10^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 24 February 2007

Individual Round: General Test, Part 1

1. [**2**] Michael has 16 white socks, 3 blue socks, and 6 red socks in a drawer. Ever the lazy college student, he has overslept and is late for his favorite team's season-opener. Because he is now in such a rush to get from Harvard to Foxborough, he randomly takes socks from the drawer (one at a time) until he has a pair of the same color. What is the largest number of socks he could possibly withdraw in this fashion?

2. [**2**] Rectangle $ABCD$ has side lengths $AB = 12$ and $BC = 5$. Let P and Q denote the midpoints of segments AB and DP, respectively. Determine the area of triangle CDQ.

3. [**3**] A, B, C, and D are points on a circle, and segments $\overline{AC}$ and $\overline{BD}$ intersect at P, such that $AP = 8$, $PC = 1$, and $BD = 6$. Find BP, given that $BP < DP$.

4. [**3**] Let a and b be integer solutions to $17a + 6b = 13$. What is the smallest possible positive value for $a - b$?

5. [**4**] Find the smallest positive integer that is twice a perfect square and three times a perfect cube.

6. [**4**] The positive integer n is such that the numbers 2^n and 5^n start with the same digit when written in decimal notation; determine this common leading digit.

7. [**4**] Jack, Jill, and John play a game in which each randomly picks and then replaces a card from a standard 52 card deck, until a spades card is drawn. What is the probability that Jill draws the spade? (Jack, Jill, and John draw in that order, and the game repeats if no spade is drawn.)

8. [**5**] Determine the largest positive integer n such that there exist positive integers x, y, z so that

$$n^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx + 3x + 3y + 3z - 6$$

9. [**6**] I have four distinct rings that I want to wear on my right hand hand (five distinct fingers.) One of these rings is a Canadian ring that must be worn on a finger by itself, the rest I can arrange however I want. If I have two or more rings on the same finger, then I consider different orders of rings along the same finger to be different arrangements. How many different ways can I wear the rings on my fingers?

10. [**7**] $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the complex roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$