

Team Round Solutions
Harvard-MIT Math Tournament
March 3, 2001

1. How many digits are in the base two representation of $10!$ (factorial)?

2. On a certain unidirectional highway, trucks move steadily at 60 miles per hour spaced $1/4$ of a mile apart. Cars move steadily at 75 miles per hour spaced 3 seconds apart. A lone sports car weaving through traffic at a steady forward speed passes two cars between each truck it passes. How quickly is it moving in miles per hour?

3. What is the 18th digit after the decimal point of $\frac{10000}{9899}$?

4. P is a polynomial. When P is divided by $x - 1$, the remainder is -4 . When P is divided by $x - 2$, the remainder is -1 . When P is divided by $x - 3$, the remainder is 4 . Determine the remainder when P is divided by $x^3 - 6x^2 + 11x - 6$.

5. Find all x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that $1 - \sin^4 x - \cos^2 x = \frac{1}{16}$.

6. What is the radius of the smallest sphere in which 4 spheres of radius 1 will fit?

7. The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. The Lucas numbers are defined by $L_1 = 1$, $L_2 = 2$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$.

Calculate $\frac{\prod_{n=1}^{15} \frac{F_{2n}}{F_n}}{\prod_{n=1}^{13} L_n}$.

8. Express $\frac{\sin 10 + \sin 20 + \sin 30 + \sin 40 + \sin 50 + \sin 60 + \sin 70 + \sin 80}{\cos 5 \cos 10 \cos 20}$ without using trigonometric functions.

9. Compute $\sum_{i=1}^{\infty} \frac{ai}{a^i}$ for $a > 1$.

10. Define a monic irreducible polynomial with integral coefficients to be a polynomial with leading coefficient 1 that cannot be factored, and the prime factorization of a polynomial with leading coefficient 1 as the factorization into monic irreducible polynomials. How many

not necessarily distinct monic irreducible polynomials are there in the prime factorization of $(x^8 + x^4 + 1)(x^8 + x + 1)$ (for instance, $(x + 1)^2$ has two prime factors)?

11. Define $a^? = (a - 1)/(a + 1)$ for $a \neq -1$. Determine all real values N for which $(N^?)^? = \tan 15$.

12. All subscripts in this problem are to be considered modulo 6, that means for example that ω_7 is the same as ω_1 . Let $\omega_1, \dots, \omega_6$ be circles of radius r , whose centers lie on a regular hexagon of side length 1. Let P_i be the intersection of ω_i and ω_{i+1} that lies further from the center of the hexagon, for $i = 1, \dots, 6$. Let $Q_i, i = 1 \dots 6$, lie on ω_i such that Q_i, P_i, Q_{i+1} are colinear. Find the number of possible values of r .