

IXth Annual Harvard-MIT Mathematics Tournament
Saturday 25 February 2006

Individual Round: Algebra Test

1. Larry can swim from Harvard to MIT (with the current of the Charles River) in 40 minutes, or back (against the current) in 45 minutes. How long does it take him to *row* from Harvard to MIT, if he rows the return trip in 15 minutes? (Assume that the speed of the current and Larry's swimming and rowing speeds relative to the current are all constant.) Express your answer in the format mm:ss.
2. Find all real solutions (x, y) of the system $x^2 + y = 12 = y^2 + x$.
3. The train schedule in Hummut is hopelessly unreliable. Train A will enter Intersection X from the west at a random time between 9:00 am and 2:30 pm; each moment in that interval is equally likely. Train B will enter the same intersection from the north at a random time between 9:30 am and 12:30 pm, independent of Train A; again, each moment in the interval is equally likely. If each train takes 45 minutes to clear the intersection, what is the probability of a collision today?
4. Let a_1, a_2, \dots be a sequence defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Find

$$\sum_{n=1}^{\infty} \frac{a_n}{4^{n+1}}.$$

5. Tim has a working analog 12-hour clock with two hands that run continuously (instead of, say, jumping on the minute). He also has a clock that runs really slow—at half the correct rate, to be exact. At noon one day, both clocks happen to show the exact time. At any given instant, the hands on each clock form an angle between 0° and 180° inclusive. At how many times during that day are the angles on the two clocks equal?
6. Let a, b, c be the roots of $x^3 - 9x^2 + 11x - 1 = 0$, and let $s = \sqrt{a} + \sqrt{b} + \sqrt{c}$. Find $s^4 - 18s^2 - 8s$.
7. Let

$$f(x) = x^4 - 6x^3 + 26x^2 - 46x + 65.$$

Let the roots of $f(x)$ be $a_k + ib_k$ for $k = 1, 2, 3, 4$. Given that the a_k, b_k are all integers, find $|b_1| + |b_2| + |b_3| + |b_4|$.

8. Solve for all complex numbers z such that $z^4 + 4z^2 + 6 = z$.
9. Compute the value of the infinite series

$$\sum_{n=2}^{\infty} \frac{n^4 + 3n^2 + 10n + 10}{2^n \cdot (n^4 + 4)}$$

10. Determine the maximum value attained by

$$\frac{x^4 - x^2}{x^6 + 2x^3 - 1}$$

over real numbers $x > 1$.