10^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 24 February 2007

Team Round: A Division

Σ, τ, and You: Fun at Fraternities? [270]

A *number theoretic function* is a function whose domain is the set of positive integers. A *multiplicative* number theoretic function is a number theoretic function f such that $f(mn) = f(m)f(n)$ for all pairs of relatively prime positive integers m and n. Examples of multiplicative number theoretic functions include σ, τ, ϕ, and μ, defined as follows. For each positive integer n,

- The *sum-of-divisors function*, $\sigma(n)$, is the sum of all positive integer divisors of n. If $p_1, \ldots, p_i$ are distinct primes and $e_1, \ldots, e_i$ are positive integers,

$$\sigma\left(p_1^{e_1} \cdots p_i^{e_i}\right) = \prod_{k=1}^{i} \left(1 + p_k + \cdots + p_k^{e_k}\right) = \prod_{k=1}^{i} \frac{p_k^{e_k+1} - 1}{p_k - 1}.$$

- The *divisor function*, $\tau(n)$, is the number of positive integer divisors of n. It can be computed by the formula

$$\tau\left(p_1^{e_1} \cdots p_i^{e_i}\right) = (e_1 + 1) \cdots (e_i + 1),$$

  where $p_1, \ldots, p_i$ and $e_1, \ldots, e_i$ are as above.

- Euler's *totient* function, $\phi(n)$, is the number of positive integers $k \leq n$ such that k and n are relatively prime. For $p_1, \ldots, p_i$ and $e_1, \ldots, e_i$ as above, the phi function satisfies

$$\phi\left(p_1^{e_1} \cdots p_i^{e_i}\right) = \prod_{k=1}^{i} p_k^{e_k - 1}(p_k - 1).$$

- The *Möbius function*, $\mu(n)$, is equal to either 1, -1, or 0. An integer is called *square-free* if it is not divisible by the square of any prime. If n is a square-free positive integer having an even number of distinct prime divisors, $\mu(n) = 1$. If n is a square-free positive integer having an odd number of distinct prime divisors, $\mu(n) = -1$. Otherwise, $\mu(n) = 0$.

The Möbius function has a number of peculiar properties. For example, if f and g are number theoretic functions such that

$$g(n) = \sum_{d|n} f(d),$$

for all positive integers n, then

$$f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right).$$

This is known as *Möbius inversion.* In proving the following problems, *you may use any of the preceding assertions without proving them. You may also cite the results of previous problems, even if you were unable to prove them.*

1. [**15**] Evaluate the functions $\phi(n), \sigma(n)$, and $\tau(n)$ for $n = 12, n = 2007$, and $n = 2^{2007}$.

   Solution. For $n = 12 = 2^2 \cdot 3^1$,

$$\phi(12) = 2(2-1)(3-1) = 4, \quad \sigma(12) = (1+2+4)(1+3) = 28, \quad \tau(12) = (2+1)(1+1) = 6;$$

   for $n = 2007 = 3^2 \cdot 223$,

$$\phi(2007) = 3(3-1)(223-1) = 1332, \quad \sigma(2007) = (1+3+9)(1+223) = 2912, \quad \tau(2007) = (2+1)(1+1) = 6;$$

   and for $n = 2^{2007}$,

$$\phi\left(2^{2007}\right) = 2^{2006}, \quad \sigma\left(2^{2007}\right) = \left(1 + 2 + \cdots + 2^{2007}\right) = 2^{2008} - 1, \quad \tau\left(2^{2007}\right) = 2007 + 1 = 2008. \square$$

2. [**20**] Solve for the positive integer(s) n such that $\phi(n^2) = 1000\phi(n)$.

Answer: $\boxed{\mathbf{1000}}$.

Solution. The unique solution is $n = 1000$. For, $\phi(pn) = p\phi(n)$ for every prime p dividing n, so that $\phi(n^2) = n\phi(n)$ for all positive integers n. $\square$

3. [**25**] Prove that for every integer n greater than 1,

$$\sigma(n)\phi(n) \leq n^2 - 1.$$

When does equality hold?

Solution. Note that

$$\sigma(mn)\phi(mn) = \sigma(m)\phi(m)\sigma(n)\phi(n) \leq (m^2-1)(n^2-1) = (mn)^2 - (m^2+n^2-1) < (mn)^2 - 1$$

for any pair of relatively prime positive integers (m, n) other than $(1, 1)$. Now, for p a prime and k a positive integer, $\sigma\left(p^k\right) = 1 + p + \cdots + p^k = \frac{p^{k+1}-1}{p-1}$ and $\phi\left(p^k\right) = p^k - \frac{1}{p} \cdot p^k = (p-1)p^{k-1}$. Thus,

$$\sigma\left(p^k\right)\phi\left(p^k\right) = \frac{p^{k+1}-1}{p-1} \cdot (p-1)p^{k-1} = (p^{k+1}-1)p^{k-1} = p^{2k} - p^{k-1} \leq p^{2k} - 1,$$

with equality where $k = 1$. It follows that equality holds in the given inequality if and only if n is prime. $\square$

4. [**25**] Let F and G be two multiplicative functions, and define for positive integers n,

$$H(n) = \sum_{d|n} F(d)G\left(\frac{n}{d}\right).$$

The number theoretic function H is called the *convolution* of F and G. Prove that H is multiplicative.

Solution. Let m and n be relatively prime positive integers. We have

$$\begin{aligned} H(m)H(n) &= \left(\sum_{d|m} F(d)G\left(\frac{m}{d}\right)\right)\left(\sum_{d'|n} F(d')G\left(\frac{n}{d'}\right)\right) \\ &= \sum_{d|m,\, d'|n} F(d)F(d')G\left(\frac{m}{d'}\right)G\left(\frac{n}{d}\right) = \sum_{d|m,\, d'|n} F(dd')G\left(\frac{mn}{dd'}\right) \\ &= \sum_{d|mn} F(d)G\left(\frac{mn}{d}\right) = H(mn). \square \end{aligned}$$

5. [**30**] Prove the identity

$$\sum_{d|n} \tau(d)^3 = \left(\sum_{d|n} \tau(d)\right)^2.$$

Solution. Note that τ^3 is multiplicative; in light of the convolution property just shown, it follows that both sides of the posed equality are multiplicative. Thus, it would suffice to prove the claim for n a power of a prime. So, write $n = p^k$ where p is a prime and k is a nonnegative integer. Then

$$\begin{aligned} \sum_{d|n} \tau(d)^3 &= \sum_{i=0}^{k} \tau\left(p^i\right)^3 = \sum_{i=0}^{k} (i+1)^3 \\ &= 1^3 + \cdots + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4} = \left(\frac{(k+1)(k+2)}{2}\right)^2 \\ &= \left(\sum_{i=0}^{k} \tau\left(p^i\right)\right)^2 = \left(\sum_{d|n} \tau(d)\right)^2, \end{aligned}$$

as required. $\square$

6. [**25**] Show that for positive integers n,

$$\sum_{d|n} \phi(d) = n.$$

Solution. Both sides are multiplicative functions of n, the right side trivially and the left because for relatively prime positive integers n and n',

$$\left(\sum_{d|n} \phi(d)\right)\left(\sum_{d'|n'} \phi(d')\right) = \sum_{d|n,\, d'|n'} \phi(d)\phi(d'),$$

and $\phi(d)\phi(d') = \phi(dd')$. The identity is then easy to check; since $\phi(p^k) = p^{k-1}(p-1)$ for positive integers k and $\phi(1) = 1$, we have $\phi(1) + \phi(p) + \cdots + \phi(p^k) = 1 + (p-1) + (p^2 - p) + \cdots + \left(p^k - p^{k-1}\right) = p^k$, as desired. $\square$

7. [**25**] Show that for positive integers n,

$$\sum_{d|n} \frac{\mu(d)}{d} = \frac{\phi(n)}{n}.$$

Solution. On the grounds of the previous problem, Möbius inversion with $f(k) = \phi(k)$ and $g(k) = k$ gives:

$$\phi(n) = f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right) = \sum_{d'|n} g\left(\frac{n}{d'}\right)\mu(d') = \sum_{d'|n} \frac{n}{d'}\mu(d'). \square$$

Alternatively, one uses the convolution of the functions $f(k) = n$ and $g(k) = \frac{\mu(d)}{d}$. The strategy is the same as the previous convolution proof. For $n = p^k$ with k a positive integer, we have $\phi(n) = p^k - p^{k-1}$, while the series reduces to $p^k \cdot \mu(1) + p^k \cdot \mu(p)/p = p^k - p^{k-1}$.

8. [**30**] Determine with proof, a simple closed form expression for

$$\sum_{d|n} \phi(d)\tau\left(\frac{n}{d}\right).$$

Solution. We claim the series reduces to $\sigma(n)$. The series counts the ordered triples (d, x, y) with $d|n; x|d; 0 < y \le n/d$; and $(y, n/d) = 1$. To see this, write

$$\sum_{d|n} \phi(d)\tau\left(\frac{n}{d}\right) = \sum_{d'|n} \phi(\frac{n}{d'})\tau(d'),$$

so that for a given $d'|n$ we may choose x and y as described above. On the other hand, we can count these triples by groups sharing a given x. Fixing x as a divisor of n fixes an integer $\frac{n}{x}$. Then d varies such that $\frac{n}{d}$ is a divisor of $\frac{n}{x}$. For each divisor $\frac{n}{d}$ of $\frac{n}{x}$ there are precisely $\phi\left(\frac{n}{d}\right)$ choices y, so that by the lemma from the previous problem, there are $\frac{n}{x}$ triples (d, x, y) for a given x. It follows that there are precisely $\sigma(n)$ such triples (d, x, y). $\square$

Again, an alternative is to use the multiplicativity of the convolution, although it is now a little more difficult. Write $n = p^k$ so that

$$\begin{aligned}\sum_{d|n} \phi(d)\tau\left(\frac{n}{d}\right) &= \sum_{m=0}^{k} \phi\left(p^m\right)\tau\left(p^{k-m}\right) = k + 1 + \sum_{m=1}^{k} p^{m-1}(p-1)(k-m+1) \\ &= k + 1 + \left(\sum_{m=1}^{k} p^m(k-m+1)\right) - \left(\sum_{m=1}^{k} p^{m-1}(k-m+1)\right) = k + 1 + p^k - k + \sum_{m'=1}^{k-1} p^{m'} \\ &= 1 + p + \cdots + p^k = \sigma(p^k).\end{aligned}$$

9. [**35**] Find all positive integers n such that

$$\sum_{k=1}^{n} \phi(k) = \frac{3n^2+5}{8}.$$

Answer: $\boxed{\mathbf{1, 3, 5}}$.

Solution. We contend that the proper relation is

$$\sum_{k=1}^{n} \phi(k) \le \frac{3n^2+5}{8}. \qquad (*)$$

Let $\Phi(k)$ denote the left hand side of $(*)$. It is trivial to see that for $n \le 7$ the posed inequality holds, has equality where $n = 1, 3, 5$, and holds strictly for $n = 7$. Note that $\phi(2k) \le k$ and $\phi(2k+1) \le 2k$, the former because $2, 4, \ldots, 2k$ share a common divisor. It follows that $\phi(2k)+\phi(2k+1) \le 3k$. Suppose for the sake of induction that $\Phi(2k-1) < \frac{3(2k-1)^2+5}{8}$. Then

$$\Phi(2k+1) = \Phi(2k-1) + \phi(2k) + \phi(2k+1) < \frac{3(2k-1)^2+5}{8} + 3k = \frac{3(2k+1)^2+5}{8}.$$

To complete the proof, it is enough to note that for a positive integer k,

$$\frac{3(2k-1)^2+5}{8} + k < \frac{3(2k)^2+5}{8}. \square$$

10. [**40**] Find all pairs (n, k) of positive integers such that

$$\sigma(n)\phi(n) = \frac{n^2}{k}.$$

Answer: $\boxed{(\mathbf{1}, \mathbf{1})}$.

Solution. It is clear that for a given integer n, there is at most one integer k for which the equation holds. For $n = 1$ this is $k = 1$. But, for $n > 1$, problem 1 asserts that $\sigma(n)\phi(n) \le n^2 - 1 < n^2$, so that $k \ge 2$. We now claim that $2 > \frac{n^2}{\sigma(n)\phi(n)}$. Write $n = p_1^{e_1} \cdots p_k^{e_k}$, where the p_i are distinct primes and $e_i \ge 1$ for all i, and let $q_1 < q_2 < \cdots$ be the primes in ascending order. Then

$$\begin{aligned}
\frac{n^2}{\sigma(n)\phi(n)} &= \prod_{i=1}^{k} \frac{p_i^{2e_i}}{\frac{p_i^{e_i+1}-1}{p_i-1} \cdot (p_i-1)p_i^{e_i-1}} = \prod_{i=1}^{k} \frac{p_i^{2e_i}}{p_i^{2e_i} - p_i^{e_i-1}} \\
&= \prod_{i=1}^{k} \frac{1}{1 - p_i^{-1-e_i}} \le \prod_{i=1}^{k} \frac{1}{1-p_i^{-2}} < \prod_{i=1}^{\infty} \frac{1}{1-q_i^{-2}} \\
&= \prod_{i=1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{q_i^{2j}} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&< 1 + \sum_{n=2}^{\infty} \frac{1}{n^2-1} = 1 + \frac{1}{2}\left(\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots \right) = \frac{7}{4} < 2.
\end{aligned}$$

It follows that there can be no solutions to $k = \frac{n^2}{\sigma(n)\phi(n)}$ other than $n = k = 1$. $\square$

Grab Bag - Miscellaneous Problems [130]

11. [**30**] Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$\begin{aligned} f(x)f(y) &= f(x) + f(y) - f(xy) \\ 1 + f(x+y) &= f(xy) + f(x)f(y) \end{aligned}$$

for all rational numbers x, y.

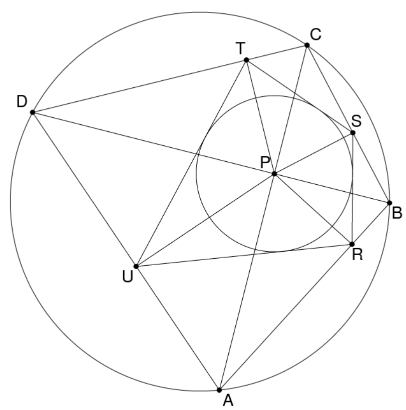
Answer: $\boxed{\mathbf{f(x) = 1\,\forall x, \text{ and } f(x) = 1 - x\,\forall x.}}$

Solution. Considering the first equation, either side of the second equation is equal to $f(x) + f(y)$. Now write $g(x) = 1 - f(x)$, so that

$$\begin{aligned} g(xy) &= 1 - f(xy) = 1 - f(x) - f(y) + f(x)f(y) = (1 - f(x))(1 - f(y)) = g(x)g(y) \\ g(x+y) &= 1 - f(x+y) = 1 - f(x) + 1 - f(y) = g(x) + g(y), \end{aligned}$$

By induction, $g(nx) = ng(x)$ for all integers n, so that $g(p/q) = (p/q)g(1)$ for integers p and q with q nonzero; i.e., $g(x) = xg(1)$. As g is multiplicative, $g(1) = g(1)^2$, so the only possibilities are $g(1) = 1$ and $g(1) = 0$. These give $g(x) = x$ and $g(x) = 0$, or $f(x) = 1 - x$ and $f(x) = 1$, respectively. One easily checks that these functions are satisfactory. □

12. [**30**] Let $ABCD$ be a cyclic quadrilateral, and let P be the intersection of its two diagonals. Points R, S, T, and U are feet of the perpendiculars from P to sides AB, BC, CD, and AD, respectively. Show that quadrilateral $RSTU$ is bicentric if and only if $AC \perp BD$. (Note that a quadrilateral is called *inscriptible* if it has an incircle; a quadrilateral is called *bicentric* if it is both cyclic and inscriptible.)



Solution. First we show that $RSTU$ is always inscriptible. Note that in addition to $ABCD$, we have cyclic quadrilaterals $ARPU$ and $BSPR$. Thus,

$$\angle PRU = \angle PAU = \angle CAD = \angle CBD = \angle SBP = \angle SRP,$$

and it follows that P lies on the bisector of $\angle SRU$. Analogously, P lies on the bisectors of $\angle TSR$ and $\angle UTS$, so is equidistant from lines UR, RS, ST, and TU, and $RSTU$ is inscriptible having incenter P. Now we show that $RSTU$ is cyclic if and only if the diagonals of $ABCD$ are orthogonal. We have

$$\begin{aligned} \angle APB &= \pi - \angle BAP - \angle PBA = \pi - \angle RAP - \angle PBR = \pi - \angle RUP - \angle PSR \\ &= \pi - \frac{1}{2}(\angle RUT + \angle TSR). \end{aligned}$$

It follows that $\angle APB = \frac{\pi}{2}$ if and only if $\angle RUT + \angle TSR = \pi$, as desired. □

13. [**30**] Find all nonconstant polynomials $P(x)$, with real coefficients and having only real zeros, such that $P(x+1)P(x^2 - x + 1) = P(x^3 + 1)$ for all real numbers x.

Answer: $\boxed{\{\mathbf{P}(\mathbf{x}) = \mathbf{x}^{\mathbf{k}} \,|\, \mathbf{k} \in \mathbb{Z}^+\}.}$

Solution. Note that if $P(\alpha) = 0$, then by setting $x = \alpha - 1$ in the given equation, we find $0 = P(x^3 + 1) = P(\alpha^3 - 3\alpha^2 + 3\alpha)$. Because P is nonconstant, it has at least one zero. Because P has finite degree, there exist minimal and maximal roots of P. Writing $\alpha^3 - 3\alpha^2 + 3\alpha \geq \alpha \iff \alpha(\alpha - 1)(\alpha - 2) \geq 0$, we see that the largest zero of P cannot exceed 2. Likewise, the smallest zero cannot be negative, so all of the zeroes of P lie in $[0, 2]$. Moreover, if $\alpha \notin \{0, 1, 2\}$ is a zero of P, then $\alpha' = \alpha^3 - 3\alpha^2 + 3\alpha$ is another zero of P that lies strictly between α and 1. Because P has only finitely many zeroes, all of its zeroes must lie in $\{0, 1, 2\}$. Now write $P(x) = kx^p(x - 1)^q(x - 2)^r$ for nonnegative integers p, q and r having a positive sum. The given equation becomes

$$\begin{aligned} k^2(x+1)^p x^q (x-1)^r (x^2 - x + 1)^p (x^2 - x)^q (x^2 - x - 1)^r &= P(x+1)P(x^2 - x + 1) \\ = P(x^3 + 1) = k(x^3 + 1)^p x^{3q} (x^3 - 1)^r. \quad (*) \end{aligned}$$

For the leading coefficients to agree, we require $k = k^2$. Because the leading coefficient is nonzero, P must be monic. In $(*)$, r must be zero lest $P(x^3 + 1) = 0$ have complex roots. Then q must be zero as well. For, if q is positive, then $P(x^2 - x + 1) = 0$ has 1 as a root while $P(x^3 + 1)$ does not. Finally, the remaining possibilities are $P(x) = x^p$ for p an arbitrary positive integer. It is easily seen that these polynomials are satisfactory. □

14. [**40**] Find an explicit, closed form formula for

$$\sum_{k=1}^{n} \frac{k \cdot (-1)^k \cdot \binom{n}{k}}{n + k + 1}.$$

Answer: $\boxed{\frac{-1}{\binom{2n+1}{n}} \text{ or } -\frac{n!(n+1)!}{(2n+1)!} \text{ or obvious equivalent.}}$

Solution. Consider the interpolation of the polynomial $P(x) = x \cdot n!$ at $x = 0, 1, \ldots, n$. We obtain the identity

$$\begin{aligned} P(x) = x \cdot n! &= \sum_{k=0}^{n} k \cdot n! \prod_{j \neq k} \frac{x - j}{k - j} \\ &= \sum_{k=0}^{n} k \cdot n! \cdot \frac{x(x-1)\cdots(x-k+1)(x-k-1)\cdots(x-n)}{k!(n-k)!(-1)^{n-k}} \\ &= \sum_{k=1}^{n} k \cdot (-1)^{n-k} \cdot \binom{n}{k} \cdot x(x-1)\cdots(x-k+1)(x-k-1)\cdots(x-n). \end{aligned}$$

This identity is valid for all complex numbers x, but, to extract a factor $\frac{1}{n+k+1}$ from the valid product of each summand, we set $x = -n - 1$, so that

$$-(n+1)! = \sum_{k=1}^{n} k(-1)^{n-k} \binom{n}{k} (-n-1)\cdots(-n-k)(-n-k-2)\cdots(-2n-1) = \sum_{k=1}^{n} \frac{k(-1)^k \binom{n}{k} (2n+1)!}{n!(n+k+1)}.$$

Finally,

$$\sum_{k=1}^{n} \frac{k \cdot (-1)^k \cdot \binom{n}{k}}{n + k + 1} = \frac{-n!(n+1)!}{(2n+1)!} = \frac{-1}{\binom{2n+1}{n}}. \square$$