12^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 21 February 2009

Individual Round: Combinatorics Test

1. [**3**] How many ways can the integers from -7 to 7 inclusive be arranged in a sequence such that the absolute value of the numbers in the sequence does not decrease?

2. [**3**] Two jokers are added to a 52 card deck and the entire stack of 54 cards is shuffled randomly. What is the expected number of cards that will be strictly between the two jokers?

3. [**4**] How many rearrangements of the letters of "HMMTHMMT" do not contain the substring "HMMT"? (For instance, one such arrangement is HMMHMTMT.)

4. [**4**] How many functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ satisfy $f(f(x)) = f(x)$ for all $x \in \{1, 2, 3, 4, 5\}$?

5. [**4**] Let $s(n)$ denote the number of 1's in the binary representation of n. Compute
$$\frac{1}{255} \sum_{0 \leq n < 16} 2^n (-1)^{s(n)}.$$

6. [**5**] How many sequences of 5 positive integers (a, b, c, d, e) satisfy $abcde \leq a + b + c + d + e \leq 10$?

7. [**5**] Paul fills in a 7×7 grid with the numbers 1 through 49 in a random arrangement. He then erases his work and does the same thing again, to obtain two different random arrangements of the numbers in the grid. What is the expected number of pairs of numbers that occur in either the same row as each other or the same column as each other in both of the two arrangements?

8. [**7**] There are 5 students on a team for a math competition. The math competition has 5 subject tests. Each student on the team must choose 2 distinct tests, and each test must be taken by exactly two people. In how many ways can this be done?

9. [**7**] The squares of a 3×3 grid are filled with positive integers such that 1 is the label of the upper-leftmost square, 2009 is the label of the lower-rightmost square, and the label of each square divides the one directly to the right of it and the one directly below it. How many such labelings are possible?

10. [**8**] Given a rearrangement of the numbers from 1 to n, each pair of consecutive elements a and b of the sequence can be either increasing (if $a < b$) or decreasing (if $b < a$). How many rearrangements of the numbers from 1 to n have exactly two increasing pairs of consecutive elements? Express your answer in terms of n.