10^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 24 February 2007

Team Round: B Division

Compute $(x-a)(x-b)\cdots(x-z)$ - Short Answer [200]

For this section, your team should give only the anwers to the problems.

1. [**20**] Find the sum of the positive integer divisors of 2^{2007}.

2. [**20**] The four sides of quadrilateral $ABCD$ are equal in length. Determine the perimeter of $ABCD$ given that it has area 120 and $AC = 10$.

3. [**20**] Five people are crowding into a booth against a wall at a noisy restaurant. If at most three can fit on one side, how many seating arrangements accomodate them all?

4. [**20**] Thomas and Michael are just two people in a large pool of well qualified candidates for appointment to a problem writing committee for a prestigious college math contest. It is 40 times more likely that both will serve if the size of the committee is increased from its traditional 3 members to a whopping n members. Determine n. (Each person in the pool is equally likely to be chosen.)

5. [**20**] The curves $y = x^2(x-3)^2$ and $y = (x^2-1)(x-2)$ intersect at a number of points in the real plane. Determine the sum of the x-coordinates of these points of intersection.

6. [**20**] Andrew has a fair six sided die labeled with 1 through 6 as usual. He tosses it repeatedly, and on every third roll writes down the number facing up as long as it is not the 6. He stops as soon as the last two numbers he has written down are squares or one is a prime and the other is a square. What is the probability that he stops after writing squares consecutively?

7. [**20**] Three positive reals x, y, and z are such that

$$\begin{aligned} x^2 + 2(y-1)(z-1) &= 85 \\ y^2 + 2(z-1)(x-1) &= 84 \\ z^2 + 2(x-1)(y-1) &= 89. \end{aligned}$$

   Compute $x + y + z$.

8. [**20**] Find the *positive* real number(s) x such that $\frac{1}{2}\left(3x^2-1\right) = \left(x^2-50x-10\right)\left(x^2+25x+5\right)$.

9. [**20**] Cyclic quadrilateral $ABCD$ has side lengths $AB = 1, BC = 2, CD = 3$, and $AD = 4$. Determine AC/BD.

10. [**20**] A positive real number x is such that

$$\sqrt[3]{1-x^3} + \sqrt[3]{1+x^3} = 1.$$

   Find x^2.

Adult Acorns - Gee, I'm a Tree! [200]

In this section of the team round, your team will derive some basic results concerning *tangential* quadrilaterals. Tangential quadrilaterals have an *incircle*, or a circle lying within them that is tangent to all four sides. If a quadrilateral has an incircle, then the center of this circle is the *incenter* of the quadrilateral. As you shall see, tangential quadrilaterals are related to cyclic quadrilaterals. For reference, a review of cyclic quadrilaterals is given at the end of this section.

Your answers for this section of the team test should be proofs. *Note that you may use any standard facts about cyclic quadrilaterals, such as those listed at the end of this test, without proving them. Additionally, you may cite the results of previous problems, even if you were unable to prove them.*

For these problems, $ABCD$ is a tangential quadrilateral having incenter I. For the first three problems, the point P is constructed such that triangle PAB is similar to triangle IDC and lies outside $ABCD$.

1. [**30**] Show that $PAIB$ is cyclic by proving that $\angle IAP$ is supplementary to $\angle PBI$.
2. [**40**] Show that triangle PAI is similar to triangle BIC. Then conclude that

$$PA = \frac{PI}{BC} \cdot BI.$$

3. [**25**] Deduce from the above that

$$\frac{BC}{AD} \cdot \frac{AI}{BI} \cdot \frac{DI}{CI} = 1.$$

4. [**25**] Show that $AB + CD = AD + BC$. Use the above to conclude that for some positive number α,

$$AB = \alpha \cdot \left(\frac{AI}{CI} + \frac{BI}{DI} \right) \qquad BC = \alpha \cdot \left(\frac{BI}{DI} + \frac{CI}{AI} \right)$$
$$CD = \alpha \cdot \left(\frac{CI}{AI} + \frac{DI}{BI} \right) \qquad DA = \alpha \cdot \left(\frac{DI}{BI} + \frac{AI}{CI} \right).$$

5. [**40**] Show that

$$AB \cdot BC = BI^2 + \frac{AI \cdot BI \cdot CI}{DI}.$$

6. [**40**] Let the incircle of $ABCD$ be tangent to sides AB, BC, CD, and AD at points P, Q, R, and S, respectively. Show that $ABCD$ is cyclic if and only if $PR \perp QS$.

A brief review of cyclic Quadrilaterals.

The following discussion of cyclic quadrilaterals is included for reference. Any of the results given here may be cited without proof in your writeups.

A *cyclic quadrilateral* is a quadrilateral whose four vertices lie on a circle called the *circumcircle* (the circle is unique if it exists.) If a quadrilateral has a circumcircle, then the center of this circumcircle is called the *circumcenter* of the quadrilateral. For a convex quadrilateral $ABCD$, the following are equivalent:

- Quadrilateral $ABCD$ is cyclic;
- $\angle ABD = \angle ACD$ (or $\angle BCA = \angle BDA$, etc.);
- Angles $\angle ABC$ and $\angle CDA$ are *supplementary*, that is, $m\angle ABC + m\angle CDA = 180°$ (or angles $\angle BCD$ and $\angle BAD$ are supplementary);

Cyclic quadrilaterals have a number of interesting properties. A cyclic quadrilateral $ABCD$ satisfies

$$AC \cdot BD = AB \cdot CD + AD \cdot BC,$$

a result known as *Ptolemy's theorem.* Another result, typically called *Power of a Point*, asserts that given a circle ω, a point P anywhere in the plane of ω, and a line ℓ through P intersecting ω at points A and B, the value of $AP \cdot BP$ is independent of ℓ; i.e., if a second line ℓ' through P intersects ω at A' and B', then $AP \cdot BP = A'P \cdot B'P$. This second theorem is proved via similar triangles. Say P lies outside of ω, that ℓ and ℓ' are as before and that A and A' lie on segments BP and $B'P$ respectively. Then triangle $AA'P$ is similar to triangle $B'BP$ because the triangles share an angle at P and we have

$$m\angle AA'P = 180° - m\angle B'A'A = m\angle ABB' = m\angle PBB'.$$

The case where $A = B$ is valid and describes the tangents to ω. A similar proof works for P inside ω.