

**Calculus Test**  
**Harvard-MIT Math Tournament**  
March 3, 2001

1. A sequence of ants walk from  $(0, 0)$  to  $(1, 0)$  in the plane. The  $n$ th ant walks along  $n$  semicircles of radius  $\frac{1}{n}$  with diameters lying along the line from  $(0, 0)$  to  $(1, 0)$ . Let  $L_n$  be the length of the path walked by the  $n$ th ant. Compute  $\lim_{n \rightarrow \infty} L_n$ .

2. The polynomial  $3x^5 - 250x^3 + 735x$  is interesting because it has the maximum possible number of relative extrema and points of inflection at integer lattice points for a quintic polynomial. What is the sum of the  $x$ -coordinates of these points?

3. A balloon that blows up in the shape of a perfect cube is being blown up at a rate such that at time  $t$  fortnights, it has surface area  $6t$  square furlongs. At how many cubic furlongs per fortnight is the air being pumped in when the surface area is 144 square furlongs?

4. What is the size of the largest rectangle that can be drawn inside of a 3-4-5 right triangle with one of the rectangle's sides along one of the legs of the triangle?

5. Same as question 4, but now we want one of the rectangle's sides to be along the hypotenuse.

6. The graph of  $x^2 - (y - 1)^2 = 1$  has one tangent line with positive slope that passes through  $(x, y) = (0, 0)$ . If the point of tangency is  $(a, b)$ , find  $\sin^{-1}(\frac{a}{b})$  in radians.

7. Find the coefficient of  $x^{12}$  in the Maclaurin series (i.e. Taylor series around  $x = 0$ ) for  $\frac{1}{1-3x+2x^2}$ .

8. Evaluate  $\sum_{n=0}^{\infty} \cot^{-1}(n^2 + n + 1)$ .

9. On the planet Lemniscate, the people use the elliptic table of elements, a far more advanced version of our periodic table. They're not very good at calculus, though, so they've asked for your help. They know that Kr is somewhat radioactive and deteriorates into Pl, a very unstable element that deteriorates to form the stable element As. They started with a block of Kr of size 10 and nothing else. (Their units don't translate into English, sorry.) and

nothing else. At time  $t$ , they let  $x(t)$  be the amount of Kr,  $y(t)$  the amount of Pl, and  $z(t)$  the amount of As. They know that  $x'(t) = -x$ , and that, in the absence of Kr,  $y'(t) = -2y$ . Your job is to find at what time  $t$  the quantity of Pl will be largest. You should assume that the entire amount of Kr that deteriorates has turned into Pl.

10. Evaluate the definite integral  $\int_{-1}^{+1} \frac{2u^{332} + u^{998} + 4u^{1664} \sin u^{691}}{1 + u^{666}} du$ .