10^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 24 February 2007

Team Round: A Division

Σ, τ, and You: Fun at Fraternities? [270]

A *number theoretic function* is a function whose domain is the set of positive integers. A *multiplicative* number theoretic function is a number theoretic function f such that $f(mn) = f(m)f(n)$ for all pairs of relatively prime positive integers m and n. Examples of multiplicative number theoretic functions include σ, τ, ϕ, and μ, defined as follows. For each positive integer n,

- The *sum-of-divisors function*, $\sigma(n)$, is the sum of all positive integer divisors of n. If $p_1, \ldots, p_i$ are distinct primes and $e_1, \ldots, e_i$ are positive integers,

$$\sigma\left(p_1^{e_1} \cdots p_i^{e_i}\right) = \prod_{k=1}^{i} \left(1 + p_k + \cdots + p_k^{e_k}\right) = \prod_{k=1}^{i} \frac{p_k^{e_k+1} - 1}{p_k - 1}.$$

- The *divisor function*, $\tau(n)$, is the number of positive integer divisors of n. It can be computed by the formula
$$\tau\left(p_1^{e_1} \cdots p_i^{e_i}\right) = (e_1 + 1) \cdots (e_i + 1),$$
where $p_1, \ldots, p_i$ and $e_1, \ldots, e_i$ are as above.

- Euler's *totient* function, $\phi(n)$, is the number of positive integers $k \leq n$ such that k and n are relatively prime. For $p_1, \ldots, p_i$ and $e_1, \ldots, e_i$ as above, the phi function satisfies

$$\phi\left(p_1^{e_1} \cdots p_i^{e_i}\right) = \prod_{k=1}^{i} p_k^{e_k - 1}(p_k - 1).$$

- The *Möbius function*, $\mu(n)$, is equal to either 1, -1, or 0. An integer is called *square-free* if it is not divisible by the square of any prime. If n is a square-free positive integer having an even number of distinct prime divisors, $\mu(n) = 1$. If n is a square-free positive integer having an odd number of distinct prime divisors, $\mu(n) = -1$. Otherwise, $\mu(n) = 0$.

The Möbius function has a number of peculiar properties. For example, if f and g are number theoretic functions such that
$$g(n) = \sum_{d|n} f(d),$$
for all positive integers n, then
$$f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right).$$
This is known as *Möbius inversion.* In proving the following problems, *you may use any of the preceding assertions without proving them. You may also cite the results of previous problems, even if you were unable to prove them.*

1. [**15**] Evaluate the functions $\phi(n), \sigma(n)$, and $\tau(n)$ for $n = 12, n = 2007$, and $n = 2^{2007}$.

2. [**20**] Solve for the positive integer(s) n such that $\phi(n^2) = 1000\phi(n)$.

3. [**25**] Prove that for every integer n greater than 1,
$$\sigma(n)\phi(n) \leq n^2 - 1.$$
When does equality hold?

4. [**25**] Let F and G be two multiplicative functions, and define for positive integers n,

$$H(n) = \sum_{d|n} F(d)G\left(\frac{n}{d}\right).$$

The number theoretic function H is called the *convolution* of F and G. Prove that H is multiplicative.

5. [**30**] Prove the identity

$$\sum_{d|n} \tau(d)^3 = \left(\sum_{d|n} \tau(d)\right)^2.$$

6. [**25**] Show that for positive integers n,

$$\sum_{d|n} \phi(d) = n.$$

7. [**25**] Show that for positive integers n,

$$\sum_{d|n} \frac{\mu(d)}{d} = \frac{\phi(n)}{n}.$$

8. [**30**] Determine with proof, a simple closed form expression for

$$\sum_{d|n} \phi(d)\tau\left(\frac{n}{d}\right).$$

9. [**35**] Find all positive integers n such that

$$\sum_{k=1}^{n} \phi(k) = \frac{3n^2+5}{8}.$$

10. [**40**] Find all pairs (n, k) of positive integers such that

$$\sigma(n)\phi(n) = \frac{n^2}{k}.$$

Grab Bag - Miscellaneous Problems [130]

11. [**30**] Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$\begin{aligned} f(x)f(y) &= f(x) + f(y) - f(xy) \\ 1 + f(x+y) &= f(xy) + f(x)f(y) \end{aligned}$$

for all rational numbers x, y.

12. [**30**] Let $ABCD$ be a cyclic quadrilateral, and let P be the intersection of its two diagonals. Points R, S, T, and U are feet of the perpendiculars from P to sides AB, BC, CD, and AD, respectively. Show that quadrilateral $RSTU$ is bicentric if and only if $AC \perp BD$. (Note that a quadrilateral is called *inscriptible* if it has an incircle; a quadrilateral is called *bicentric* if it is both cyclic and inscriptible.)

13. [**30**] Find all nonconstant polynomials $P(x)$, with real coefficients and having only real zeros, such that $P(x+1)P(x^2-x+1) = P(x^3+1)$ for all real numbers x.

14. [**40**] Find an explicit, closed form formula for

$$\sum_{k=1}^{n} \frac{k \cdot (-1)^k \cdot \binom{n}{k}}{n+k+1}.$$