10^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 24 February 2007

Guts Round

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10^{th} HARVARD-MIT MATHEMATICS TOURNAMENT, 24 FEBRUARY 2007 — GUTS ROUND

Note that there are just 36 problems in the Guts round this year.

1. [**5**] Define the sequence of positive integers a_n recursively by $a_1 = 7$ and $a_n = 7^{a_{n-1}}$ for all $n \geq 2$. Determine the last two digits of a_{2007}.

2. [**5**] A candy company makes 5 colors of jellybeans, which come in equal proportions. If I grab a random sample of 5 jellybeans, what is the probability that I get exactly 2 distinct colors?

3. [**5**] The equation $x^2 + 2x = i$ has two complex solutions. Determine the product of their real parts.

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4. [**6**] A sequence consists of the digits $122333444455555\ldots$ such that the each positive integer n is repeated n times, in increasing order. Find the sum of the 4501st and 4052nd digits of this sequence.

5. [**6**] Compute the largest positive integer such that $\frac{2007!}{2007^n}$ is an integer.

6. [**6**] There are three video game systems: the Paystation, the WHAT, and the ZBoz2π, and none of these systems will play games for the other systems. Uncle Riemann has three nephews: Bernoulli, Galois, and Dirac. Bernoulli owns a Paystation and a WHAT, Galois owns a WHAT and a ZBoz2π, and Dirac owns a ZBoz2π and a Paystation. A store sells 4 different games for the Paystation, 6 different games for the WHAT, and 10 different games for the ZBoz2π. Uncle Riemann does not understand the difference between the systems, so he walks into the store and buys 3 random games (not necessarily distinct) and randomly hands them to his nephews. What is the probability that each nephew receives a game he can play?

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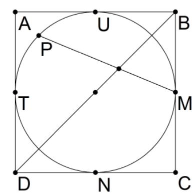
7. [**7**] A student at Harvard named Kevin
Was counting his stones by 11
He messed up n times
And instead counted 9s
And wound up at 2007.

How many values of n could make this limerick true?

8. [**7**] A circle inscribed in a square,
Has two chords as shown in a pair.
It has radius 2,
And P bisects TU.
The chords' intersection is where?



Answer the question by giving the distance of the point of intersection from the center of the circle.

9. [**7**] I ponder some numbers in bed,
All products of three primes I've said,
Apply ϕ they're still fun:
now Elev'n cubed plus one.
What numbers could be in my head?

$n = 37^2 \cdot 3 \ldots$
$\phi(n) =$
$11^3 + 1$?

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10. [**8**] Let A_{12} denote the answer to problem 12. There exists a unique triple of digits (B, C, D) such that $10 > A_{12} > B > C > D > 0$ and

$$\overline{A_{12}BCD} - \overline{DCBA_{12}} = \overline{BDA_{12}C},$$

where $\overline{A_{12}BCD}$ denotes the four digit base 10 integer. Compute $B + C + D$.

11. [**8**] Let A_{10} denote the answer to problem 10. Two circles lie in the plane; denote the lengths of the internal and external tangents between these two circles by x and y, respectively. Given that the product of the radii of these two circles is $15/2$, and that the distance between their centers is A_{10}, determine $y^2 - x^2$.

12. [**8**] Let A_{11} denote the answer to problem 11. Determine the smallest prime p such that the arithmetic sequence $p, p + A_{11}, p + 2A_{11}, \ldots$ begins with the largest possible number of primes.

There is just one triple of possible (A_{10}, A_{11}, A_{12}) of answers to these three problems. Your team will receive credit only for answers matching these. (So, for example, submitting a wrong answer for problem 11 will not alter the correctness of your answer to problem 12.)

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13. [**9**] Determine the largest integer n such that $7^{2048}-1$ is divisible by 2^n.

14. [**9**] We are given some similar triangles. Their areas are $1^2, 3^2, 5^2 \ldots,$ and 49^2. If the smallest triangle has a perimeter of 4, what is the sum of all the triangles' perimeters?

15. [**9**] Points $A, B,$ and C lie in that order on line ℓ, such that $AB = 3$ and $BC = 2$. Point H is such that CH is perpendicular to ℓ. Determine the length CH such that $\angle AHB$ is as large as possible.

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16. [**10**] Let ABC be a triangle with $AB = 7, BC = 9,$ and $CA = 4$. Let D be the point such that $AB \parallel CD$ and $CA \parallel BD$. Let R be a point within triangle BCD. Lines ℓ and m going through R are parallel to CA and AB respectively. Line ℓ meets AB and BC at P and P' respectively, and m meets CA and BC at Q and Q' respectively. If S denotes the largest possible sum of the areas of triangles $BPP', RP'Q'$, and CQQ', determine the value of S^2.

17. [**10**] During the regular season, Washington Redskins achieve a record of 10 wins and 6 losses. Compute the probability that their wins came in three streaks of consecutive wins, assuming that all possible arrangements of wins and losses are equally likely. (For example, the record LLWWWWWLWWL-WWWLL contains three winning streaks, while WWWWWWWLLLLLLWWW has just two.)

18. [**10**] Convex quadrilateral $ABCD$ has right angles $\angle A$ and $\angle C$ and is such that $AB = BC$ and $AD = CD$. The diagonals AC and BD intersect at point M. Points P and Q lie on the circumcircle of triangle AMB and segment CD, respectively, such that points $P, M,$ and Q are collinear. Suppose that $m\angle ABC = 160^\circ$ and $m\angle QMC = 40^\circ$. Find $MP \cdot MQ$, given that $MC = 6$.

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19. [**10**] Define $x \star y = \frac{\sqrt{x^2+3xy+y^2-2x-2y+4}}{xy+4}$. Compute

$$((\cdots((2007 \star 2006) \star 2005) \star \cdots) \star 1).$$

20. [**10**] For a a positive real number, let x_1, x_2, x_3 be the roots of the equation $x^3 - ax^2 + ax - a = 0$. Determine the smallest possible value of $x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3$.

21. [**10**] Bob the bomb-defuser has stumbled upon an active bomb. He opens it up, and finds the red and green wires conveniently located for him to cut. Being a seasoned member of the bomb-squad, Bob quickly determines that it is the green wire that he should cut, and puts his wirecutters on the green wire. But just before he starts to cut, the bomb starts to count down, ticking every second. Each time the bomb ticks, starting at time $t = 15$ seconds, Bob panics and has a certain chance to move his wirecutters to the other wire. However, he is a rational man even when panicking, and has a $\frac{1}{2t^2}$ chance of switching wires at time t, regardless of which wire he is about to cut. When the bomb ticks at $t = 1$, Bob cuts whatever wire his wirecutters are on, without switching wires. What is the probability that Bob cuts the green wire?

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22. [**12**] The sequence $\{a_n\}_{n\geq 1}$ is defined by $a_{n+2} = 7a_{n+1} - a_n$ for positive integers n with initial values $a_1 = 1$ and $a_2 = 8$. Another sequence, $\{b_n\}$, is defined by the rule $b_{n+2} = 3b_{n+1} - b_n$ for positive integers n together with the values $b_1 = 1$ and $b_2 = 2$. Find $\gcd(a_{5000}, b_{501})$.

23. [**12**] In triangle ABC, $\angle ABC$ is obtuse. Point D lies on side AC such that $\angle ABD$ is right, and point E lies on side AC between A and D such that BD bisects $\angle EBC$. Find CE, given that $AC = 35, BC = 7$, and $BE = 5$.

24. [**12**] Let x, y, n be positive integers with $n > 1$. How many ordered triples (x, y, n) of solutions are there to the equation $x^n - y^n = 2^{100}$?

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25. [**12**] Two real numbers x and y are such that $8y^4 + 4x^2y^2 + 4xy^2 + 2x^3 + 2y^2 + 2x = x^2 + 1$. Find all possible values of $x + 2y^2$.

26. [**12**] $ABCD$ is a cyclic quadrilateral in which $AB = 4, BC = 3, CD = 2$, and $AD = 5$. Diagonals AC and BD intersect at X. A circle ω passes through A and is tangent to BD at X. ω intersects AB and AD at Y and Z respectively. Compute YZ/BD.

27. [**12**] Find the number of 7-tuples $(n_1, \ldots, n_7)$ of integers such that

$$\sum_{i=1}^{7} n_i^6 = 96957.$$

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28. [**15**] Compute the circumradius of cyclic hexagon $ABCDEF$, which has side lengths $AB = BC = 2, CD = DE = 9$, and $EF = FA = 12$.

29. [**15**] A sequence $\{a_n\}_{n\geq 1}$ of positive reals is defined by the rule $a_{n+1}a_{n-1}^5 = a_n^4 a_{n-2}^2$ for integers $n > 2$ together with the initial values $a_1 = 8$ and $a_2 = 64$ and $a_3 = 1024$. Compute

$$\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}}$$

30. [**15**] $ABCD$ is a cyclic quadrilateral in which $AB = 3, BC = 5, CD = 6$, and $AD = 10$. M, I, and T are the feet of the perpendiculars from D to lines AB, AC, and BC respectively. Determine the value of MI/IT.

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31. [**18**] A sequence $\{a_n\}_{n\geq 0}$ of real numbers satisfies the recursion $a_{n+1} = a_n^3 - 3a_n^2 + 3$ for all positive integers n. For how many values of a_0 does $a_{2007} = a_0$?

32. [**18**] Triangle ABC has $AB = 4, BC = 6$, and $AC = 5$. Let O denote the circumcenter of ABC. The circle Γ is tangent to and surrounds the circumcircles of triangles AOB, BOC, and AOC. Determine the diameter of Γ.

33. [**18**] Compute

$$\int_1^2 \frac{9x+4}{x^5+3x^2+x}dx.$$

(No, your TI-89 doesn't know how to do this one. Yes, the end is near.)

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34. [?] *The Game.* Eric and Greg are watching their new favorite TV show, *The Price is Right.* Bob Barker recently raised the intellectual level of his program, and he begins the latest installment with bidding on following question: How many Carmichael numbers are there less than 100,000?

Each team is to list one nonnegative integer not greater than 100,000. Let X denote the answer to Bob's question. The teams listing N, a maximal bid (of those submitted) not greater than X, will receive N points, and all other teams will neither receive nor lose points. (A Carmichael number is an odd composite integer n such that n divides $a^{n-1} - 1$ for all integers a relatively prime to n with $1 < a < n$.)

35. [$\leq$ **25**] *The Algorithm.* There are thirteen broken computers situated at the following set S of thirteen points in the plane:

$A = (1, 10)$	$B = (976, 9)$	$C = (666, 87)$
$D = (377, 422)$	$E = (535, 488)$	$F = (775, 488)$
$G = (941, 500)$	$H = (225, 583)$	$I = (388, 696)$
$J = (3, 713)$	$K = (504, 872)$	$L = (560, 934)$
	$M = (22, 997)$	

At time $t = 0$, a repairman begins moving from one computer to the next, traveling continuously in straight lines at unit speed. Assuming the repairman begins and A and fixes computers instantly, what path does he take to minimize the *total downtime* of the computers? List the points he visits in order. Your score will be $\lfloor \frac{N}{40} \rfloor$, where

$$N = 1000 + \lfloor \text{the optimal downtime} \rfloor - \lfloor \text{your downtime} \rfloor,$$

or 0, whichever is greater. By total downtime we mean the sum

$$\sum_{P \in S} t_P,$$

where t_P is the time at which the repairman reaches P.

36. [**25**] *The Marathon.* Let ω denote the incircle of triangle ABC. The segments BC, CA, and AB are tangent to ω at D, E, and F, respectively. Point P lies on EF such that segment PD is perpendicular to BC. The line AP intersects BC at Q. The circles ω_1 and ω_2 pass through B and C, respectively, and are tangent to AQ at Q; the former meets AB again at X, and the latter meets AC again at Y. The line XY intersects BC at Z. Given that $AB = 15, BC = 14$, and $CA = 13$, find $\lfloor XZ \cdot YZ \rfloor$.