Exercises on orthogonal vectors and subspaces

Problem 16.1: (4.1 #7. *Introduction to Linear Algebra:* Strang) For every system of m equations with no solution, there are numbers $y_1, ..., y_m$ that multiply the equations so they add up to $0 = 1$. This is called *Fredholm's Alternative*:

> Exactly one of these problems has a solution:
> $A\mathbf{x} = \mathbf{b}$ OR $A^T\mathbf{y} = \mathbf{0}$ with $\mathbf{y}^T\mathbf{b} = 1$.

If $\mathbf{b}$ is not in the column space of A it is not orthogonal to the nullspace of A^T. Multiply the equations $x_1 - x_2 = 1$, $x_2 - x_3 = 1$ and $x_1 - x_3 = 1$ by numbers y_1, y_2 and y_3 chosen so that the equations add up to $0 = 1$.

et $y_1 = 1$, $y_2 = 1$ and $y_3 = -1$. Then the left-hand side of the sum of the equations is:

$$(x_1 - x_2) + (x_2 - x_3) - (x_1 - x_3) = x_1 - x_2 + x_2 - x_3 + x_3 - x_1 = 0,$$

and the right-hand side verifies that $\mathbf{y}^T\mathbf{b} = 1$:

$$1 + 1 - 1 = 1.$$

Problem 16.2: (4.1#32.) Suppose I give you four nonzero vectors $\mathbf{r}$, $\mathbf{n}$, $\mathbf{c}$ and $\mathbf{l}$ in $\mathbb{R}^2$.

a) What are the conditions for those to be bases for the four fundamental subspaces $C(A^T)$, $N(A)$, $C(A)$, and $N(A^T)$ of a 2 by 2 matrix?

b) What is one possible matrix A?