

10th Annual Harvard-MIT Mathematics Tournament
Saturday 24 February 2007

Individual Round: Calculus Test

1. [3] Compute:

$$\lim_{x \rightarrow 0} \frac{x^2}{1 - \cos(x)}$$

Answer: [2] Since $\sin^2(x) = 1 - \cos^2(x)$, we multiply the numerator and denominator by $1 + \cos(x)$ and use the fact that $x/\sin(x) \rightarrow 1$, obtaining

$$\lim_{x \rightarrow 0} \frac{x^2}{1 - \cos(x)} = \lim_{x \rightarrow 0} \frac{x^2(1 + \cos(x))}{1 - \cos^2(x)} = \lim_{x \rightarrow 0} \left(\frac{x}{\sin(x)} \right)^2 \cdot 2 = 2$$

Remarks. Another solution, using *L'Hôpital's rule*, is possible: $\lim_{x \rightarrow 0} \frac{x^2}{1 - \cos(x)} = \lim_{x \rightarrow 0} \frac{2x}{\sin(x)} = 2$.

2. [3] Determine the real number a having the property that $f(a) = a$ is a relative minimum of $f(x) = x^4 - x^3 - x^2 + ax + 1$.

Answer: [1] Being a relative minimum, we have $0 = f'(a) = 4a^3 - 3a^2 - 2a + a = a(4a + 1)(a - 1)$. Then $a = 0, 1, -1/4$ are the only possibilities. However, it is easily seen that $a = 1$ is the only value satisfying $f(a) = a$.

3. [4] Let a be a positive real number. Find the value of a such that the definite integral

$$\int_a^{a^2} \frac{dx}{x + \sqrt{x}}$$

achieves its smallest possible value.

Answer: [3 - 2√2] Let $F(a)$ denote the given definite integral. Then

$$F'(a) = \frac{d}{da} \int_a^{a^2} \frac{dx}{x + \sqrt{x}} = 2a \cdot \frac{1}{a^2 + \sqrt{a^2}} - \frac{1}{a + \sqrt{a}}.$$

Setting $F'(a) = 0$, we find that $2a + 2\sqrt{a} = a + 1$ or $(\sqrt{a} + 1)^2 = 2$. We find $\sqrt{a} = \pm\sqrt{2} - 1$, and because $\sqrt{a} > 0$, $a = (\sqrt{2} - 1)^2 = 3 - 2\sqrt{2}$.

4. [4] Find the real number α such that the curve $f(x) = e^x$ is tangent to the curve $g(x) = \alpha x^2$.

Answer: [e²/4] Suppose tangency occurs at $x = x_0$. Then $e^{x_0} = \alpha x_0^2$ and $f'(x_0) = 2\alpha x_0$. On the other hand, $f'(x) = f(x)$, so $\alpha x_0^2 = 2\alpha x_0$. Clearly, $\alpha = 0$ and $x_0 = 0$ are impossible, so it must be that $x_0 = 2$. Then $\alpha = e^{x_0}/(x_0^2) = e^2/4$.

5. [5] The function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $f(x^2)f''(x) = f'(x)f'(x^2)$ for all real x . Given that $f(1) = 1$ and $f'''(1) = 8$, determine $f'(1) + f''(1)$.

Answer: [6] Let $f'(1) = a$ and $f''(1) = b$. Then setting $x = 1$ in the given equation, $b = a^2$. Differentiating the given yields

$$2xf'(x^2)f''(x) + f(x^2)f'''(x) = f''(x)f'(x^2) + 2xf'(x)f''(x^2).$$

Plugging $x = 1$ into this equation gives $2ab + 8 = ab + 2ab$, or $ab = 8$. Then because a and b are real, we obtain the solution $(a, b) = (2, 4)$.

Remarks. A priori, the function needn't exist, but one possibility is $f(x) = e^{2x-2}$.

6. [5] The elliptic curve $y^2 = x^3 + 1$ is tangent to a circle centered at $(4, 0)$ at the point (x_0, y_0) . Determine the sum of all possible values of x_0 .

Answer: $\boxed{\frac{1}{3}}$. Note that $y^2 \geq 0$, so $x^3 \geq -1$ and $x \geq -1$. Let the circle be defined by $(x - 4)^2 + y^2 = c$ for some $c \geq 0$. Now differentiate the equations with respect to x , obtaining $2y \frac{dy}{dx} = 3x^2$ from the given and $2y \frac{dy}{dx} = -2x + 8$ from the circle. For tangency, the two expressions $\frac{dy}{dx}$ must be equal if they are well-defined, and this is almost always the case. Thus, $-2x_0 + 8 = 3x_0^2$ so $x_0 = -2$ or $x_0 = 4/3$, but only the latter corresponds to a point on $y^2 = x^3 + 1$. Otherwise, $y_0 = 0$, and this gives the trivial solution $x_0 = -1$.

7. [5] Compute

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1) \cdot (n+1)!}.$$

Answer: $\boxed{3 - e}$. We write

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1) \cdot (n+1)!} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \frac{1}{(n+1)!} = \sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1)!} - \sum_{n=1}^{\infty} \frac{1}{(n+1) \cdot (n+1)!} \\ &= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{n \cdot (n+1)!} - \sum_{n=1}^{\infty} \frac{1}{(n+1) \cdot (n+1)!} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+1) \cdot (n+2)!} - \frac{1}{(n+1) \cdot (n+1)!} \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 - (n+2)}{(n+1) \cdot (n+2)!} = \frac{1}{2} - \left(\frac{1}{3!} + \frac{1}{4!} + \dots \right) = 3 - \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots \right) = 3 - e. \end{aligned}$$

Alternatively, but with considerably less motivation, we can induce telescoping by adding and subtracting $e - 2 = 1/2! + 1/3! + \dots$, obtaining

$$\begin{aligned} 2 - e + \sum_{n=1}^{\infty} \frac{n(n+1) + 1}{n \cdot (n+1) \cdot (n+1)!} &= 2 - e + \sum_{n=1}^{\infty} \frac{(n+1)^2 - n}{n \cdot (n+1) \cdot (n+1)!} \\ 2 - e + \sum_{n=1}^{\infty} \frac{1}{n \cdot n!} - \frac{1}{(n+1) \cdot (n+1)!} &= 3 - e. \end{aligned}$$

8. [6] Suppose that ω is a primitive 2007th root of unity. Find $(2^{2007} - 1) \sum_{j=1}^{2006} \frac{1}{2 - \omega^j}$.

For this problem only, you may express your answer in the form $m \cdot n^k + p$, where m, n, k , and p are positive integers. Note that a number z is a *primitive n^{th} root of unity* if $z^n = 1$ and n is the smallest number amongst $k = 1, 2, \dots, n$ such that $z^k = 1$.

Answer: $\boxed{2005 \cdot 2^{2006} + 1}$. Note that

$$\begin{aligned} \frac{1}{z - \omega} + \dots + \frac{1}{z - \omega^{2006}} &= \frac{\sum_{j=1}^{2006} \prod_{i \neq j} (z - \omega^i)}{(z - \omega) \dots (z - \omega^{2006})} \\ &= \frac{\frac{d}{dz} [z^{2006} + z^{2005} + \dots + 1]}{z^{2006} + z^{2005} + \dots + 1} = \frac{2006z^{2005} + 2005z^{2004} + \dots + 1}{z^{2006} + z^{2005} + \dots + 1} \cdot \frac{z - 1}{z - 1} \\ &= \frac{2006z^{2006} - z^{2005} - z^{2004} - \dots - 1}{z^{2007} - 1} \cdot \frac{z - 1}{z - 1} = \frac{2006z^{2007} - 2007z^{2006} + 1}{(z^{2007} - 1)(z - 1)}. \end{aligned}$$

Plugging in $z = 2$ gives $\frac{2005 \cdot 2^{2006} + 1}{2^{2007} - 1}$; whence the answer.

9. [7] g is a twice differentiable function over the positive reals such that

$$g(x) + 2x^3 g'(x) + x^4 g''(x) = 0 \quad \text{for all positive reals } x. \quad (1)$$

$$\lim_{x \rightarrow \infty} xg(x) = 1 \quad (2)$$

Find the real number $\alpha > 1$ such that $g(\alpha) = 1/2$.

Answer: $\frac{6}{\pi}$. In the first equation, we can convert the expression $2x^3g'(x) + x^4g''(x)$ into the derivative of a product, and in fact a second derivative, by writing $y = 1/x$. Specifically,

$$\begin{aligned} 0 = g(x) + 2x^3g'(x) + x^4g''(x) &= g\left(\frac{1}{y}\right) + 2y^{-3}g'\left(\frac{1}{y}\right) + y^{-4}g''\left(\frac{1}{y}\right) \\ &= g\left(\frac{1}{y}\right) + \frac{d}{dy}\left[-y^{-2}g'\left(\frac{1}{y}\right)\right] \\ &= g\left(\frac{1}{y}\right) + \frac{d^2}{dy^2}\left[g\left(\frac{1}{y}\right)\right] \end{aligned}$$

Thus $g\left(\frac{1}{y}\right) = c_1 \cos(y) + c_2 \sin(y)$ or $g(x) = c_1 \cos(1/x) + c_2 \sin(1/x)$. Now the second condition gives

$$1 = \lim_{x \rightarrow \infty} c_1x + c_2 \cdot \frac{\sin(1/x)}{1/x} = c_2 + \lim_{x \rightarrow \infty} c_1x$$

It must be that $c_1 = 0, c_2 = 1$. Now since $0 < 1/\alpha < 1$, the value of α such that $g(\alpha) = \sin(1/\alpha) = 1/2$ is given by $1/\alpha = \pi/6$ and so $\alpha = 6/\pi$.

10. [8] Compute

$$\int_0^\infty \frac{e^{-x} \sin(x)}{x} dx$$

Answer: $\frac{\pi}{4}$. We can compute the integral by introducing a parameter and exchanging the order of integration:

$$\begin{aligned} \int_0^\infty e^{-x} \left(\frac{\sin(x)}{x}\right) dx &= \int_0^\infty e^{-x} \left(\int_0^1 \cos(ax) da\right) dx = \int_0^1 \left(\int_0^\infty e^{-x} \cos(ax) dx\right) da \\ &= \int_0^1 \operatorname{Re} \left[\int_0^\infty e^{(-1+ai)x} dx\right] da = \int_0^1 \operatorname{Re} \left[\frac{e^{(-1+ai)x}}{-1+ai} \Big|_{x=0}^\infty\right] da \\ &= \int_0^1 \operatorname{Re} \left[\frac{1}{1-ai}\right] da = \int_0^1 \operatorname{Re} \left[\frac{1+ai}{1+a^2}\right] da \\ &= \int_0^1 \frac{1}{1+a^2} da = \tan^{-1}(a) \Big|_{a=0}^1 = \frac{\pi}{4} \end{aligned}$$