1998 HMMT Advanced Topics Event

1. Evaluate $\sin(1998°+237°)\sin(1998°-1653°)$.

2. How many values of x, $-19 < x < 98$ satisfy $\cos^2 x + 2\sin^2 x = 1$?

3. Find the sum of the infinite series $1 + 2(\frac{1}{1998}) + 3(\frac{1}{1998})^2 + 4(\frac{1}{1998})^3 + \ldots$

4. Find the range of $f(A) = \dfrac{\sin A(3\cos^2 A + \cos^4 A + 3\sin^2 A + \sin^2 A\cos^2 A)}{\tan A(\sec A - \sin A\tan A)}$ if $A \neq \dfrac{n\pi}{2}$.

5. How many positive integers less than 1998 are relatively prime to 1547? (Two integers are relatively prime if they have no common factors besides 1.)

6. In the diagram below, how many distinct paths are there from January 1 to December 31, moving from one adjacent dot to the next either to the right, down, or diagonally down to the right?

Jan. 1 ->	*	*	*	*	*	*	*	*	*	*	
	*	*		*	*		*	*		*	
	*	*	*	*	*	*	*	*	*	*	
	*			*			*	*		*	
	*	*	*	*	*	*	*	*	*	*	<-Dec. 31

7. The Houson Association of Mathematics Educators decides to hold a grand forum on mathematics education and invites a number of politicians from around the United States to participate. Around lunch time the politicians decide to play a game. In this game, players can score 19 points for pegging the coordinator of the gathering with a spit ball, 9 points for downing an entire cup of the forum's interpretation of coffee, or 8 points for quoting more than three consecutive words from the speech Senator Bobbo delivered before lunch. What is the product of the two greatest scores that a player *cannot* score in this game?

8. Given any two positive real numbers x and y, then $x \Diamond y$ is a positive real number defined in terms of x and y by some fixed rule. Suppose the operation $x \Diamond y$ satisfies the equations

$(x \cdot y)\Diamond y = x(y\Diamond y)$ and $(x\Diamond 1)\Diamond x = x\Diamond 1$ for all $x, y > 0$. Given that $1\Diamond 1 = 1$, find $19\Diamond 98$.

9. Bob's Rice ID number has six digits, each a number from 1 to 9, and any digit can be used any number of times. The ID number satifies the following property: the first two digits is a number divisible by 2, the first three digits is a number divisible by 3, etc. so that the ID number itself is divisible by 6. One ID number that satifies this condition is 123252. How many different possibilities are there for Bob's ID number?

10. In the fourth annual Swirled Series, the Oakland Alphas are playing the San Francisco Gammas. The first game is played in San Francisco and succeeding games alternate in location. San Francisco has a 50% chance of winning their home games, while Oakland has a probability of 60% of winning at home. Normally, the series will stretch on forever until one team gets a three game lead, in which case they are declared the winners. However, after each game in San Francisco there is a 50% chance of an earthquake, which will cause the series to end with the team that has won more games declared the winner. What is the probability that the Gammas will win?