10^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 24 February 2007

Individual Round: Algebra Test

1. [**3**] Compute
$$\left\lfloor \frac{2007! + 2004!}{2006! + 2005!} \right\rfloor.$$
(Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

Answer: $\boxed{\mathbf{2006}.}$ We have
$$\left\lfloor \frac{2007! + 2004!}{2006! + 2005!} \right\rfloor = \left\lfloor \frac{\left(2007 \cdot 2006 + \frac{1}{2005}\right) \cdot 2005!}{(2006+1) \cdot 2005!} \right\rfloor = \left\lfloor \frac{2007 \cdot 2006 + \frac{1}{2005}}{2007} \right\rfloor = \left\lfloor 2006 + \frac{1}{2005 \cdot 2007} \right\rfloor.$$

2. [**3**] Two reals x and y are such that $x - y = 4$ and $x^3 - y^3 = 28$. Compute xy.

Answer: $\boxed{\mathbf{-3.}}$ We have $28 = x^3 - y^3 = (x-y)(x^2+xy+y^2) = (x-y)((x-y)^2+3xy) = 4 \cdot (16+3xy)$, from which $xy = -3$.

3. [**4**] Three real numbers x, y, and z are such that $(x+4)/2 = (y+9)/(z-3) = (x+5)/(z-5)$. Determine the value of x/y.

Answer: $\boxed{\mathbf{1/2.}}$ Because the first and third fractions are equal, adding their numerators and denominators produces another fraction equal to the others: $((x+4)+(x+5))/(2+(z-5)) = (2x+9)/(z-3)$. Then $y + 9 = 2x + 9$, etc.

4. [**4**] Compute
$$\frac{2^3-1}{2^3+1} \cdot \frac{3^3-1}{3^3+1} \cdot \frac{4^3-1}{4^3+1} \cdot \frac{5^3-1}{5^3+1} \cdot \frac{6^3-1}{6^3+1}.$$

Answer: $\boxed{\mathbf{\frac{43}{63}}.}$ Use the factorizations $n^3 - 1 = (n-1)(n^2+n+1)$ and $n^3+1 = (n+1)(n^2-n+1)$ to write
$$\frac{1 \cdot 7}{3 \cdot 3} \cdot \frac{2 \cdot 13}{4 \cdot 7} \cdot \frac{3 \cdot 21}{5 \cdot 13} \cdot \frac{4 \cdot 31}{6 \cdot 21} \cdot \frac{5 \cdot 43}{7 \cdot 31} = \frac{1 \cdot 2 \cdot 43}{3 \cdot 6 \cdot 7} = \frac{43}{63}.$$

5. [**5**] A convex quadrilateral is determined by the points of intersection of the curves $x^4 + y^4 = 100$ and $xy = 4$; determine its area.

Answer: $\boxed{\mathbf{4\sqrt{17}}.}$ By symmetry, the quadrilateral is a rectangle having $x = y$ and $x = -y$ as axes of symmetry. Let (a, b) with $a > b > 0$ be one of the vertices. Then the desired area is
$$\left(\sqrt{2}(a-b)\right) \cdot \left(\sqrt{2}(a+b)\right) = 2(a^2-b^2) = 2\sqrt{a^4-2a^2b^2+b^4} = 2\sqrt{100-2\cdot 4^2} = 4\sqrt{17}.$$

6. [**5**] Consider the polynomial $P(x) = x^3 + x^2 - x + 2$. Determine all real numbers r for which there exists a complex number z not in the reals such that $P(z) = r$.

Answer: $\boxed{\mathbf{r > 3, r < \frac{49}{27}}.}$ Because such roots to polynomial equations come in conjugate pairs, we seek the values r such that $P(x) = r$ has just one real root x. Considering the shape of a cubic, we are interested in the boundary values r such that $P(x) - r$ has a repeated zero. Thus, we write
$$P(x) - r = x^3 + x^2 - x + (2-r) = (x-p)^2(x-q) = x^3 - (2p+q)x^2 + p(p+2q)x - p^2q.$$
Then $q = -2p - 1$ and $1 = p(p+2q) = p(-3p-2)$ so that $p = 1/3$ or $p = -1$. It follows that the graph of $P(x)$ is horizontal at $x = 1/3$ (a maximum) and $x = -1$ (a minimum), so the desired values r are $r > P(-1) = 3$ and $r < P(1/3) = 1/27 + 1/9 - 1/3 + 2 = 49/27$.

7. [**5**] An infinite sequence of positive real numbers is defined by $a_0 = 1$ and $a_{n+2} = 6a_n - a_{n+1}$ for $n = 0, 1, 2, \cdots$ Find the possible value(s) of a_{2007}.

   Answer: $\boxed{\mathbf{2^{2007}}.}$ The characteristic equation of the linear homogeneous equation is $m^2 + m - 6 = (m + 3)(m - 2) = 0$ with solutions $m = -3$ and $m = 2$. Hence the general solution is given by $a_n = A(2)^n + B(-3)^n$ where A and B are constants to be determined. Then we have $a_n > 0$ for $n \geq 0$, so necessarily $B = 0$, and $a_0 = 1 \Rightarrow A = 1$. Therefore, the unique solution to the recurrence is $a_n = 2^n$ for all n.

8. [**6**] Let $A := \mathbb{Q} \setminus \{0, 1\}$ denote the set of all rationals other than 0 and 1. A function $f : A \to \mathbb{R}$ has the property that for all $x \in A$,

$$f(x) + f\left(1 - \frac{1}{x}\right) = \log |x|.$$

   Compute the value of $f(2007)$.

   Answer: $\boxed{\log\left(\mathbf{2007/2006}\right).}$ Let $g : A \to A$ be defined by $g(x) := 1 - 1/x$; the key property is that

$$g(g(g(x))) = 1 - \frac{1}{1 - \frac{1}{1-\frac{1}{x}}} = x.$$

   The given equation rewrites as $f(x) + f(g(x)) = \log |x|$. Substituting $x = g(y)$ and $x = g(g(z))$ gives the further equations $f(g(y)) + f(g(g(y))) = \log |g(x)|$ and $f(g(g(z))) + f(z) = \log |g(g(x))|$. Setting y and z to x and solving the system of three equations for $f(x)$ gives

$$f(x) = \frac{1}{2} \cdot (\log |x| - \log |g(x)| + \log |g(g(x))|).$$

   For $x = 2007$, we have $g(x) = \frac{2006}{2007}$ and $g(g(x)) = \frac{-1}{2006}$, so that

$$f(2007) = \frac{\log |2007| - \log \left|\frac{2006}{2007}\right| + \log \left|\frac{-1}{2006}\right|}{2} = \log(2007/2006).$$

9. [**7**] The complex numbers $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the four distinct roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$

   Answer: $\boxed{\{\mathbf{1 \pm \sqrt{5}, -2}\}.}$ Employing the elementary symmetric polynomials ($s_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -2$, $s_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 = 0$, $s_3 = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \alpha_3\alpha_4\alpha_1 + \alpha_4\alpha_1\alpha_2 = 0$, and $s_4 = \alpha_1\alpha_2\alpha_3\alpha_4 = 2$) we consider the polynomial

$$P(x) = (x - (\alpha_1\alpha_2 + \alpha_3\alpha_4))(x - (\alpha_1\alpha_3 + \alpha_2\alpha_4))(x - (\alpha_1\alpha_4 + \alpha_2\alpha_3))$$

   Because P is symmetric with respect to $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, we can express the coefficients of its expanded form in terms of the elementary symmetric polynomials. We compute

$$\begin{aligned} P(x) &= x^3 - s_2x^2 + (s_3s_1 - 4s_4)x + (-s_3^2 - s_4s_1^2 + s_4s_2) \\ &= x^3 - 8x - 8 \\ &= (x + 2)(x^2 - 2x - 4) \end{aligned}$$

   The roots of $P(x)$ are -2 and $1 \pm \sqrt{5}$, so the answer is $\{1 \pm \sqrt{5}, -2\}$.

   Remarks. It is easy to find the coefficients of x^2 and x by expansion, and the constant term can be computed without the complete expansion and decomposition of $(\alpha_1\alpha_2 + \alpha_3\alpha_4)(\alpha_1\alpha_3 + \alpha_2\alpha_4)(\alpha_1\alpha_4 + \alpha_2\alpha_3)$ by noting that the only nonzero 6th degree expressions in s_1, s_2, s_3, and s_4 are s_1^6 and $s_4s_1^2$. The general polynomial P constructed here is called the *cubic resolvent* and arises in Galois theory.

10. [**8**] The polynomial $f(x) = x^{2007} + 17x^{2006} + 1$ has distinct zeroes $r_1, \ldots, r_{2007}$. A polynomial P of degree 2007 has the property that $P\left(r_j + \frac{1}{r_j}\right) = 0$ for $j = 1, \ldots, 2007$. Determine the value of $P(1)/P(-1)$.

    Answer: $\boxed{\mathbf{\frac{289}{259}}.}$ For some constant k, we have

$$P(z) = k \prod_{j=1}^{2007} \left(z - \left(r_j + \frac{1}{r_j}\right)\right).$$

Now writing $\omega^3 = 1$ with $\omega \neq 1$, we have $\omega^2 + \omega = -1$. Then

$$\begin{aligned} P(1)/P(-1) &= \frac{k \prod_{j=1}^{2007} \left(1 - \left(r_j + \frac{1}{r_j}\right)\right)}{k \prod_{j=1}^{2007} \left(-1 - \left(r_j + \frac{1}{r_j}\right)\right)} = \prod_{j=1}^{2007} \frac{r_j^2 - r_j + 1}{r_j^2 + r_j + 1} = \prod_{j=1}^{2007} \frac{(-\omega - r_j)(-\omega^2 - r_j)}{(\omega - r_j)(\omega^2 - r_j)} \\ &= \frac{f(-\omega) f(-\omega^2)}{f(\omega) f(\omega^2)} = \frac{\left(-\omega^{2007} + 17\omega^{2006} + 1\right)\left(-(\omega^2)^{2007} + 17(\omega^2)^{2006} + 1\right)}{(\omega^{2007} + 17\omega^{2006} + 1)((\omega^2)^{2007} + 17(\omega^2)^{2006} + 1)} = \frac{(17\omega^2)(17\omega)}{(2 + 17\omega^2)(2 + 17\omega)} \\ &= \frac{289}{4 + 34(\omega + \omega^2) + 289} = \frac{289}{259}. \end{aligned}$$