1998 Harvard/MIT Math Tournament

CALCULUS Answer Sheet

Name: ______________________________

School: ______________________ **Grade:** ______

1	______________	**6**	______________
2	______________	**7**	______________
3	______________	**8**	______________
4	______________	**9**	______________
5	______________	**10**	______________

TOTAL: ________

CALCULUS

Question One. [3 points]

Farmer Tim is lost in the densely-forested Cartesian plane. Starting from the origin he walks a sinusoidal path in search of home; that is, after t minutes he is at position $(t, \sin t)$.

Five minutes after he sets out, Alex enters the forest at the origin and sets out in search of Tim. He walks in such a way that after he has been in the forest for m minutes, his position is $(m, \cos t)$.

What is the greatest distance between Alex and Farmer Tim while they are walking in these paths?

Question Two. [3 points]

A cube with sides 1m in length is filled with water, and has a tiny hole through which the water drains into a cylinder of radius 1m.

If the water level in the cube is falling at a rate of $1\, {}^{cm}\!/_{s}$, at what rate is the water level in the cylinder rising?

Question Three. [4 points]

Find the area of the region bounded by the graphs of $y = x^2$, $y = x$, and $x = 2$.

Question Four. [4 points]

Let $f(x) = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \ldots$, for $-1 \le x \le 1$. Find $\sqrt{e^{\int_0^1 f(x)\,dx}}$.

Question Five. [5 points]

Evaluate $\lim_{x\to 1} x^{\frac{x}{\sin(1-x)}}$.

Question Six. [5 points]

Edward, the author of this test, had to escape from prison to work in the grading room today. He stopped to rest at a place 1,875 feet from the prison and was spotted by a guard with a crossbow.

The guard fired an arrow with an initial velocity of $10\,{}^{ft}\!/_{s}$. At the same time, Edward started running away with an acceleration of $1\,{}^{ft}\!/_{s^2}$. Assuming that air resistance causes the arrow to decelerate at ${}^{ft}\!/_{s^2}$ and that it does hit Edward, how fast was the arrow moving at the moment of impact (in ${}^{ft}\!/_{s}$)?

Question Seven. [5 points]

A parabola is inscribed in equilateral triangle *ABC* of side length 1 in the sense that *AC* and *BC* are tangent to the parabola at *A* and *B*, respectively:

Find the area between *AB* and the parabola.

Question Eight. [6 points]

Find the slopes of all lines passing through the origin and tangent to the curve $y^2 = x^3 + 39x - 35$.

Question Nine. [7 points]

Evaluate $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^{n-1}}$

Question Ten. [8 points]

Let *S* be the locus of all points (x, y) in the first quadrant such that $\frac{x}{t} + \frac{y}{1-t} = 1$ for some t with $0<t<1$. Find the area of *S*.