Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Calculus Subject Test — Solutions

1. A point is chosen randomly with uniform distribution in the interior of a circle of radius 1. What is its expected distance from the center of the circle?

   Solution: $\boxed{2/3}$

   The probability of the point falling between a distance r and $r + dr$ from the center is the ratio of the area of the corresponding annulus to the area of the whole circle: $\frac{\pi[(r+dr)^2-r^2]}{\pi} \to \frac{2\pi r\,dr}{\pi} = 2r\,dr$ for small values of dr. Then the expected distance is $\int_0^1 r \cot 2r\,dr = \frac{2}{3}$.

2. A particle moves along the x-axis in such a way that its velocity at position x is given by the formula $v(x) = 2 + \sin x$. What is its acceleration at $x = \frac{\pi}{6}$?

   Solution: $\boxed{5\sqrt{3}/4}$

   Acceleration is given by $a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v = \cos x \cdot (2 + \sin x) = 5\sqrt{3}/4$.

3. What is the area of the region bounded by the curves $y = x^{2003}$ and $y = x^{1/2003}$ and lying above the x-axis?

   Solution: $\boxed{1001/1002}$

   The two curves intersect at $(0, 0)$ and $(1, 1)$, so the desired area is

   $$\int_0^1 \left(x^{1/2003} - x^{2003}\right) dx = \left[\frac{x^{2004/2003}}{2004/2003} - \frac{x^{2004}}{2004}\right]_0^1 = \frac{1001}{1002}.$$

4. The sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies $\lim_{n\to\infty}(x_{2n} + x_{2n+1}) = 315$ and $\lim_{n\to\infty}(x_{2n} + x_{2n-1}) = 2003$. Evaluate $\lim_{n\to\infty}(x_{2n}/x_{2n+1})$.

   Solution: $\boxed{-1}$

   We have $\lim_{n\to\infty}(x_{2n+1} - x_{2n-1}) = \lim_{n\to\infty}[(x_{2n} + x_{2n+1}) - (x_{2n} + x_{2n-1})] = 315 - 2003 = -1688$; it follows that $x_{2n+1} \to -\infty$ as $n \to \infty$. Then

   $$\lim_{n\to\infty} \frac{x_{2n}}{x_{2n+1}} = \lim_{n\to\infty} \frac{x_{2n} + x_{2n+1}}{x_{2n+1}} - 1 = -1,$$

   since $x_{2n} + x_{2n+1} \to 315$ while $x_{2n+1} \to -\infty$.

5. Find the minimum distance from the point $(0, 5/2)$ to the graph of $y = x^4/8$.

   Solution: $\boxed{\sqrt{17}/2}$

   We want to minimize $x^2 + (x^4/8 - 5/2)^2 = x^8/64 - 5x^4/8 + x^2 + 25/4$, which is equivalent to minimizing $z^4/4 - 10z^2 + 16z$, where we have set $z = x^2$. The derivative of this expression is $z^3 - 20z + 16$, which is seen on inspection to have 4 as a root, leading to the factorization $(z - 4)(z + 2 - 2\sqrt{2})(z + 2 - 2\sqrt{2})$. Since $z = x^2$ ranges over $[0, \infty)$, the possible minima are at $z = 0, z = -2 + 2\sqrt{2}$, and $z = 4$. However, the derivative is positive on $(0, -2 + 2\sqrt{2})$, so this leaves only 0 and 4 to be tried. We find that the minimum is in fact achieved at $z = 4$, so the closest point on the graph is given by $x = \pm 2$, with distance $\sqrt{2^2 + (2^4/8 - 5/2)^2} = \sqrt{17}/2$.

6. For n an integer, evaluate

$$\lim_{n\to\infty}\Big(\frac{1}{\sqrt{n^2-0^2}}+\frac{1}{\sqrt{n^2-1^2}}+\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\Big).$$

Solution: $\boxed{\pi/2}$

Note that $\frac{1}{\sqrt{n^2-i^2}}=\frac{1}{n}\cdot\frac{1}{\sqrt{1-(\frac{i}{n})^2}}$, so that the sum we wish to evaluate is just a Riemann sum. Then,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\sqrt{1-(\frac{i}{n})^2}}\right)=\int_0^1\frac{1}{\sqrt{1-x^2}}\,dx=[\sin^{-1}x]_0^1=\frac{\pi}{2}.$$

7. For what value of $a>1$ is

$$\int_a^{a^2}\frac{1}{x}\log\frac{x-1}{32}\,dx$$

minimum?

Solution: $\boxed{3}$

Let $f(a)=\int_a^{a^2}\frac{1}{x}\log\frac{x-1}{32}\,dx$. Then we want $\frac{df}{da}=0$; by the Fundamental Theorem of Calculus and the chain rule, this implies that

$$2a\left(\frac{1}{a^2}\log\frac{a^2-1}{32}\right)-\frac{1}{a}\log\frac{a-1}{32}=\frac{d}{da}\left(\int_c^{a^2}\frac{1}{x}\log\frac{x-1}{32}\,dx-\int_c^{a}\frac{1}{x}\log\frac{x-1}{32}\,dx\right)=0,$$

where c is any constant with $1<c<a$. Then $2\log\frac{a^2-1}{32}=\log\frac{a-1}{32}$, so that $(\frac{a^2-1}{32})^2=\frac{a-1}{32}$. After canceling factors of $(a-1)/32$ (since $a>1$), this simplifies to $(a^2-1)(a+1)=32\Rightarrow a^3+a^2-a-33=0$, which in turn factors as $(a-3)(a^2+4a+11)=0$. The quadratic factor has no real solutions, so this leaves only $a=3$. However, we have that $a>1$, and we can check that $f(1)=0$, $\lim_{a\to\infty}f(a)>0$, and $f(3)<0$, so the global minimum does occur at $a=3$.

8. A right circular cone with a height of 12 inches and a base radius of 3 inches is filled with water and held with its vertex pointing downward. Water flows out through a hole at the vertex at a rate in cubic inches per second numerically equal to the height of the water in the cone. (For example, when the height of the water in the cone is 4 inches, water flows out at a rate of 4 cubic inches per second.) Determine how many seconds it will take for all of the water to flow out of the cone.

Solution: $\boxed{9\pi/2}$

When the water in the cone is h inches high, it forms a cone similar to the original, so that its base has radius $h/4$ and its volume is hence $\pi h^3/48$. The given condition then states that

$$\frac{d}{dt}\left(\frac{\pi h^3}{48}\right)=-h\Rightarrow\frac{\pi h^2}{16}\cdot\frac{dh}{dt}=-h\Rightarrow 2h\cdot\frac{dh}{dt}=-\frac{32}{\pi}.$$

Integrating with respect to t, we get that $h^2=-32t/\pi+C$; setting $t=0$, $h=12$, we get $C=144$. The cone empties when $h=0$, so $0=-32t/\pi+144\Rightarrow t=9\pi/2$.

9. Two differentiable real functions $f(x)$ and $g(x)$ satisfy

$$\frac{f'(x)}{g'(x)} = e^{f(x)-g(x)}$$

for all x, and $f(0) = g(2003) = 1$. Find the largest constant c such that $f(2003) > c$ for all such functions f, g.

Solution: $\boxed{1 - \ln 2}$

Rearranging the given equation gives $f'(x)e^{-f(x)} = g'(x)e^{-g(x)}$ for all x, so $\frac{d}{dx}(e^{-f(x)} - e^{-g(x)}) = -f'(x)e^{-f(x)} + g'(x)e^{-g(x)} = 0$. Thus, $e^{-f(x)} - e^{-g(x)}$ is a constant, and it must be less than $e^{-f(0)} = e^{-1}$. Thus, $e^{-f(2003)} < e^{-g(2003)} + e^{-1} = 2e^{-1} = e^{\ln 2 - 1} \Rightarrow f(2003) > 1 - \ln 2$. On the other hand, we can find positive-valued functions $e^{-f(x)}, e^{-g(x)}$ that take on the required values at 0 and 2003 and have constant difference arbitrarily close to e^{-1}. For example, for arbitrarily large t, we can set $e^{-f(x)} = e^{-(t(2003-x)+1)} + e^{-1} - e^{-(2003t+1)}$ and $e^{-g(x)} = e^{-(t(2003-x)+1)}$, and we can check that the resulting functions f, g satisfy the required conditions. Thus, we can make $f(2003)$ arbitrarily close to $1 - \ln 2$, so this is the answer.

10. Evaluate

$$\int_{-\infty}^{\infty} \frac{1 - x^2}{1 + x^4}\, dx.$$

Solution: $\boxed{0}$

Let $S = \int_0^\infty 1/(x^4 + 1)\, dx$; note that the integral converges absolutely. Substituting $x = 1/u$, so that $dx = -1/u^2\, du$, we have

$$S = \int_0^\infty \frac{1}{1 + x^4}\, dx = \int_\infty^0 \frac{1}{1 + u^{-4}} \frac{du}{-u^2} = \int_\infty^0 \frac{-u^2}{u^4 + 1}\, du$$
$$= \int_0^\infty \frac{u^2}{1 + u^4}\, du = \int_0^\infty \frac{x^2}{1 + x^4}\, dx$$

(the manipulations are justified by absolute convergence), from which we see that $\int_0^\infty (1 - x^2)/(1 + x^4)\, dx = 0$. Since the integrand is an even function, it follows that the integral from $-\infty$ to ∞ is zero as well.