10^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 24 February 2007

Individual Round: Combinatorics Test

1. [**3**] A committee of 5 is to be chosen from a group of 9 people. How many ways can it be chosen, if Biff and Jacob must serve together or not at all, and Alice and Jane refuse to serve with each other?

   Answer: $\boxed{41.}$ If Biff and Jacob are on the committee, there are $\binom{7}{3} = 35$ ways for the other members to be chosen. Amongst these 35 possibilities, we reject the $\binom{5}{1} = 5$ choices where both Alice and Jane are also serving. If Biff and Jacob are not serving, then there are $\binom{7}{5} = 21$ ways to choose the remaining 5 members. Again, we reject the $\binom{5}{3} = 10$ instances where Alice and Jane are chosen, so the total is $(35-5)+(21-10) = 41$.

2. [**3**] How many 5-digit numbers $\overline{abcde}$ exist such that digits b and d are each the sum of the digits to their immediate left and right? (That is, $b = a + c$ and $d = c + e$.)

   Answer: $\boxed{330.}$ Note that $a > 0$, so that $b > c$, and $e \geq 0$ so that $d \geq c$. Conversely, for each choice of (b, c, d) with $b > c$ and $d \geq c$, there exists a unique pair (a, e) such that $\overline{abcde}$ is a number having the desired property. Thus, we compute

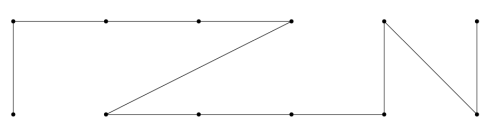
$$\sum_{c=0}^{9}(9-c)(10-c) = \sum_{c=0}^{9} c^2 - 19c + 90 = 330.$$

3. [**4**] Jack, Jill, and John play a game in which each randomly picks and then *replaces* a card from a standard 52 card deck, until a spades card is drawn. What is the probability that Jill draws the spade? (Jack, Jill, and John draw in that order, and the game repeats if no spade is drawn.)

   Answer: $\boxed{\frac{12}{37}.}$ The desired probability is the relative probability that Jill draws the spade. In the first round, Jack, Jill, and John draw a spade with probability $1/4, 3/4 \cdot 1/4$, and $(3/4)^2 \cdot 1/4$ respectively. Thus, the probability that Jill draws the spade is

$$\frac{3/4 \cdot 1/4}{1/4 + 3/4 \cdot 1/4 + (3/4)^2 \cdot 1/4} = \frac{12}{37}.$$

4. [**4**] On the Cartesian grid, Johnny wants to travel from $(0, 0)$ to $(5, 1)$, and he wants to pass through all twelve points in the set $S = \{(i, j) \mid 0 \leq i \leq 1, 0 \leq j \leq 5, i, j \in \mathbb{Z}\}$. Each step, Johnny may go from one point in S to another point in S by a line segment connecting the two points. How many ways are there for Johnny to start at $(0, 0)$ and end at $(5, 1)$ so that he never crosses his own path?



   Answer: $\boxed{252.}$ Observe that Johnny needs to pass through the points $(0, 0)$, $(1, 0)$, $(2, 0)$, $\ldots$, $(5, 0)$ in that order, and he needs to pass through $(0, 1)$, $(1, 1)$, $(2, 1)$, $\ldots$, $(5, 1)$ in that order, or else he will intersect his own path. Then, the problem is equivalent to interlacing those two sequence together, so that the first term is $(0, 0)$ and the final term is $(5, 1)$. To do this, we need to select 5 positions out of 10 to have points with x-coordinate 0. Hence the answer is $\binom{10}{5} = 252$.

5. [**5**] Determine the number of ways to select a positive number of squares on an 8×8 chessboard such that no two lie in the same row or the same column and no chosen square lies to the left of and below another chosen square.

Answer: $\boxed{\mathbf{12869 = \binom{16}{8} - 1.}}$ If k is the number of squares chosen, then there are $\binom{8}{k}$ ways to choose k columns, and $\binom{8}{k}$ ways to choose k rows, and this would uniquely determine the set of squares selected. Thus the answer is

$$\sum_{k=1}^{8}\binom{8}{k}\binom{8}{k} = -1 + \sum_{k=0}^{8}\binom{8}{k}\binom{8}{k} = -1 + \binom{16}{8} = 12869$$

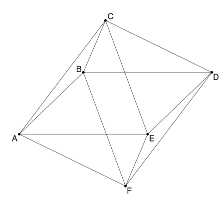
6. [**5**] Kevin has four red marbles and eight blue marbles. He arranges these twelve marbles randomly, in a ring. Determine the probability that no two red marbles are adjacent.

   Answer: $\boxed{\mathbf{\frac{7}{33}}.}$ Select any blue marble and consider the remaining eleven marbles, arranged in a line. The proportion of arrangement for which no two red marbles are adjacent will be the same as for the original twelve marbles, arranged in a ring. The total number of ways of arranging 4 red marbles out of 11 is $\binom{11}{4} = 330$. To count the number of arrangements such that no two red marbles are adjacent, there must be one red marble between each two would-be adjacent red marbles. Having fixed the positions of three blue marbles we have four blue marbles to play with. So that we can arrange the remaining four marbles is $\binom{8}{4} = 70$ ways. This yields a probability of $70/330 = 7/33$ as our final answer.

7. [**5**] Forty two cards are labeled with the natural numbers 1 through 42 and randomly shuffled into a stack. One by one, cards are taken off of the top of the stack until a card labeled with a prime number is removed. How many cards are removed on average?

   Answer: $\boxed{\mathbf{\frac{43}{14}}.}$ Note that there are 13 prime numbers amongst the cards. We may view these as separating the remaining 29 cards into 14 groups of nonprimes - those appearing before the first prime, between the first and second, etc. Each of these groups is equally likely to appear first, so 29/14 nonprimes are removed on average. We are done since exactly one prime is always drawn.

8. [**6**] A set of six edges of a regular octahedron is called *Hamiltonian cycle* if the edges in some order constitute a single continuous loop that visits each vertex exactly once. How many ways are there to partition the twelve edges into two Hamiltonian cycles?



   Answer: $\boxed{\mathbf{6}.}$ Call the octahedron $ABCDEF$, where A, B, and C are opposite D, E, and F, respectively. Note that each Hamiltonian cycle can be described in terms of the order it visits vertices in exactly 12 different ways. Conversely, listing the six vertices in some order determines a Hamiltonian cycle precisely when no pair of opposite vertices are listed consecutively or first-and-last. Suppose we begin with AB. If D is listed third, then the final three letters are CEF or FEC. Otherwise, C or F is listed next, and each gives three possibilities for the final three. For example ABC is be followed by DEF, DFE, or EDF. Thus, there are $6 \cdot 4 \cdot (2+3+3) = 192$ listings. These correspond to $192/12 = 16$ Hamiltonian cycles. Finally, the complement of all but four Hamiltonian cycles is a Hamiltonian cycle. For, each vertex has degree four, so is an endpoint of two edges in the complement of a Hamiltonian cycle, so is also a Hamiltonian cycle unless it describes two opposite faces. It follows that there are six pairs of disjoint Hamiltonian cycles.

9. [**7**] Let S denote the set of all triples (i, j, k) of positive integers where $i + j + k = 17$. Compute

$$\sum_{(i,j,k)\in S} ijk.$$

Answer: $\boxed{\mathbf{11628 = \binom{19}{5}}.}$ We view choosing five objects from a row of 19 objects in an unusual way. First, remove two of the chosen objects, the second and fourth, which are not adjacent nor at either end, forming three nonempty groups of consecutive objects. We then have i, j, and k choices for the first, third, and fifth objects. Because this is a reversible process taking a triple (i,j,k) to ijk choices, the answer is $\binom{19}{5} = 11628$.

A simple generating functions argument is also possible. Let $s_n = \sum_{i+j+k=n} ijk$. Then

$$\sum_{n\geq 0} s_n x^n = \left(\sum_{n\geq 0} n x^n\right)^3 = \left(\frac{x}{(1-x)^2}\right)^3 = \frac{x^3}{(1-x)^6},$$

and so $s_n = \left(\!\binom{6}{n-3}\!\right) = \binom{n+2}{5}$, yielding $s_{17} = \binom{19}{5}$.

10. [**8**] A subset S of the nonnegative integers is called *supported* if it contains 0, and $k+8, k+9 \in S$ for all $k \in S$. How many supported sets are there?

Answer: $\boxed{\mathbf{1430}.}$ Note that every supported set S contains 0, 8, 9, 16, 17, 18, 24-27, 32-36, 40-45, 48-54, and all $n \geq 55$. Now define $\overline{S} := \mathbb{Z}^+ \setminus S$, which is a subset of $\{1-7, 10-15, 19-23, 28-31, 37, 38, 39, 46, 47, 55\}$ satisfying the opposite property that $k \in \overline{S} \Longrightarrow k-8, k-9 \in \overline{S}$.

								55								
							46		47							
						37		38		39						
					28		29		30		31					
				19		20		21		22		23				
			10		11		12		13		14		15			
		1		2		3		4		5		6		7		
	*		*		*		*		*		*		*		*	
(0,0) *		*		*		*		*		*		*		*		* (16,0)

Consider the above arrangement after removing the numbers not in $\overline{S}$. The condition that S be supported ensures that sets $\overline{S}$ are in bijective correspondence with paths from (0,0) to (16,0) consisting of discrete steps of $\langle 1,1\rangle$ and $\langle 1,-1\rangle$ and lying above the x-axis: from the modified version of the above diagram, a unique path passes through the top items left in each column. The number of such paths is the 8th Catalan number, so the answer is $C_8 = \frac{1}{8+1}\binom{8\cdot 2}{8} = \frac{12870}{9} = 1430$. (Incidentally, 16 choose 8 was computed in an earlier problem.) Without the explicit formula for Catalan numbers, the answer can be computed recursively by filling in the number of ways a path can reach (16,0) from each position in the figure. One works right to left, obtaining the following:

								1								
							8		1							
						35		7		1						
					110		27		6		1					
				275		75		20		5		1				
			572		165		48		14		4		1			
		1001		297		90		28		9		3		1		
	1430		429		132		42		14		5		2		1	
1430		429		132		42		14		5		2		1		1

One can exploit symmetry and, having determined the middle column, sum the squares: $1^2 + 7^2 + 20^2 + 28^2 + 14^2 = 1430$.