

10th Annual Harvard-MIT Mathematics Tournament
Saturday 24 February 2007

Individual Round: Algebra Test

1. [3] Compute

$$\left\lfloor \frac{2007! + 2004!}{2006! + 2005!} \right\rfloor.$$

(Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x .)

2. [3] Two reals x and y are such that $x - y = 4$ and $x^3 - y^3 = 28$. Compute xy .
3. [4] Three real numbers x, y , and z are such that $(x+4)/2 = (y+9)/(z-3) = (x+5)/(z-5)$. Determine the value of x/y .
4. [4] Compute

$$\frac{2^3 - 1}{2^3 + 1} \cdot \frac{3^3 - 1}{3^3 + 1} \cdot \frac{4^3 - 1}{4^3 + 1} \cdot \frac{5^3 - 1}{5^3 + 1} \cdot \frac{6^3 - 1}{6^3 + 1}.$$

5. [5] A convex quadrilateral is determined by the points of intersection of the curves $x^4 + y^4 = 100$ and $xy = 4$; determine its area.
6. [5] Consider the polynomial $P(x) = x^3 + x^2 - x + 2$. Determine all real numbers r for which there exists a complex number z not in the reals such that $P(z) = r$.
7. [5] An infinite sequence of positive real numbers is defined by $a_0 = 1$ and $a_{n+2} = 6a_n - a_{n+1}$ for $n = 0, 1, 2, \dots$. Find the possible value(s) of a_{2007} .
8. [6] Let $A := \mathbb{Q} \setminus \{0, 1\}$ denote the set of all rationals other than 0 and 1. A function $f : A \rightarrow \mathbb{R}$ has the property that for all $x \in A$,

$$f(x) + f\left(1 - \frac{1}{x}\right) = \log|x|.$$

Compute the value of $f(2007)$.

9. [7] The complex numbers $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the four distinct roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$

10. [8] The polynomial $f(x) = x^{2007} + 17x^{2006} + 1$ has distinct zeroes r_1, \dots, r_{2007} . A polynomial P of degree 2007 has the property that $P\left(r_j + \frac{1}{r_j}\right) = 0$ for $j = 1, \dots, 2007$. Determine the value of $P(1)/P(-1)$.