10^{th} Annual Harvard-MIT Mathematics Tournament
Saturday 24 February 2007

Individual Round: Calculus Test

1. [**3**] Compute:
$$\lim_{x\to 0} \quad \frac{x^2}{1-\cos(x)}$$

2. [**3**] Determine the real number a having the property that $f(a) = a$ is a relative minimum of $f(x) = x^4 - x^3 - x^2 + ax + 1$.

3. [**4**] Let a be a positive real number. Find the value of a such that the definite integral
$$\int_a^{a^2} \frac{\mathrm{d}x}{x+\sqrt{x}}$$
achieves its smallest possible value.

4. [**4**] Find the real number α such that the curve $f(x) = e^x$ is tangent to the curve $g(x) = \alpha x^2$.

5. [**5**] The function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x^2)f''(x) = f'(x)f'(x^2)$ for all real x. Given that $f(1) = 1$ and $f'''(1) = 8$, determine $f'(1) + f''(1)$.

6. [**5**] The elliptic curve $y^2 = x^3 + 1$ is tangent to a circle centered at $(4, 0)$ at the point (x_0, y_0). Determine the sum of all possible values of x_0.

7. [**5**] Compute
$$\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1) \cdot (n+1)!}.$$

8. [**6**] Suppose that ω is a primitive 2007^{th} root of unity. Find $\left(2^{2007} - 1\right) \sum_{j=1}^{2006} \frac{1}{2-\omega^j}$.

   For this problem only, you may express your answer in the form $m \cdot n^k + p$, where m, n, k, and p are positive integers. Note that a number z is a *primitive n^{th} root of unity* if $z^n = 1$ and n is the smallest number amongst $k = 1, 2, \ldots, n$ such that $z^k = 1$.

9. [**7**] g is a twice differentiable function over the positive reals such that
$$g(x) + 2x^3 g'(x) + x^4 g''(x) = 0 \quad \text{for all positive reals } x. \qquad (1)$$
$$\lim_{x\to\infty} x g(x) = 1 \qquad (2)$$
Find the real number $\alpha > 1$ such that $g(\alpha) = 1/2$.

10. [**8**] Compute
$$\int_0^{\infty} \frac{e^{-x}\sin(x)}{x} dx$$