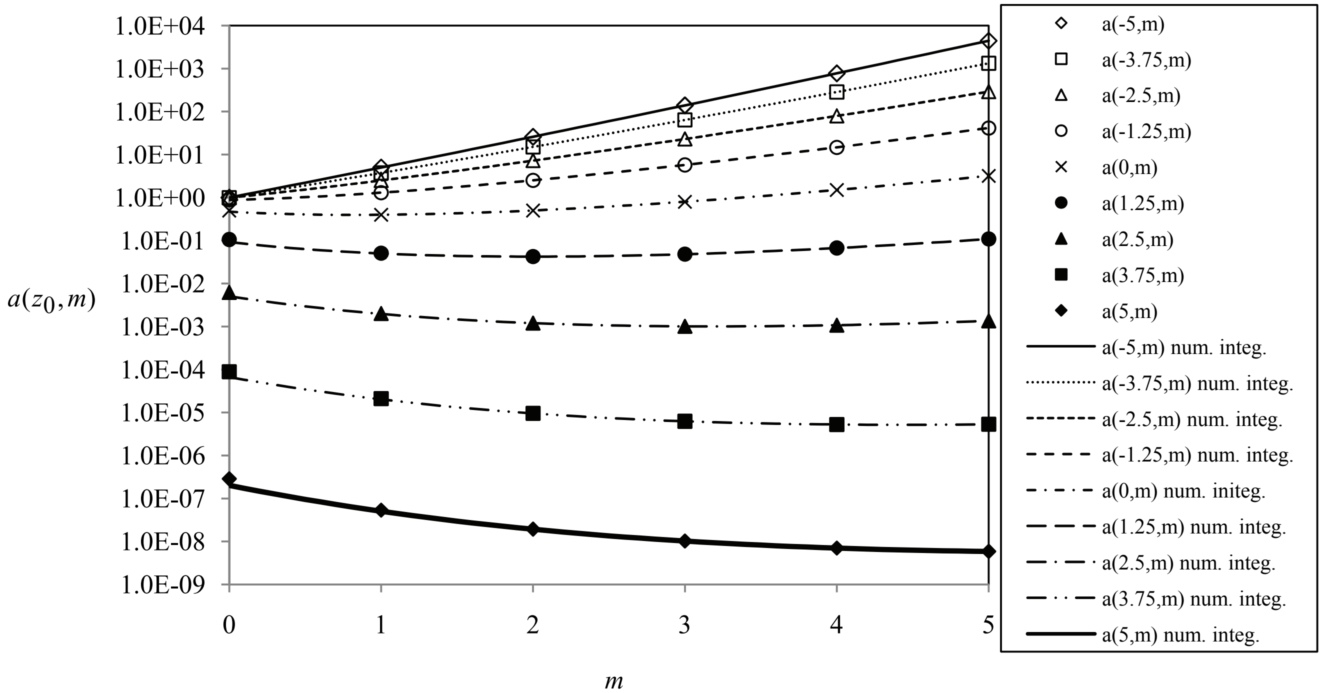

Figure 12: Comparison of 500 Step Trapezoid Integration Results with Equation (20)

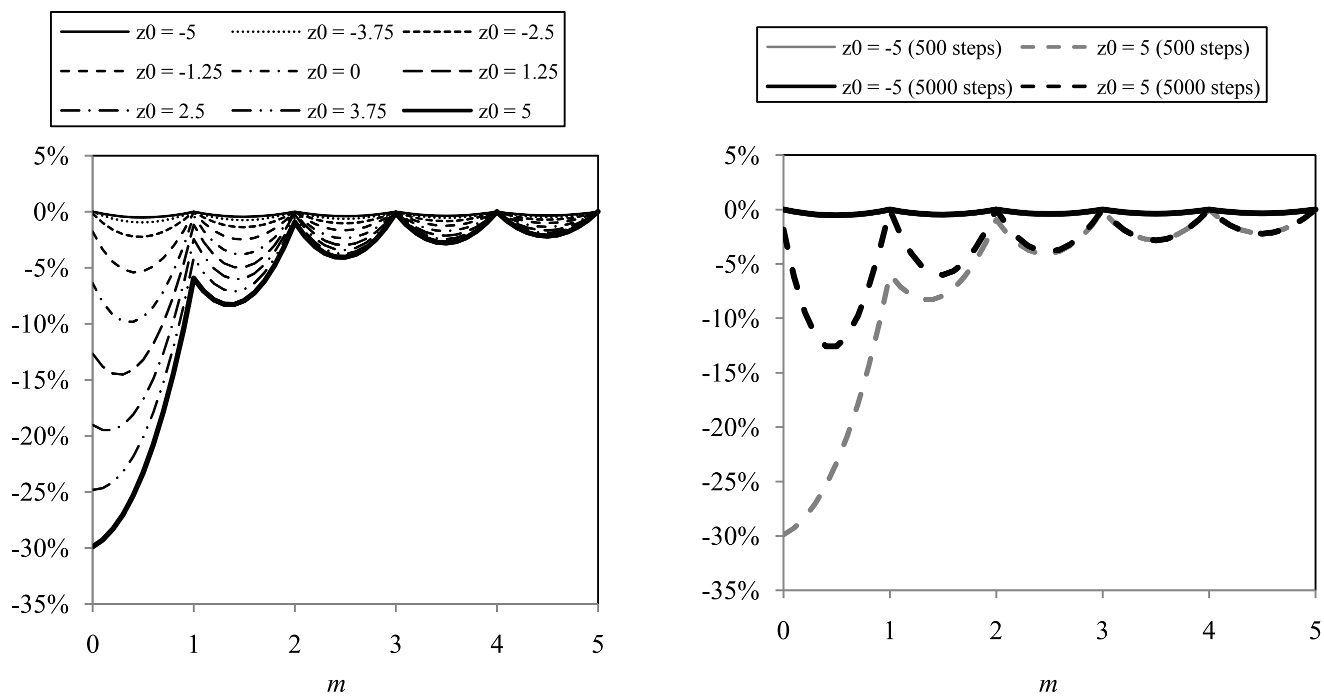

Figure 13: Percent Difference Between Trapezoid Integration and Semi-Log Interpolation per Equation (21)

It should be noted that the current implementation of a and b functions is known to exhibit a loss of precision due to taking differences within the algorithm, such as the negative function return reported in the footnote to Table 3. Although these errors have only been noticed at extremely low evaluations of the a and b functions where the terms may safely be neglected, alternate implementations could be pursued for greater numerical precision. It is noted however that the current precision of a and b function evaluation for integer values of m exceeds that available from Monte Carlo methods.

Readers may desire to evaluate so-called inverse a and b functions. The (forward) functions presented herein are useful for calculating life for a given reliability, as demonstrated in Table 3. However, one may wish to find the reliability for a given life. Pending future development of inverse function evaluations, such problems require an iterative method where reliability is adjusted until the target life is obtained. Although the speed of evaluation makes iterative methods practical, methods to directly evaluate the inverse functions could be pursued.

Finally, as discussed above, further work would be required before applying a and b function methods to the case of variable usage. Variable usage has historically been an important factor in considering reliability, as discussed in [7]. Concepts such as composite-worst-case usage have traditionally been incorporated into the design usage spectrum in order to handle such variation in usage. It is noted, however, that current concepts for Condition Based Maintenance (CBM) of helicopter systems routinely include regime recognition components intended to accurately record usage. As stated above, the deterministic usage methods discussed in the present work are directly applicable to all cases of known usage, including the case of assessing component damage accumulated to date due to accurately monitored past usage.

CONCLUSIONS

Novel analytical methods have been derived, namely the $a(z_0,m)$ and $b(w_0,m,\beta)$ functions, which are useful in solving cumulative-damage fatigue problems with Normal- or Weibull-Distributed Random-Independent Stress and Random-Fixed Strength. The classical analytical functions $\phi(\bullet)$, $erfc(\bullet)$, and $\Gamma(\bullet,\bullet)$ have been used to evaluate the a and b functions in the special case where the material constant m is restricted to be an integer. The advantages of the resulting methods have been demonstrated on realistic example problems, namely high speed and precision. In fact, the methods derived herein show great promise for use in computation intensive design optimization schemes. Possible future extensions to the present work have also been presented.

APPENDIX A

The upper incomplete gamma function

$$\Gamma(a,x) = \int_x^{\infty} t^{a-1}\exp(-t)dt$$

may be calculated easily using standard mathematical software packages, for example:

MS Excel®:
EXP(GAMMALN(A1)*(1-GAMMADIST(A2,A1,1,TRUE))
where cell A1 contains “a” and cell A2 contains “x”

Mathematica®: Gamma [a, x]

MATLAB®: gamma(a)*gammainc(x,a,'upper')

APPENDIX B

The following “BentonABVer1.bas” text file has been implemented in *MS Excel®* using the *Visual Basic®* editor. It is provided “as-is” (without software support, warranty, indemnity, or any other form of guarantee – “user beware”) with hopes that readers would find it useful in verifying and extending the results of this paper. Pending further study of precision related issues, it is recommended to investigate the significance and source of any negative a and b function evaluations fully before attributing the error to precision issues.

```
Attribute VB_Name = "BentonABVer1"
Function BentonA(z0 As Double, m As Double) As Variant
'  Benton "a function" for calculating Miner's rule fatigue damage
'     with normal load
'
'  z0   = (alpha*E-mu)/sigma, where alpha, E, and m are fatigue curve
'         parameters and mu and sigma describe the normal load
'         distribution
'  m    = a fatigue curve parameter, note that in this version of the
'         function, m is an integer between 0 and 5
'
'  REB 15 Jan 08, version 1.0
'
   Dim em As Integer, pm(5) As Double, qm(5) As Double
   em = Int(m)
   pm(0) = 1#
   qm(0) = 0#
   pm(1) = -z0
   qm(1) = 1#
   pm(2) = z0 ^ 2 + 1#
   qm(2) = -z0
   pm(3) = -1# * z0 ^ 3 - 3# * z0
   qm(3) = z0 ^ 2 + 2#
   pm(4) = z0 ^ 4 + 6# * z0 ^ 2 + 3#
   qm(4) = -1# * z0 ^ 3 - 5# * z0
   pm(5) = -1# * z0 ^ 5 - 10# * z0 ^ 3 - 15# * z0
   qm(5) = z0 ^ 4 + 9# * z0 ^ 2 + 8#
   If m <> em Or em > 5 Then
      BentonA = "error -- must use integer m < 6"
   Else
      BentonA = pm(em) * (1 - Application.WorksheetFunction.NormSDist(z0)) _
      + qm(em) * 1 / Sqr(2 * Application.WorksheetFunction.Pi) * Exp(-0.5 * z0 ^ 2)
   End If
End Function
```

```
Function BentonB(w0 As Double, m As Double, beta As Double) As Variant
'  Benton "b function" for calculating Miner's rule fatigue damage
'     with Weibull load
'
'  w0   = (alpha*E-delta)/(eta-delta), where alpha, E, and m are
'         fatigue curve parameters and beta, eta, and delta describe
'         the Weibull load distribution (shape, scale, and location),
'         note that we define the three parameter Weibull cumulative
'         distribution function as
'         F=1-exp{-[(x-delta)/(eta-delta)]^beta}
'  m    = a fatigue curve parameter, note that in this version of the
'         function, m is an integer greater or equal to 0
'  beta = the shape factor for the Weibull load distribution
'
'  REB 15 Jan 08, version 1.0
'
   Dim total As Double, l As Integer, em As Integer, term As Double
   em = Int(m)
   If m <> em Then
      BentonB = "error -- must use integer m"
   Else
      total = 0#
      For l = 0 To em
         If w0 > 0 Then
            term = Exp(Application.WorksheetFunction.GammaLn((l + beta) / beta)) _
            * (1 - Application.WorksheetFunction.GammaDist(w0 ^ beta, _
            (l + beta) / beta, 1, True))
         Else
            term = Exp(Application.WorksheetFunction.GammaLn((l + beta) / beta))
         End If
         total = total + (Application.WorksheetFunction.Fact(em) _
         / Application.WorksheetFunction.Fact(l) _
         / Application.WorksheetFunction.Fact(em - l)) * (-w0) ^ (em - l) * term
      Next
      BentonB = total
   End If
End Function
```

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This research is intended to enhance the safety and effectiveness of US Army soldiers protecting our freedom. The author is grateful for these heroes and humbled by their sacrifices. The author also wishes to thank co-workers for their inspiration and encouragement in this effort, especially Dr. Jung-Hua Chang, Mr. Ed Martin, Mr. Martin Rogers, and Mr. Carlton Smith.

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