Harvard-MIT Mathematics Tournament

February 28, 2004

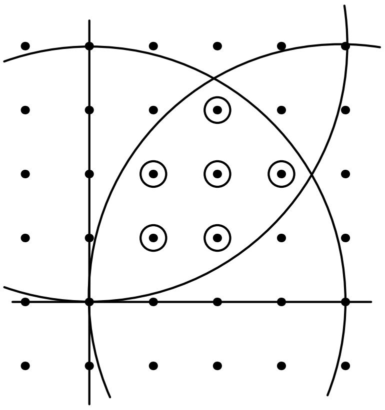
Individual Round: Algebra Subject Test — Solutions

1. How many ordered pairs of integers (a,b) satisfy all of the following inequalities?

$$\begin{aligned} a^2+b^2 &< 16 \\ a^2+b^2 &< 8a \\ a^2+b^2 &< 8b \end{aligned}$$

Solution: $\boxed{6}$

This is easiest to see by simply graphing the inequalities. They correspond to the (strict) interiors of circles of radius 4 and centers at $(0,0),(4,0),(0,4)$, respectively. So we can see that there are 6 lattice points in their intersection (circled in the figure).



2. Find the largest number n such that $(2004!)!$ is divisible by $((n!)!)!$.

Solution: $\boxed{6}$

For positive integers a,b, we have

$$a! \mid b! \quad \Leftrightarrow \quad a! \le b! \quad \Leftrightarrow \quad a \le b.$$

Thus,

$$((n!)!)! \mid (2004!)! \quad \Leftrightarrow \quad (n!)! \le 2004! \quad \Leftrightarrow \quad n! \le 2004 \quad \Leftrightarrow \quad n \le 6.$$

3. Compute:

$$\left\lfloor \frac{2005^3}{2003 \cdot 2004} - \frac{2003^3}{2004 \cdot 2005} \right\rfloor.$$

Solution: $\boxed{8}$

Let $x = 2004$. Then the expression inside the floor brackets is

$$\frac{(x+1)^3}{(x-1)x} - \frac{(x-1)^3}{x(x+1)} = \frac{(x+1)^4-(x-1)^4}{(x-1)x(x+1)} = \frac{8x^3+8x}{x^3-x} = 8 + \frac{16x}{x^3-x}.$$

Since x is certainly large enough that $0 < 16x/(x^3-x) < 1$, the answer is 8.

4. Evaluate the sum

$$\frac{1}{2\lfloor\sqrt{1}\rfloor+1}+\frac{1}{2\lfloor\sqrt{2}\rfloor+1}+\frac{1}{2\lfloor\sqrt{3}\rfloor+1}+\cdots+\frac{1}{2\lfloor\sqrt{100}\rfloor+1}.$$

   Solution: $\boxed{190/21}$

   The first three terms all equal $1/3$, then the next five all equal $1/5$; more generally, for each $a = 1, 2, \ldots, 9$, the terms $1/(2\left\lfloor\sqrt{a^2}\right\rfloor + 1)$ to $1/(2\left\lfloor\sqrt{a^2+2a}\right\rfloor + 1)$ all equal $1/(2a + 1)$, and there are $2a + 1$ such terms. Thus our terms can be arranged into 9 groups, each with sum 1, and only the last term $1/(2\left\lfloor\sqrt{100}\right\rfloor + 1)$ remains, so the answer is $9 + 1/21 = 190/21$.

5. There exists a positive real number x such that $\cos(\tan^{-1}(x)) = x$. Find the value of x^2.

   Solution: $\boxed{(-1+\sqrt{5})/2}$

   Draw a right triangle with legs $1, x$; then the angle θ opposite x is $\tan^{-1} x$, and we can compute $\cos(\theta) = 1/\sqrt{x^2+1}$. Thus, we only need to solve $x = 1/\sqrt{x^2+1}$. This is equivalent to $x\sqrt{x^2+1} = 1$. Square both sides to get $x^4 + x^2 = 1 \Rightarrow x^4 + x^2 - 1 = 0$. Use the quadratic formula to get the solution $x^2 = (-1+\sqrt{5})/2$ (unique since x^2 must be positive).

6. Find all real solutions to $x^4 + (2 - x)^4 = 34$.

   Solution: $\boxed{1 \pm \sqrt{2}}$

   Let $y = 2 - x$, so $x + y = 2$ and $x^4 + y^4 = 34$. We know

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = x^4 + y^4 + 2xy(2x^2 + 2y^2 + 3xy).$$

   Moreover, $x^2+y^2 = (x+y)^2-2xy$, so the preceding equation becomes $2^4 = 34+2xy(2\cdot 2^2-xy)$, or $(xy)^2-8xy-9 = 0$. Hence $xy = 9$ or -1. Solving $xy = 9, x + y = 2$ produces complex solutions, and solving $xy = -1, x + y = 2$ produces $(x, y) = (1 + \sqrt{2}, 1 - \sqrt{2})$ or $(1 - \sqrt{2}, 1 + \sqrt{2})$. Thus, $x = 1 \pm \sqrt{2}$.

7. If x, y, k are positive reals such that

$$3 = k^2\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)+k\left(\frac{x}{y}+\frac{y}{x}\right),$$

   find the maximum possible value of k.

   Solution: $\boxed{(-1+\sqrt{7})/2}$

   We have $3 = k^2(x^2/y^2 + y^2/x^2) + k(x/y + y/x) \geq 2k^2 + 2k$, hence $7 \geq 4k^2 + 4k + 1 = (2k+1)^2$, hence $k \leq (\sqrt{7}-1)/2$. Obviously k can assume this value, if we let $x = y = 1$.

8. Let x be a real number such that $x^3 + 4x = 8$. Determine the value of $x^7 + 64x^2$.

   Solution: $\boxed{128}$

For any integer $n \geq 0$, the given implies $x^{n+3} = -4x^{n+1} + 8x^n$, so we can rewrite any such power of x in terms of lower powers. Carrying out this process iteratively gives

$$\begin{aligned} x^7 &= -4x^5 + 8x^4 \\ &= 8x^4 + 16x^3 - 32x^2 \\ &= 16x^3 - 64x^2 + 64x \\ &= -64x^2 + 128. \end{aligned}$$

Thus, our answer is 128.

9. A sequence of positive integers is defined by $a_0 = 1$ and $a_{n+1} = a_n^2 + 1$ for each $n \geq 0$. Find $\gcd(a_{999}, a_{2004})$.

   Solution: $\boxed{677}$

   If d is the relevant greatest common divisor, then $a_{1000} = a_{999}^2 + 1 \equiv 1 = a_0 \pmod{d}$, which implies (by induction) that the sequence is periodic modulo d, with period 1000. In particular, $a_4 \equiv a_{2004} \equiv 0$. So d must divide a_4. Conversely, we can see that $a_5 = a_4^2 + 1 \equiv 1 = a_0$ modulo a_4, so (again by induction) the sequence is periodic modulo a_4 with period 5, and hence a_{999}, a_{2004} are indeed both divisible by a_4. So the answer is a_4, which we can compute directly; it is 677.

10. There exists a polynomial P of degree 5 with the following property: if z is a complex number such that $z^5 + 2004z = 1$, then $P(z^2) = 0$. Calculate the quotient $P(1)/P(-1)$.

    Solution: $\boxed{-2010012/2010013}$

    Let $z_1, \ldots, z_5$ be the roots of $Q(z) = z^5 + 2004z - 1$. We can check these are distinct (by using the fact that there's one in a small neighborhood of each root of $z^5 + 2004z$, or by noting that $Q(z)$ is relatively prime to its derivative). And certainly none of the roots of Q is the negative of another, since $z^5 + 2004z = 1$ implies $(-z)^5 + 2004(-z) = -1$, so their squares are distinct as well. Then, $z_1^2, \ldots, z_5^2$ are the roots of P, so if we write C for the leading coefficient of P, we have

$$\begin{aligned} \frac{P(1)}{P(-1)} &= \frac{C(1-z_1^2)\cdots(1-z_5^2)}{C(-1-z_1^2)\cdots(-1-z_5^2)} \\ &= \frac{[(1-z_1)\cdots(1-z_5)]\cdot[(1+z_1)\cdots(1+z_5)]}{[(i-z_1)\cdots(i-z_5)]\cdot[(i+z_1)\cdots(i+z_5)]} \\ &= \frac{[(1-z_1)\cdots(1-z_5)]\cdot[(-1-z_1)\cdots(-1-z_5)]}{[(i-z_1)\cdots(i-z_5)]\cdot[(-i-z_1)\cdots(-i-z_5)]} \\ &= \frac{(1^5+2004\cdot 1-1)(-1^5+2004\cdot(-1)-1)}{(i^5+2004\cdot i-1)(-i^5+2004\cdot(-i)-1)} \\ &= \frac{(2004)(-2006)}{(-1+2005i)(-1-2005i)} \\ &= -\frac{2005^2-1}{2005^2+1} \\ &= -4020024/4020026 = -2010012/2010013. \end{aligned}$$

Alternative Solution: In fact, we can construct the polynomial P explicitly (up to multiplication by a constant). We write $P(z^2)$ as a polynomial in z; it must use only even powers of z and be divisible by $z^5+2004z-1$, so we are inspired to try a difference of squares,

$$P(z^2) = (z^5 + 2004z - 1)(z^5 + 2004z + 1) = (z^5 + 2004z)^2 - 1^2 = z^2(z^4 + 2004)^2 - 1,$$

giving

$$P(z) = z(z^2 + 2004)^2 - 1.$$

Now plugging in $z = 1$ and $z = -1$ rapidly gives $(2005^2 - 1)/(-2005^2 - 1)$ as before.