

1998 Power Question Solutions

I. Graphs, total of 20 points

a. completely correct gets 1 point, total of 6 points

- i. yes. vertices A,B,C,D, edges AB,AC,AD,BD
- ii. no. A and B are connected twice
- iii. yes. vertices A,B,C, edges ABC
- iv. yes. vertices A,B,C,D,E, edges AB,AD,AE,BE,CD,CE
- v. no. the edge from B does not connect to another vertex
- vi. no. E is connected to itself

b. completely correct gets 1 point, total of 2 points

- i. BL,BE,BU,EL,EU
- ii. $A \leftrightarrow B$, $B \leftrightarrow L$, $C \leftrightarrow U$, $D \leftrightarrow E$

c. 1 point each for i-iii, 3 for iv-vi, total of 12 points

- i. no
- ii. no
- iii. no
- iv. yes
- v. no
- vi. yes

note that the second graph in v has more vertices of degree 3 than the first

II. Planar Graphs, total of 60 points

a. 1 point each for v, e, f, total of 6 points

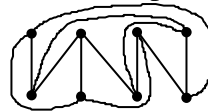
- i. $v=9$, $e=16$, $f=9$
- ii. $v=7$, $e=14$, $f=9$

note that $v-e+f=2$ by Euler's formula

b. 3 points each, total of 12 points

- i. yes
- ii. yes
- iii. no
- iv. yes

note that this is a planar drawing of ii:



for iii, note that if we remove the two vertices of degree two and make the vertices they were adjacent to adjacent to each other then we get a graph isomorphic to the one in d.ii.

c. total of 15 points

partial credit: up to 5 for effort, 5 for insight, otherwise as described below

[1 point] If G has 3 vertices and two edges then the assertion is easy since $f=1$.

[6 points] Otherwise each face is bounded by 3 or more edges, hence counting the number of edges bounding each faces and summing, then using the fact that we at most double counted [-2 for asserting we exactly double counted] each face we get $3f \leq 2e$, thus proving the left inequality.

[8 points] Now $f \leq (2/3)e$ and we want to eliminate f , so we can add $v-e$ to both sides to get $v-e+f \leq v-e+(2/3)e$, thus by Euler's formula $2 \leq v-e/3$, and rearranging yields the right inequality.

d. 5 points for i, 10 for ii, total of 15 points

partial credit: if Jordan curve thm is used then 3 points for i, 5 for ii.

on part ii: 3 for effort, 3 for insight, otherwise as described

i. Assume it is planar. $v=5, e=10, 3v-6=9 < e$, contradicting c, thus nonplanar

ii. [1 point] Note that $v=6, e=9$, so $3v-6=12 > e$ is not a contradiction

[2 points] There are no triangles in this graph, thus every face is bordered by at least 4 edges. This will allow us to prove a better inequality as in c with the assumption the graph is planar, and that will provide the contradiction.

[2 points] Thus as in the proof of c we get $4f \leq 2e$.

[4 points] Hence $f \leq e/2$, and again we want to eliminate f so adding $v-e$ to both sides and applying Euler's formula yields $2 \leq v-e/2$. (or $e \leq 2v-4$)

[1 point] $6-9/2 < 2$, contradicting the inequality just proved, thus the graph is not planar.

e. total of 12 points

partial credit: 4 for effort, 4 for insight, otherwise as described

[2 points] Without loss of generality we may assume the graph is connected.

[2 points] If the graph has fewer than 3 vertices then the assertion is obvious.

[5 points] Otherwise we can apply the inequality in c. First assume every vertex has degree at least 6, then adding the degrees of each vertex double counts the edges so $6v \leq 2e$.

[3 points] Now by c. $e \leq 3v-6$, and combining these inequalities we get $6v \leq 6v-12$, a contradiction, thus the graph must have at least one vertex of degree 5 or less.

III. Coloring, total of 56 points

a. 3 points each, total of 12 points

i. 4

ii. 5

iii. 4

iv. 3

b. total of 14 points

partial credit: 5 for effort, 5 for insight, otherwise as described

[8 points] Number the colors used in G $1, 2, \dots, \chi$. Let v_1 be the number of vertices colored with color 1. Then since no pair of them are adjacent, they are all adjacent in the new graph and thus $\bar{\chi} \geq v_1$. Repeating this procedure for v_2, \dots, v_χ and adding we get $\chi \bar{\chi} \geq v$.

[6 points] By the AM-GM inequality $\chi + \bar{\chi} \geq 2\sqrt{\chi \bar{\chi}} \geq 2\sqrt{v}$.

c. 5 points for i and iii, 20 for ii, total of 30 points

partial credit: as described below

i. By II.e. G must have a vertex of degree 5 or less

ii. [2 points] We want to remove the vertex V without decreasing the number of colors needed. In the cases where V has degree < 5 this is obviously possible.

[10 points] In the case where V has degree 5, color the adjacent vertices with colors 1, 2, 3, 4, 5 in clockwise order. If the vertices colored 1 and 3 are not connected by a walk with all vertices colored 1 or 3 then change the vertex colored 3 to 1, change any 1 vertices adjacent to it to 3, then change any 3 vertices adjacent to those to 1, and so on. This recoloring does not affect any of the other vertices adjacent to V and thus allows us to make V color 3, i.e. if a sixth color is needed for G then it is still needed after we remove V .

[8 points] If the vertices colored 1 and 3 are connected by a walk of vertices colored 1 or 3 then look at the vertices colored 2 and 4. If they are connected by a walk then by the Jordan Curve Theorem [-4 for not using this] this walk must cross the closed curve formed by the 1-3 walk and the edges from V to the vertices colored 1 and 3. The graph is planar, so we can assume it is drawn without edge crossings, thus the walk from the vertex colored 2 to the one colored 4 must cross this curve at a vertex, hence they are not connected by a walk with all vertices colored 2 or 4, and by the previous argument we can change the vertex colored 2 to 4 and make V color 2, so it can be removed without removing the need for a sixth color.

iii. [4 points] We have shown that for any graph with $\chi > 5$ we can find one with fewer vertices, and since the proof did not depend on the number of vertices we can keep doing this until we get a graph with 5 vertices, which can clearly be colored with only 5 colors, thus G could not require more than 5 colors.

[1 points] To make this rigorous some mention should be made of the well-ordering principal or infinite descent.

Historical notes and inspiration for further study of the subject:

The Four Color theorem was first conjectured in 1852. Many “proofs” were given, including one in 1879 by A. B. Kempe that was believed to be correct until P. J. Heawood found a flaw in 1890. The proof of the Four Color theorem, given in 1976 by Haken and Appel, uses a similar technique to what we just used, but instead of using 6 possible subgraphs it uses 1482. The way these theorems relate to coloring maps is that we can consider the *dual graph*, formed by putting a vertex in every face and connecting the vertices whose faces share an edge. Coloring the dual graph is the same as coloring the faces of the original graph in such a way that no two faces sharing an edge have the same color, which is what one does when coloring a map.

While it may seem that we used many definitions in this power question, there are many more that had to be omitted. Some of the other interesting definitions used in graph theory are *expansion*, which is what you get if you add vertices along an edge of a graph, and *supergraph*, which is what you get if you add new vertices anywhere and connect them arbitrarily to the other vertices. Kuratowski’s Theorem states that every nonplanar graph is a supergraph of an expansion of one of the two graphs in II.d. A corollary of this is that a graph is nonplanar iff it is a supergraph of an expansion of one of those two graphs, thus we can classify planar and nonplanar graphs in terms of just two particular graphs.

A planar graph can also be described as one that can be drawn on a sphere with no edge crossings. A torus is essentially the surface of a solid sphere with a hole drilled all the way through it. The *genus* g of a graph is the minimum number of holes that must be drilled in a sphere in order to be able to draw it without edge crossings. In other words a graph of genus g can be drawn on a g holed torus without edge crossings, but not on a $g-1$ holed torus. The Heawood Coloring Theorem states that if G has genus $g > 0$ then G can be colored with $\left\lceil \frac{7 + \sqrt{1 + 48g}}{2} \right\rceil$ colors. Note that plugging in $g=0$ yields 4, but no proof

of this theorem is known that does not depend on the condition $g > 0$ (of course we can combine the proof for $g>0$ with the four color theorem to conclude that the result holds for all g , but it would be nice to have just one proof that works equally well for all g).

Another thing we can study about graphs is what kind of walks exist in them. An *eulerian walk*, named after the great Swiss mathematician Leonhard Euler (1707-1783, pronounced Oiler), is one that uses every edge exactly once. A *hamiltonian walk*, named after the Irish mathematician Sir William Rowan Hamilton (1805-1865), is one that uses every vertex exactly once. Try to classify which graphs have such walks, with or without the condition that the starting and ending points must be the same (in the case of a hamiltonian walk the end vertex is thus counted twice and it is called a *closed hamiltonian walk* to distinguish from the *open walk*).

This power question was barely an introduction to graph theory. It is a very broad field of mathematics, closely related to topology and knot theory. I hope you enjoyed this test, learned something from it, and that you will continue your studies of mathematics for many years.

Edward Early
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