

10th Annual Harvard-MIT Mathematics Tournament
Saturday 24 February 2007

Guts Round

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10th HARVARD-MIT MATHEMATICS TOURNAMENT, 24 FEBRUARY 2007 — GUTS ROUND

Note that there are just 36 problems in the Guts round this year.

1. [5] Define the sequence of positive integers a_n recursively by $a_1 = 7$ and $a_n = 7^{a_{n-1}}$ for all $n \geq 2$. Determine the last two digits of a_{2007} .

Answer: 43. Note that the last two digits of 7^4 are 01. Also, $a_{2006} = 7^{a_{2005}} = (-1)^{a_{2005}} = -1 = 3 \pmod{4}$ since a_{2005} is odd. Therefore, $a_{2007} = 7^{a_{2006}} = 7^3 = 43 \pmod{100}$.

2. [5] A candy company makes 5 colors of jellybeans, which come in equal proportions. If I grab a random sample of 5 jellybeans, what is the probability that I get exactly 2 distinct colors?

Answer: $\frac{12}{125}$. There are $\binom{5}{2} = 10$ possible pairs of colors. Each pair of colors contributes $2^5 - 2 = 30$ sequences of beans that use *both* colors. Thus, the answer is $10 \cdot 30/5^5 = 12/125$.

3. [5] The equation $x^2 + 2x = i$ has two complex solutions. Determine the product of their real parts.

Answer: $\frac{1-\sqrt{2}}{2}$. Complete the square by adding 1 to each side. Then $(x+1)^2 = 1+i = e^{i\pi/4}\sqrt{2}$, so $x+1 = \pm e^{i\pi/8}\sqrt[4]{2}$. The desired product is then

$$\left(-1 + \cos\left(\frac{\pi}{8}\right)\sqrt[4]{2}\right)\left(-1 - \cos\left(\frac{\pi}{8}\right)\sqrt[4]{2}\right) = 1 - \cos^2\left(\frac{\pi}{8}\right)\sqrt{2} = 1 - \frac{(1 + \cos(\frac{\pi}{4}))}{2}\sqrt{2} = \frac{1 - \sqrt{2}}{2}.$$

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4. [6] A sequence consists of the digits 122333444455555... such that the each positive integer n is repeated n times, in increasing order. Find the sum of the 4501st and 4052nd digits of this sequence.

Answer: 13. Note that n contributes $n \cdot d(n)$ digits, where $d(n)$ is the number of digits of n . Then because $1 + \dots + 99 = 4950$, we know that the digits of interest appear amongst copies of two digit numbers. Now for $10 \leq n \leq 99$, the number of digits in the subsequence up to the last copy of n is

$$1 + 2 + 3 + \dots + 9 + 2 \cdot (10 + \dots + n) = 2 \cdot (1 + \dots + n) - 45 = n^2 + n - 45.$$

Since $67^2 + 67 - 45 = 4511$, the two digits are 6 and 7 in some order, so have sum 13.

5. [6] Compute the largest positive integer such that $\frac{2007!}{2007^n}$ is an integer.

Answer: 9. Note that $2007 = 3^2 \cdot 223$. Using the fact that the number of times a prime p divides $n!$ is given by

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots,$$

it follows that the answer is 9.

6. [6] There are three video game systems: the Paystation, the WHAT, and the ZBoz2 π , and none of these systems will play games for the other systems. Uncle Riemann has three nephews: Bernoulli, Galois, and Dirac. Bernoulli owns a Paystation and a WHAT, Galois owns a WHAT and a ZBoz2 π , and Dirac owns a ZBoz2 π and a Paystation. A store sells 4 different games for the Paystation, 6 different games for the WHAT, and 10 different games for the ZBoz2 π . Uncle Riemann does not understand the

difference between the systems, so he walks into the store and buys 3 random games (not necessarily distinct) and randomly hands them to his nephews. What is the probability that each nephew receives a game he can play?

Answer: $\frac{7}{25}$. Since the games are not necessarily distinct, probabilities are independent. Multiplying the odds that each nephew receives a game he can play, we get $10/20 \cdot 14/20 \cdot 16/20 = 7/25$.

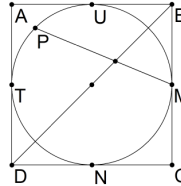
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7. [7] A student at Harvard named Kevin
 Was counting his stones by 11
 He messed up n times
 And instead counted 9s
 And wound up at 2007.

How many values of n could make this limerick true?

Answer: $\boxed{21}$. The mathematical content is that $9n + 11k = 2007$, for some nonnegative integers n and k . As $2007 = 9 \cdot 223$, k must be divisible by 9. Using modulo 11, we see that n is 3 more than a multiple of 11. Thus, the possibilities are $n = 223, 212, 201, \dots, 3$, which are 21 in number.

8. [7] A circle inscribed in a square,
 Has two chords as shown in a pair.
 It has radius 2,
 And P bisects TU .
 The chords' intersection is where?



Answer the question by giving the distance of the point of intersection from the center of the circle.

Answer: $\boxed{2\sqrt{2} - 2}$. Let OB intersect the circle at X and Y , and the chord PM at Q , such that O lies between X and Q . Then $MNXQ$ is a parallelogram. For, $OB \parallel NM$ by homothety at C and $PM \parallel NX$ because $MNXQ$ is an isosceles trapezoid. It follows that $QX = MN$. Considering that the center of the circle together with points M, C , and N determines a square of side length 2, it follows that $MN = 2\sqrt{2}$, so the answer is $2\sqrt{2} - 2$.

9. [7] I ponder some numbers in bed,
 All products of three primes I've said,
 Apply ϕ they're still fun:
 now Elev'n cubed plus one.
 What numbers could be in my head?
- $$n = 37^2 \cdot 3 \dots$$
- $$\phi(n) = 11^3 + 1?$$

Answer: $\boxed{2007, 2738, 3122}$. The numbers expressible as a product of three primes are each of the form p^3 , p^2q , or pqr , where p, q , and r are distinct primes. Now, $\phi(p^3) = p^2(p-1)$, $\phi(p^2q) = p(p-1)(q-1)$, and $\phi(pqr) = (p-1)(q-1)(r-1)$. We require $11^3 + 1 = 12 \cdot 111 = 2^2 \cdot 3^2 \cdot 37$. The first case is easy to rule out, since necessarily $p = 2$ or $p = 3$, which both fail. The second case requires $p = 2$, $p = 3$, or $p = 37$. These give $q = 667, 223$, and 2 , respectively. As $667 = 23 \cdot 29$, we reject $2^2 \cdot 667$, but $3^2 \cdot 223 = 2007$ and $37^2 \cdot 2 = 2738$. In the third case, exactly one of the primes is 2, since all other primes are odd. So say $p = 2$. There are three possibilities for (q, r) : $(2 \cdot 1 + 1, 2 \cdot 3^2 \cdot 37 + 1)$, $(2 \cdot 3 + 1, 2 \cdot 3 \cdot 37 + 1)$, and $(2 \cdot 3^2 + 1, 2 \cdot 37 + 1)$. Those are $(3, 667)$, $(7, 223)$, and $(19, 75)$, respectively, of which only $(7, 223)$ is a pair of primes. So the third and final possibility is $2 \cdot 7 \cdot 223 = 3122$.

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10. [8] Let A_{12} denote the answer to problem 12. There exists a unique triple of digits (B, C, D) such that $10 > A_{12} > B > C > D > 0$ and

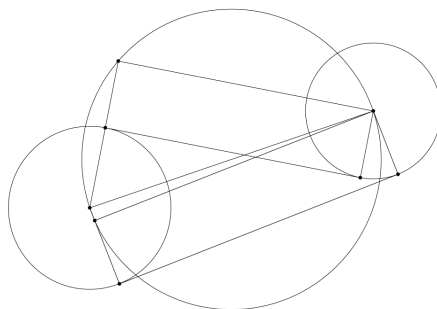
$$\overline{A_{12}BCD} - \overline{DCBA_{12}} = \overline{BDA_{12}C},$$

where $\overline{A_{12}BCD}$ denotes the four digit base 10 integer. Compute $B + C + D$.

Answer: [11.] Since $D < A_{12}$, when A is subtracted from D we must carry over from C . Thus, $D + 10 - A_{12} = C$. Next, since $C - 1 < C < B$, we must carry over from the tens digit, so that $(C - 1 + 10) - B = A_{12}$. Now $B > C$ so $B - 1 \geq C$, and $(B - 1) - C = D$. Similarly, $A_{12} - D = B$. Solving this system of four equations produces $(A_{12}, B, C, D) = (7, 6, 4, 1)$.

11. [8] Let A_{10} denote the answer to problem 10. Two circles lie in the plane; denote the lengths of the internal and external tangents between these two circles by x and y , respectively. Given that the product of the radii of these two circles is $15/2$, and that the distance between their centers is A_{10} , determine $y^2 - x^2$.

Answer: [30.] Suppose the circles have radii r_1 and r_2 . Then using the tangents to build right triangles, we have $x^2 + (r_1 + r_2)^2 = A_{10}^2 = y^2 + (r_1 - r_2)^2$. Thus, $y^2 - x^2 = (r_1 + r_2)^2 - (r_1 - r_2)^2 = 4r_1r_2 = 30$.



12. [8] Let A_{11} denote the answer to problem 11. Determine the smallest prime p such that the arithmetic sequence $p, p + A_{11}, p + 2A_{11}, \dots$ begins with the largest possible number of primes.

Answer: [7.] First, note that the maximal number of initial primes is bounded above by the smallest prime not dividing A_{11} , with equality possible only if p is this prime. For, if q is the smallest prime not dividing A_{11} , then the first q terms of the arithmetic sequence determine a complete residue class modulo q , and the multiple of q is nonprime unless it equals q . If $q < A_{11}$, then q must appear first in the sequence, and thus divide the $(q + 1)$ st term. If $q > A_{11}$, then $A_{11} = 2$ and $q = 3$ by Bertrand's postulate, so q must appear first by inspection.

Now since $A_{11} = 30$, the bound is 7. In fact, 7, 37, 67, 97, 127, and 157 are prime, but 187 is not. Then on the one hand, our bound of seven initial primes is not realizable. On the other hand, this implies an upper bound of six, and this bound is achieved by $p = 7$. Smaller primes p yield only one initial prime, so 7 is the answer.

Remarks. A number of famous theorems are concerned with the distribution of prime numbers. For two relatively prime positive integers a and b , the arithmetic progression $a, a + b, a + 2b, \dots$ contains infinitely many primes, a result known as *Dirichlet's theorem*. It was shown recently (c. 2004) that there exist arbitrarily long arithmetic progressions consisting of primes only. *Bertrand's postulate* states that for any positive integer n , there exists a prime p such that $n < p \leq 2n$. This is an unfortunate misnomer, as the statement is known to be true. As with many theorems concerning the distributions of primes, these results are easily stated in elementary terms, concealing elaborate proofs.

There is just one triple of possible (A_{10}, A_{11}, A_{12}) of answers to these three problems. Your team will receive credit only for answers matching these. (So, for example, submitting a wrong answer for problem 11 will not alter the correctness of your answer to problem 12.)

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13. [9] Determine the largest integer n such that $7^{2048} - 1$ is divisible by 2^n .

Answer: 14. We have

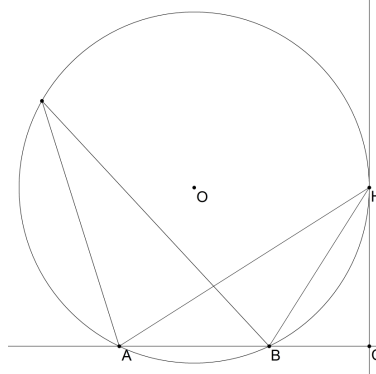
$$7^{2048} - 1 = (7 - 1)(7 + 1)(7^2 + 1)(7^4 + 1) \cdots (7^{1024} + 1).$$

In the expansion, the eleven terms other than $7 + 1$ are divisible by 2 exactly once, as can be checked easily with modulo 4.

14. [9] We are given some similar triangles. Their areas are $1^2, 3^2, 5^2, \dots$, and 49^2 . If the smallest triangle has a perimeter of 4, what is the sum of all the triangles' perimeters?

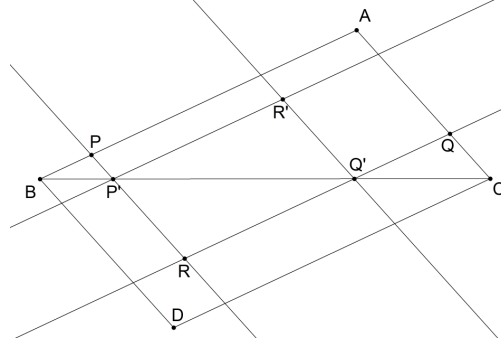
Answer: 2500. Because the triangles are all similar, they all have the same ratio of perimeter squared to area, or, equivalently, the same ratio of perimeter to the square root of area. Because the latter ratio is 4 for the smallest triangle, it is 4 for all the triangles, and thus their perimeters are $4 \cdot 1, 4 \cdot 3, 4 \cdot 5, \dots, 4 \cdot 49$, and the sum of these numbers is $4(1 + 3 + 5 + \dots + 49) = 4(25^2) = 2500$.

15. [9] Points A, B , and C lie in that order on line ℓ , such that $AB = 3$ and $BC = 2$. Point H is such that CH is perpendicular to ℓ . Determine the length CH such that $\angle AHB$ is as large as possible.



Answer: $\sqrt{10}$. Let ω denote the circumcircle of triangle ABH . Since AB is fixed, the smaller the radius of ω , the bigger the angle AHB . If ω crosses the line CH in more than one point, then there exists a smaller circle that goes through A and B that crosses CH at a point H' . But angle $AH'B$ is greater than AHB , contradicting our assumption that H is the optimal spot. Thus the circle ω crosses the line CH at exactly one spot: ie, ω is tangent to CH at H . By Power of a Point, $CH^2 = CA \cdot CB = 5 \cdot 2 = 10$, so $CH = \sqrt{10}$.

16. [10] Let ABC be a triangle with $AB = 7, BC = 9$, and $CA = 4$. Let D be the point such that $AB \parallel CD$ and $CA \parallel BD$. Let R be a point within triangle BCD . Lines ℓ and m going through R are parallel to CA and AB respectively. Line ℓ meets AB and BC at P and P' respectively, and m meets CA and BC at Q and Q' respectively. If S denotes the largest possible sum of the areas of triangles $BPP', RP'Q'$, and CQQ' , determine the value of S^2 .



Answer: 180. Let R' denote the intersection of the lines through Q' and P' parallel to ℓ and m respectively. Then $[RP'Q'] = [R'P'Q']$. Triangles BPP' , $R'P'Q'$, and CQQ' lie in ABC without overlap, so that on the one hand, $S \leq ABC$. On the other, this bound is realizable by taking R to be a vertex of triangle BCD . We compute the square of the area of ABC to be $10 \cdot (10-9) \cdot (10-7) \cdot (10-4) = 180$.

17. [10] During the regular season, Washington Redskins achieve a record of 10 wins and 6 losses. Compute the probability that their wins came in three streaks of consecutive wins, assuming that all possible arrangements of wins and losses are equally likely. (For example, the record LLWWWWLWLLWWL contains three winning streaks, while WWWWWLWLLLLLWWW has just two.)

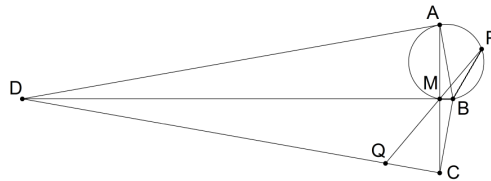
Answer: $\frac{45}{286}$. Suppose the winning streaks consist of w_1, w_2 , and w_3 wins, in chronological order, where the first winning streak is preceded by l_0 consecutive losses and the i winning streak is immediately succeeded by l_i losses. Then $w_1, w_2, w_3, l_1, l_2 > 0$ are positive and $l_0, l_3 \geq 0$ are nonnegative. The equations

$$w_1 + w_2 + w_3 = 10 \quad \text{and} \quad (l_0 + 1) + l_1 + l_2 + (l_3 + 1) = 8,$$

are independent, and have $\binom{9}{2}$ and $\binom{7}{3}$ solutions, respectively. It follows that the answer is

$$\frac{\binom{9}{2} \binom{7}{3}}{\binom{16}{6}} = \frac{315}{2002}.$$

18. [10] Convex quadrilateral $ABCD$ has right angles $\angle A$ and $\angle C$ and is such that $AB = BC$ and $AD = CD$. The diagonals AC and BD intersect at point M . Points P and Q lie on the circumcircle of triangle AMB and segment CD , respectively, such that points P, M , and Q are collinear. Suppose that $m\angle ABC = 160^\circ$ and $m\angle QMC = 40^\circ$. Find $MP \cdot MQ$, given that $MC = 6$.



Answer: 36. Note that $m\angle QPB = m\angle MPB = m\angle MAB = m\angle CAB = \angle BCA = \angle CDB$. Thus, $MP \cdot MQ = MB \cdot MD$. On the other hand, segment CM is an altitude of right triangle BCD , so $MB \cdot MD = MC^2 = 36$.

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19. [10] Define $x \star y = \frac{\sqrt{x^2 + 3xy + y^2 - 2x - 2y + 4}}{xy + 4}$. Compute

$$((\dots((2007 \star 2006) \star 2005) \star \dots) \star 1).$$

Answer: $\frac{\sqrt{15}}{9}$. Note that $x \star 2 = \frac{\sqrt{x^2+6x+4}-2x-4+4}{2x+4} = \frac{\sqrt{(x+2)^2}}{2(x+2)} = \frac{1}{2}$ for $x > -2$. Because $x \star y > 0$ if $x, y > 0$, we need only compute $\frac{1}{2} \star 1 = \frac{\sqrt{\frac{1}{4} + \frac{3}{2} + 1} - 3 + 4}{\frac{1}{2} + 4} = \frac{\sqrt{15}}{9}$.

20. [10] For a a positive real number, let x_1, x_2, x_3 be the roots of the equation $x^3 - ax^2 + ax - a = 0$. Determine the smallest possible value of $x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3$.

Answer: -4 . Note that $x_1 + x_2 + x_3 = x_1x_2 + x_2x_3 + x_3x_1 = a$. Then

$$\begin{aligned} x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 &= (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 - (x_1x_2 + x_2x_3 + x_3x_1)) \\ &= (x_1 + x_2 + x_3) \left((x_1 + x_2 + x_3)^2 - 3(x_1x_2 + x_2x_3 + x_3x_1) \right) = a \cdot (a^2 - 3a) = a^3 - 3a^2. \end{aligned}$$

The expression is negative only where $0 < a < 3$, so we need only consider these values of a . Finally, AM-GM gives $\sqrt[3]{(6-2a)(a)(a)} \leq \frac{(6-2a)+a+a}{3} = 2$, with equality where $a = 2$, and this rewrites as $(a-3)a^2 \geq -4$.

21. [10] Bob the bomb-defuser has stumbled upon an active bomb. He opens it up, and finds the red and green wires conveniently located for him to cut. Being a seasoned member of the bomb-squad, Bob quickly determines that it is the green wire that he should cut, and puts his wirecutters on the green wire. But just before he starts to cut, the bomb starts to count down, ticking every second. Each time the bomb ticks, starting at time $t = 15$ seconds, Bob panics and has a certain chance to move his wirecutters to the other wire. However, he is a rational man even when panicking, and has a $\frac{1}{2t^2}$ chance of switching wires at time t , regardless of which wire he is about to cut. When the bomb ticks at $t = 1$, Bob cuts whatever wire his wirecutters are on, without switching wires. What is the probability that Bob cuts the green wire?

Answer: $\frac{23}{30}$. Suppose Bob makes n independent decisions, with probabilities of switching p_1, p_2, \dots, p_n . Then in the expansion of the product

$$P(x) = (p_1 + (1 - p_1)x)(p_2 + (1 - p_2)x) \cdots (p_n + (1 - p_n)x),$$

the sum of the coefficients of even powers of x gives the probability that Bob makes his original decision. This is just $(P(1) + P(-1))/2$, so the probability is just

$$\frac{1 + \left(1 - \frac{1}{15 \cdot 15}\right) \left(1 - \frac{1}{14 \cdot 14}\right) \cdots \left(1 - \frac{1}{2 \cdot 2}\right)}{2} = \frac{1 + \frac{14}{15} \frac{16}{15} \frac{13}{14} \frac{15}{14} \cdots \frac{1}{2} \frac{3}{2}}{2} = \frac{1 + \frac{8}{15}}{2} = \frac{23}{30}.$$

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22. [12] The sequence $\{a_n\}_{n \geq 1}$ is defined by $a_{n+2} = 7a_{n+1} - a_n$ for positive integers n with initial values $a_1 = 1$ and $a_2 = 8$. Another sequence, $\{b_n\}$, is defined by the rule $b_{n+2} = 3b_{n+1} - b_n$ for positive integers n together with the values $b_1 = 1$ and $b_2 = 2$. Find $\gcd(a_{5000}, b_{501})$.

Answer: 89 . We show by induction that $a_n = F_{4n-2}$ and $b_n = F_{2n-1}$, where F_k is the k th Fibonacci number. The base cases are clear. As for the inductive steps, note that

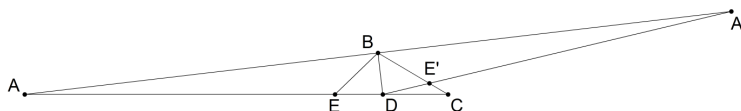
$$F_{k+2} = F_{k+1} + F_k = 2F_k + F_{k-1} = 3F_k - F_{k-2}$$

and

$$F_{k+4} = 3F_{k+2} - F_k = 8F_k + 3F_{k-2} = 7F_k - F_{k-4}.$$

We wish to compute the greatest common denominator of F_{19998} and F_{1001} . The Fibonacci numbers satisfy the property that $\gcd(F_m, F_n) = F_{\gcd(m,n)}$, which can be proven by noting that they are periodic modulo any positive integer. So since $\gcd(19998, 1001) = 11$, the answer is $F_{11} = 89$.

23. [12] In triangle ABC , $\angle ABC$ is obtuse. Point D lies on side AC such that $\angle ABD$ is right, and point E lies on side AC between A and D such that BD bisects $\angle EBC$. Find CE , given that $AC = 35$, $BC = 7$, and $BE = 5$.



Answer: 10. Reflect A and E over BD to A' and E' respectively. Note that the angle conditions show that A' and E' lie on AB and BC respectively. B is the midpoint of segment AA' and $CE' = BC - BE' = 2$. Menelaus' theorem now gives

$$\frac{CD}{DA} \cdot \frac{AA'}{A'B} \cdot \frac{BE'}{E'C} = 1,$$

from which $DA = 5CD$ or $CD = AC/6$. By the angle bisector theorem, $DE = 5CD/7$, so that $CE = 12CD/7 = 10$.

24. [12] Let x, y, n be positive integers with $n > 1$. How many ordered triples (x, y, n) of solutions are there to the equation $x^n - y^n = 2^{100}$?

Answer: 49. Break all possible values of n into the four cases: $n = 2$, $n = 4$, $n > 4$ and n odd. By Fermat's theorem, no solutions exist for the $n = 4$ case because we may write $y^4 + (2^{25})^4 = x^4$.

We show that for n odd, no solutions exist to the more general equation $x^n - y^n = 2^k$ where k is a positive integer. Assume otherwise for contradiction's sake, and suppose on the grounds of well ordering that k is the least exponent for which a solution exists. Clearly x and y must both be even or both odd. If both are odd, we have $(x - y)(x^{n-1} + \dots + y^{n-1})$. The right factor of this expression contains an odd number of odd terms whose sum is an odd number greater than 1, impossible. Similarly if x and y are even, write $x = 2u$ and $y = 2v$. The equation becomes $u^n - v^n = 2^{k-n}$. If $k - n$ is greater than 0, then our choice k could not have been minimal. Otherwise, $k - n = 0$, so that two consecutive positive integers are perfect n th powers, which is also absurd.

For the case that n is even and greater than 4, consider the same generalization and hypotheses. Writing $n = 2m$, we find $(x^m - y^m)(x^m + y^m) = 2^k$. Then $x^m - y^m = 2^a < 2^k$. By our previous work, we see that m cannot be an odd integer greater than 1. But then m must also be even, contrary to the minimality of k .

Finally, for $n = 2$ we get $x^2 - y^2 = 2^{100}$. Factoring the left hand side gives $x - y = 2^a$ and $x + y = 2^b$, where implicit is $a < b$. Solving, we get $x = 2^{b-1} + 2^{a-1}$ and $y = 2^{b-1} - 2^{a-1}$, for a total of 49 solutions. Namely, those corresponding to $(a, b) = (1, 99), (2, 98), \dots, (49, 51)$.

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25. [12] Two real numbers x and y are such that $8y^4 + 4x^2y^2 + 4xy^2 + 2x^3 + 2y^2 + 2x = x^2 + 1$. Find all possible values of $x + 2y^2$.

Answer: $\frac{1}{2}$. Writing $a = x + 2y^2$, the given quickly becomes $4y^2a + 2x^2a + a + x = x^2 + 1$. We can rewrite $4y^2a$ for further reduction to $a(2a - 2x) + 2x^2a + a + x = x^2 + 1$, or

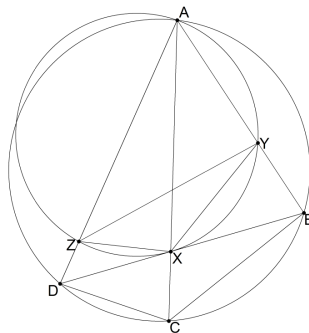
$$2a^2 + (2x^2 - 2x + 1)a + (-x^2 + x - 1) = 0. \quad (*)$$

The quadratic formula produces the discriminant

$$(2x^2 - 2x + 1)^2 + 8(x^2 - x + 1) = (2x^2 - 2x + 3)^2,$$

an identity that can be treated with the difference of squares, so that $a = \frac{-2x^2 + 2x - 1 \pm (2x^2 - 2x + 3)}{4} = \frac{1}{2}, -x^2 + x - 1$. Now a was constructed from x and y , so is not free. Indeed, the second expression flies in the face of the trivial inequality: $a = -x^2 + x - 1 < -x^2 + x \leq x + 2y^2 = a$. On the other hand, $a = 1/2$ is a bona fide solution to (*), which is identical to the original equation.

26. [12] $ABCD$ is a cyclic quadrilateral in which $AB = 4, BC = 3, CD = 2$, and $AD = 5$. Diagonals AC and BD intersect at X . A circle ω passes through A and is tangent to BD at X . ω intersects AB and AD at Y and Z respectively. Compute YZ/BD .



Answer: $\frac{115}{143}$. Denote the lengths AB, BC, CD , and DA by a, b, c , and d respectively. Because $ABCD$ is cyclic, $\triangle ABX \sim \triangle DCX$ and $\triangle ADX \sim \triangle BCX$. It follows that $\frac{AX}{DX} = \frac{BX}{CX} = \frac{a}{c}$ and $\frac{AX}{BX} = \frac{DX}{CX} = \frac{d}{b}$. Therefore we may write $AX = adk, BX = abk, CX = bck$, and $DX = cdk$ for some k . Now, $\angle XDC = \angle BAX = \angle YXB$ and $\angle DCX = \angle XBY$, so $\triangle BXY \sim \triangle CDX$. Thus, $XY = DX \cdot \frac{BX}{CD} = cdk \cdot \frac{abk}{c} = abdk^2$. Analogously, $XY = acdk^2$. Note that $XY/XZ = CB/CD$. Since $\angle YXZ = \pi - \angle ZAY = \angle BCD$, we have that $\triangle XYZ \sim \triangle CBD$. Thus, $YZ/BD = XY/CB = adk^2$. Finally, Ptolemy's theorem applied to $ABCD$ gives

$$(ad + bc)k \cdot (ab + cd)k = ac + bd$$

It follows that the answer is

$$\frac{ad(ac + bd)}{(ab + cd)(ad + bc)} = \frac{20 \cdot 23}{22 \cdot 26} = \frac{115}{143}$$

27. [12] Find the number of 7-tuples (n_1, \dots, n_7) of integers such that

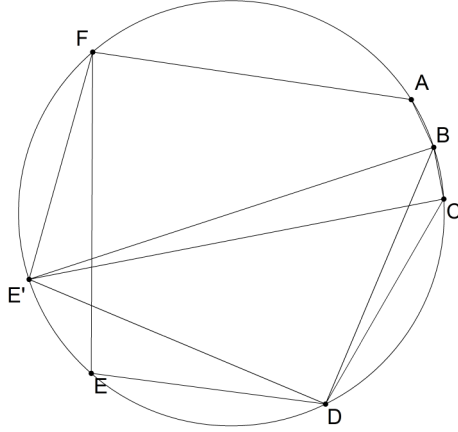
$$\sum_{i=1}^7 n_i^6 = 96957.$$

Answer: $\mathbf{2688}$. Consider the equation in modulo 9. All perfect 6th powers are either 0 or 1. Since 9 divides 96957, it must be that each n_i is a multiple of 3. Writing $3a_i = n_i$ and dividing both sides by 3^6 , we have $a_1^6 + \dots + a_7^6 = 133$. Since sixth powers are nonnegative, $|a_i| \leq 2$. Again considering modulo 9, we see that $a_i \neq 0$. Thus, $a_i^6 \in \{1, 64\}$. The only possibility is $133 = 64 + 64 + 1 + 1 + 1 + 1 + 1$, so $|a_1|, \dots, |a_7|$ consists of 2 2's and 5 1's. It follows that the answer is $\binom{7}{2} \cdot 2^7 = 2688$.

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28. [15] Compute the circumradius of cyclic hexagon $ABCDEF$, which has side lengths $AB = BC = 2, CD = DE = 9$, and $EF = FA = 12$.



Answer: 8. Construct point E' on the circumcircle of $ABCDEF$ such that $DE' = EF = 12$ and $E'F = DE = 9$; then $\overline{BE'}$ is a diameter. Let $BE' = d$. Then $CE' = \sqrt{BE'^2 - BC^2} = \sqrt{d^2 - 4}$ and $BD = \sqrt{BE'^2 - DE'^2} = \sqrt{d^2 - 144}$. Applying Ptolemy's theorem to $BCDE'$ now yields

$$9 \cdot d + 2 \cdot 12 = \sqrt{(d^2 - 4)(d^2 - 144)}$$

Squaring and rearranging, we find $0 = d^4 - 229d^2 - 432d = d(d - 16)(d^2 + 16d + 27)$. Since d is a positive real number, $d = 16$, and the circumradius is 8.

29. [15] A sequence $\{a_n\}_{n \geq 1}$ of positive reals is defined by the rule $a_{n+1}a_{n-1}^5 = a_n^4a_{n-2}^2$ for integers $n > 2$ together with the initial values $a_1 = 8$ and $a_2 = 64$ and $a_3 = 1024$. Compute

$$\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}}$$

Answer: $3\sqrt{2}$. Taking the base-2 log of the sequence $\{a_n\}$ converts the multiplicative rule to a more familiar additive rule: $\log_2(a_{n+1}) - 4\log_2(a_n) + 5\log_2(a_{n-1}) - 2\log_2(a_{n-2}) = 0$. The characteristic equation is $0 = x^3 - 4x^2 + 5x - 2 = (x - 1)^2(x - 2)$, so $\log_2(a_n)$ is of the form $a \cdot n + b + c \cdot 2^n$ and we find $a_n = 2^{2n+2^{n-1}}$. Now,

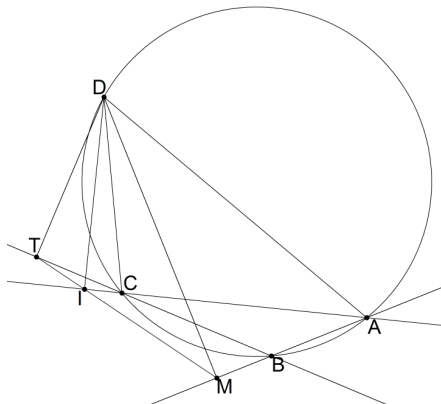
$$\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}} = \sqrt{2} \cdot \sqrt{4 + \sqrt{16 + \sqrt{64 + \cdots}}}$$

We can estimate the new nested radical expression as 3, which expands thus

$$3 = \sqrt{4 + 5} = \sqrt{4 + \sqrt{16 + 9}} = \sqrt{4 + \sqrt{16 + \sqrt{64 + 17}}} = \cdots$$

As a rigorous confirmation, we have $2^k + 1 = \sqrt{4^k + (2^{k+1} + 1)}$, as desired. It follows that the answer is $3\sqrt{2}$.

30. [15] $ABCD$ is a cyclic quadrilateral in which $AB = 3, BC = 5, CD = 6$, and $AD = 10$. M, I , and T are the feet of the perpendiculars from D to lines AB, AC , and BC respectively. Determine the value of MI/IT .



Answer: $\boxed{\frac{25}{9}}$. Quadrilaterals $AMID$ and $DICT$ are cyclic, having right angles $\angle AMD, \angle AID,$ and $\angle CID, \angle CTD$ respectively. We see that $M, I,$ and T are collinear. For, $m\angle MID = \pi - m\angle DAM = \pi - m\angle DAB = m\angle BCD = \pi - m\angle DCT = \pi - m\angle DIT$. Therefore, Menelaus' theorem applied to triangle MTB and line ICA gives

$$\frac{MI}{IT} \cdot \frac{TC}{CB} \cdot \frac{BA}{AM} = 1$$

On the other hand, triangle ADM is similar to triangle CDT since $\angle AMD \cong \angle CTD$ and $\angle DAM \cong \angle DCT$ and thus $AM/CT = AD/CD$. It follows that

$$\frac{MI}{IT} = \frac{BC \cdot AM}{AB \cdot CT} = \frac{BC \cdot AD}{AB \cdot CD} = \frac{5 \cdot 10}{3 \cdot 6} = \frac{25}{9}$$

Remarks. The line MIT , constructed in this problem by taking perpendiculars from a point on the circumcircle of ABC , is known as the *Simson line*. It is often helpful for us to use *directed angles* while angle chasing to avoid supplementary configuration issues, such as those arising while establishing the collinearity of $M, I,$ and T .

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31. [18] A sequence $\{a_n\}_{n \geq 0}$ of real numbers satisfies the recursion $a_{n+1} = a_n^3 - 3a_n^2 + 3$ for all positive integers n . For how many values of a_0 does $a_{2007} = a_0$?

Answer: $\boxed{3^{2007}}$. If x appears in the sequence, the next term $x^3 - 3x^2 + 3$ is the same if and only if $0 = x^3 - 3x^2 - x + 3 = (x - 3)(x - 1)(x + 1)$. Moreover, that next term is strictly larger if $x > 3$ and strictly smaller if $x < -1$. It follows that no values of a_0 with $|a_0 - 1| > 2$ yield $a_0 = a_{2007}$.

Now suppose $a_0 = a_{2007}$ and write $a_0 = 1 + e^{\alpha i} + e^{-\alpha i}$; the values a_0 we seek will be in bijective correspondence with solutions α where $0 \leq \alpha \leq \pi$. Then

$$a_1 = (a_0 - 1)^3 - 3a_0 + 4 = e^{3\alpha i} + 3e^{\alpha i} + 3e^{-\alpha i} + e^{-3\alpha i} - 3e^{\alpha i} - 3e^{-\alpha i} - 3 + 4 = e^{3\alpha i} + e^{-3\alpha i} + 1,$$

and an easy inductive argument gives $a_{2007} = e^{3^{2007}\alpha i} + e^{-3^{2007}\alpha i} + 1$. It follows that $a_0 = a_{2007}$ is equivalent to $\cos(\alpha) = \cos(3^{2007}\alpha)$. Now,

$$\cos(3^{2007}\alpha) - \cos(\alpha) = 2 \sin\left(\left(\frac{3^{2007} + 1}{2}\right)\alpha\right) \sin\left(\left(\frac{3^{2007} - 1}{2}\right)\alpha\right),$$

so since $\sin(kx) = 0$ for a positive integer k if and only if x is a multiple of $\frac{\pi}{k}$, the solutions α are $\{0, \frac{2\pi}{3^{2007}-1}, \frac{4\pi}{3^{2007}-1}, \dots, \pi\} \cup \{0, \frac{2\pi}{3^{2007}+1}, \dots, \pi\}$. Because our values k are consecutive, these sets overlap only at 0 and π , so there are 3^{2007} distinct α .

32. [18] Triangle ABC has $AB = 4, BC = 6$, and $AC = 5$. Let O denote the circumcenter of ABC . The circle Γ is tangent to and surrounds the circumcircles of triangles AOB, BOC , and AOC . Determine the diameter of Γ .

Answer: $\frac{256\sqrt{7}}{17}$. Denote by $\omega, \Gamma_1, \Gamma_2$, and Γ_3 the circumcenters of triangles ABC, BOC, COA , and AOB , respectively. An inversion about ω interchanges Γ_1 and line BC , Γ_2 and line CA , and Γ_3 and line AB . This inversion also preserves tangency between generalized circles, so the image of Γ is a circle tangent to AB, BC , and CA . It is the incircle of ABC because it is closer to O than these lines and ABC is acute.

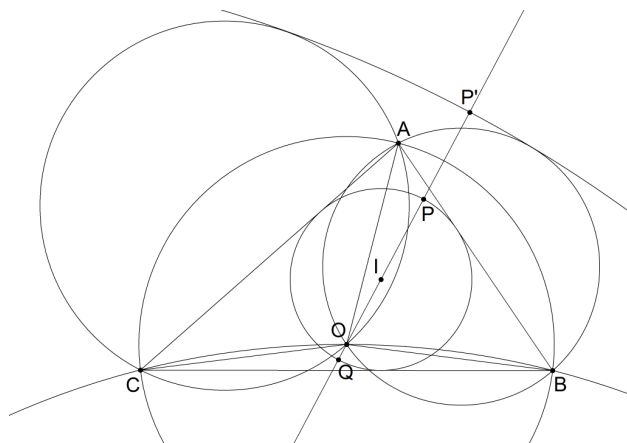
Now we run a few standard calculations. Where s, r , and R denote the semiperimeter, inradius, and circumradius of ABC , respectively, we have the following:

$$\begin{aligned} [ABC] &= \sqrt{s(s-a)(s-b)(s-c)} = \frac{15\sqrt{7}}{4}; \\ r &= [ABC]/s = \sqrt{7}/2; \\ R &= \frac{abc}{4[ABC]} = \frac{8}{\sqrt{7}} \\ OI^2 &= R(R-2r) = \frac{8}{7}. \end{aligned}$$

Let OI intersect the incircle of ABC at P and Q , with I between P and O . Then $OP = r + OI$ and $OQ = r - OI$, and \overline{PQ} is a diameter. Under the inversion, P and Q map to P' and Q' respectively. Because P, I, O , and Q are collinear in that order, P' and Q' are diametrically opposed on Γ . It follows that the diameter of Γ is

$$P'Q' = OP' + OQ' = \frac{R^2}{OP} + \frac{R^2}{OQ} = R^2 \left(\frac{1}{r+OI} + \frac{1}{r-OI} \right) = \frac{2rR^2}{r^2 - OI^2}.$$

We plug in the values found above to arrive at $\frac{256\sqrt{7}}{17}$.



33. [18] Compute

$$\int_1^2 \frac{9x+4}{x^5+3x^2+x} dx.$$

(No, your TI-89 doesn't know how to do this one. Yes, the end is near.)

Answer: $\ln \frac{80}{23}$. We break the given integral into two pieces:

$$\int_1^2 \frac{9x+4}{x^5+3x^2+x} dx = 5 \int_1^2 \frac{x^4+3x+1}{x^5+3x^2+x} dx - \int_1^2 \frac{5x^4+6x+1}{x^5+3x^2+x} dx.$$

These two new integrals are easily computed; for, the first integrand reduces to $1/x$ and the second is of the form $f'(x)/f(x)$. We obtain

$$\left[5 \ln |x| - \ln |x^5 + 3x^2 + x|\right]_1^2 = \ln 32 - \ln 46 + \ln 5 = \ln \frac{80}{23}$$

Motivation. Writing $f(x) = 9x + 4$ and $g(x) = x^5 + 3x^2 + x = x(x^4 + 3x + 1)$, we wish to find the antiderivative of $f(x)/g(x)$. It makes sense to consider other rational functions with denominator $g(x)$ that have an exact antiderivative. Clearly, if the numerator were $f_1(x) = x^4 + 3x + 1$ or a constant multiple, then we can integrate the function. Another trivial case is if the numerator were $f_2(x) = g'(x) = 5x^4 + 6x + 1$ or a constant multiple. Guessing that $f(x)$ is a linear combination of $f_1(x)$ and $f_2(x)$, we easily find that $f(x) = 9x + 4 = 5f_1(x) - f_2(x)$.

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34. [?] *The Game.* Eric and Greg are watching their new favorite TV show, *The Price is Right*. Bob Barker recently raised the intellectual level of his program, and he begins the latest installment with bidding on following question: How many Carmichael numbers are there less than 100,000?

Each team is to list one nonnegative integer not greater than 100,000. Let X denote the answer to Bob's question. The teams listing N , a maximal bid (of those submitted) not greater than X , will receive N points, and all other teams will neither receive nor lose points. (A Carmichael number is an odd composite integer n such that n divides $a^{n-1} - 1$ for all integers a relatively prime to n with $1 < a < n$.)

Answer: 16? There are 16 such numbers: 561, 1105, 1729, 2465, 2821, 6601, 8911, 10585, 15841, 29341, 41041, 46657, 52633, 62745, 63973, and 75361. The next, 101101, is too large to be counted. Their distribution is considerably more subtle than that of the primes, and it was only recently proven that there are infinitely many Carmichael numbers. For sufficiently large n , $C(n) > n^{2/7}$, although this bound has been subsequently improved. (For details, see Alford, W. R.; Granville, A.; and Pomerance, C. "There are Infinitely Many Carmichael Numbers." *Annals of Mathematics*. **139** (1994), 703-722.) The expectation is that teams are unable to determine that $X = 16$; otherwise, the obvious dominant play is listing 16. The Problem Czar, anticipating that teams will attempt to deduce X by considering the point values of the other problems in the triplet, has chosen a value X that is considerably lower. Teams may of course speculate whether this action has been taken, and to what extent, etc... On the actual TV show, many contestants win by guessing prices of 1, or other numbers dramatically lower than the actual price. This strategy is enhanced because of the show's ordered bidding, and will be more difficult here. It will be interesting to see the submissions.

35. [≤ 25] *The Algorithm.* There are thirteen broken computers situated at the following set S of thirteen points in the plane:

$A = (1, 10)$	$B = (976, 9)$	$C = (666, 87)$
$D = (377, 422)$	$E = (535, 488)$	$F = (775, 488)$
$G = (941, 500)$	$H = (225, 583)$	$I = (388, 696)$
$J = (3, 713)$	$K = (504, 872)$	$L = (560, 934)$
	$M = (22, 997)$	

At time $t = 0$, a repairman begins moving from one computer to the next, traveling continuously in straight lines at unit speed. Assuming the repairman begins at A and fixes computers instantly, what path does he take to minimize the *total downtime* of the computers? List the points he visits in order. Your score will be $\lfloor \frac{N}{40} \rfloor$, where

$$N = 1000 + \lfloor \text{the optimal downtime} \rfloor - \lfloor \text{your downtime} \rfloor,$$

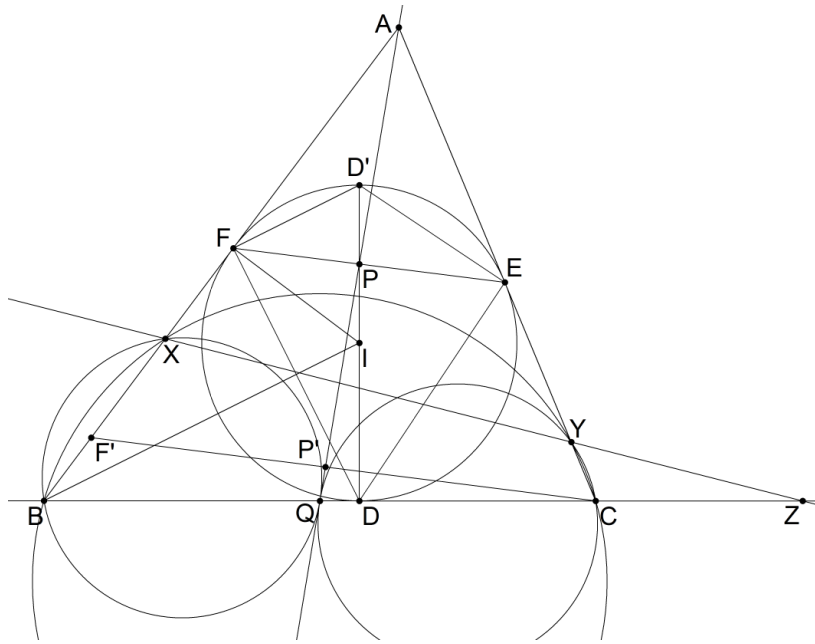
or 0, whichever is greater. By total downtime we mean the sum

$$\sum_{P \in S} t_P,$$

where t_P is the time at which the repairman reaches P .

Answer: ADHIKLEFGBCJM. This is an instance of the *minimum-latency* problem, which is at least NP-hard. There is an easy $O(n!)$ algorithm, but this is unavailable to teams on computational grounds (100MHz calculators used to seem fast...) The best strategy may be drawing an accurate picture and exercising geometric intuition. The distribution of the points somewhat resembles a short, four-pronged fork with its outermost prongs bent apart; it is plausible to assume that the optimal order respects this shape. The optimal downtime is 24113.147907, realized by ADHIKLEFGBCJM, though a number of others also receive positive marks.

36. [25] *The Marathon*. Let ω denote the incircle of triangle ABC . The segments BC, CA , and AB are tangent to ω at D, E , and F , respectively. Point P lies on EF such that segment PD is perpendicular to BC . The line AP intersects BC at Q . The circles ω_1 and ω_2 pass through B and C , respectively, and are tangent to AQ at Q ; the former meets AB again at X , and the latter meets AC again at Y . The line XY intersects BC at Z . Given that $AB = 15, BC = 14$, and $CA = 13$, find $\lfloor XZ \cdot YZ \rfloor$.



Answer: 101. Construct D' diametrically opposed to D , so that $\angle DFD'$ and $\angle DED'$ are right, and note that P lies on DD' . By standard angle chasing, $m\angle FDD' = \beta$ (half angle B) and $m\angle D'DE = \gamma$. Thus, $m\angle DD'F = 90^\circ - \beta$ and $m\angle ED'D = 90^\circ - \gamma$. Then by the law of sines, $DE : ED' : D'F : FD = \cos(\gamma) : \sin(\gamma) : \sin(\beta) : \sin(\gamma)$. Now using $\triangle DEP \sim \triangle FD'P$ and $\triangle DFP \sim \triangle ED'P$, we have

$$\frac{EP}{PF} = \frac{ED \cdot ED'}{FD \cdot FD'} = \frac{\sin(\gamma) \cos(\gamma)}{\sin(\beta) \sin(\beta)} = \frac{c}{b}.$$

Let the dilation centered at A sending E to C map P and F to P' and F' , respectively. Note that $AF' = AC$ as AE and AF are equal tangents, and $CP' : P'F' = EP : PF = c : b$ by similarity. Then by Menelaus' theorem,

$$1 = \frac{BQ}{QC} \frac{CP'}{P'F'} \frac{F'A}{AB} = \frac{BQ}{QC} \frac{cb}{bc},$$

so that $BQ = QC$ and AQ is actually a median. So, $AQ^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2) = 148$. Now by Power of a Point, $AB \cdot AX = AQ^2 = AC \cdot AY$, so $AX = 148/15$ and $AY = 148/13$. Moreover, $BXC Y$ is cyclic as $\triangle ABC \sim \triangle AYX$. Thus, $XZ \cdot YZ = BZ \cdot CZ$, and it suffices to compute BZ/CZ . Menelaus once more gives

$$1 = \frac{BZ}{ZC} \frac{CY}{YA} \frac{AX}{XB},$$

whence, $BZ/CZ = (AY/AX)(BX/CY) = (15/13)((77 \cdot 13)/(21 \cdot 15)) = 11/3$. We write $CZ = 3d$ and $BZ = 11d$. Because $AX < AB$ and $AY < AC$, Z does not lie on segment BC . Given the configuration information, $BC = 8d = 14$, so $d = 7/4$, and finally $\lfloor BZ \cdot CZ \rfloor = \lfloor 33d^2 \rfloor = \lfloor \frac{1617}{16} \rfloor = 101$.