11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Individual Round: Calculus Test

1. [**3**] Let $f(x) = 1 + x + x^2 + \cdots + x^{100}$. Find $f'(1)$.

2. [**3**] Let ℓ be the line through $(0,0)$ and tangent to the curve $y = x^3 + x + 16$. Find the slope of ℓ.

3. [**4**] Find all $y > 1$ satisfying $\displaystyle\int_1^y x \ln x \, dx = \frac{1}{4}$.

4. [**4**] Let a, b be constants such that $\displaystyle\lim_{x \to 1} \frac{(\ln(2-x))^2}{x^2 + ax + b} = 1$. Determine the pair (a, b).

5. [**4**] Let $f(x) = \sin^6\left(\frac{x}{4}\right) + \cos^6\left(\frac{x}{4}\right)$ for all real numbers x. Determine $f^{(2008)}(0)$ (i.e., f differentiated 2008 times and then evaluated at $x = 0$).

6. [**5**] Determine the value of $\displaystyle\lim_{n \to \infty} \sum_{k=0}^{n} \binom{n}{k}^{-1}$.

7. [**5**] Find p so that $\lim_{x \to \infty} x^p \left(\sqrt[3]{x+1} + \sqrt[3]{x-1} - 2\sqrt[3]{x}\right)$ is some non-zero real number.

8. [**7**] Let $T = \displaystyle\int_0^{\ln 2} \frac{2e^{3x} + e^{2x} - 1}{e^{3x} + e^{2x} - e^x + 1} \, dx$. Evaluate e^T.

9. [**7**] Evaluate the limit $\displaystyle\lim_{n \to \infty} n^{-\frac{1}{2}\left(1 + \frac{1}{n}\right)} \left(1^1 \cdot 2^2 \cdot \cdots \cdot n^n\right)^{\frac{1}{n^2}}$.

10. [**8**] Evaluate the integral $\displaystyle\int_0^1 \ln x \ln(1-x) \, dx$.