Geometry Solutions

Harvard-MIT Math Tournament

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1. $m\angle EDX = 180^\circ - m\angle LAX - m\angle ELA = 180^\circ - m\angle LAX - (180^\circ - m\angle AXE) = \boxed{80^\circ}$.

2. If the angle in radians is θ, then Anne travels $R\theta$ and Lisa travels $(R-r) + r\theta + (R-r)$. Setting these equal yields $R(\theta - 2) = r(\theta - 2)$, so $\theta = \boxed{2 \text{ radians}}$.

3. $m\angle MLD = \frac{1}{2}\overset{\frown}{AB} = \boxed{45^\circ}$.

4. The cylinder has a cross-sectional area π times greater than the cube, so the water raises $\frac{1}{\pi}$ times as quickly in the cylinder as it lowers in the cube; that is, at $\boxed{\frac{1}{\pi}\frac{\text{cm}}{\text{s}}}$.

5. If the new circle has radius r, then the distance from its center to E can be computed either as $1 + r$ or $(2 - r)\sqrt{2}$. Setting these equal yields $r = \frac{2\sqrt{2}-1}{\sqrt{2}+1} = \boxed{5 - 3\sqrt{2}}$.

6. The central circle has area $\pi\left(\frac{1}{2\sqrt{3}}\right)^2 = \frac{\pi}{12}$, and each of the three small triangles are copies of the entire figure dilated by $\frac{1}{3}$. Therefore, the total area is given by $K = \frac{\pi}{12} + 3 \cdot \left(\frac{1}{3}\right)^2 K \iff \frac{2}{3}K = \frac{\pi}{12} \iff K = \boxed{\frac{\pi}{8}}$.

7. Let $E = (a, b, 0)$, $A = (-c, b, 0)$, $R = (-c, -d, 0)$, $L = (a, -d, 0)$, $Y = (0, 0, h)$, and observe that $EY^2 + RY^2 = a^2 + b^2 + c^2 + d^2 + 2h^2 = AY^2 + LY^2$, which can only be satisfied by $EY = 1$, $AY = 4$, $RY = 8$, $LY = 7$ (or the symmetric configurations). Since EA is an integral side of a triangle whose other sides are 1 and 4, we must have $EA = 4$; similarly, $EL = 7$. Therefore, the area of rectangle $EARL$ is $\boxed{28}$. (Such a pyramid may be constructed by taking $a = \frac{1}{8}$, $b = \frac{1}{14}$, $c = \frac{31}{8}$, $d = \frac{97}{14}$, $h = \frac{\sqrt{3071}}{56}$.)

8. Since $OD \perp AC$ and $\triangle AOC$ is equilateral, we have $\angle AOD = 30^\circ$. So $AE = \frac{1}{\sqrt{3}}$, and $BH = \sqrt{AB^2 + AH^2} = \sqrt{2^2 + \left(3 - \frac{1}{\sqrt{3}}\right)^2} = \boxed{\sqrt{\frac{40}{3} - 2\sqrt{3}}} \approx 3.141533339$.

9. The dilation of ratio $-\frac{2}{3}$ about T sends C_2 to C_1, O_2 to O_1, and S to the other intersection of s with C_1, which we shall call U. We can now compute $TR \cdot TS = \frac{3}{2}TR \cdot TU = \frac{3}{2}TP^2 = \frac{3}{2}(O_1T^2 - O_1P^2) = \frac{3}{2}\left(\left(\frac{O_1O_2}{1+3/2}\right)^2 - O_1P^2\right) = \frac{3}{2}\left(\left(\frac{20}{5/2}\right)^2 - 4^2\right) = \boxed{72}$.

10. Suppose that when the ball hits a side of the table, instead of reflecting the ball's path, we reflect the entire table over this side so that the path remains straight. If we repeatedly reflect the table over its sides in all possible ways, we get a triangular grid that tiles the plane. Whenever the path crosses n lines in this grid parallel to CT, it will cross $\frac{7}{8}n$ lines parallel to CH and $\frac{15}{8}n$ lines parallel to HT. After crosssing $8 + 7 + 15 = 30$ grid lines it will have crossed three lines simultaneously again, which means that the ball will have landed in a pocket after bouncing $\boxed{27}$ times. By picturing the grid it is easy to see that the pocket in question is $\boxed{H}$. The distance the ball travels during the $\frac{1}{8}$ of its trip described in the problem is the third side of a triangle with an 120° angle between two sides 16 and 14, which is $\sqrt{16^2 + 14^2 - 2 \cdot 16 \cdot 14 \cos 120^\circ} = 26$, so length of the entire trip is $\boxed{208}$.