10^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 24 February 2007

Team Round: B Division

Compute $(x-a)(x-b)\cdots(x-z)$ - Short Answer [200]

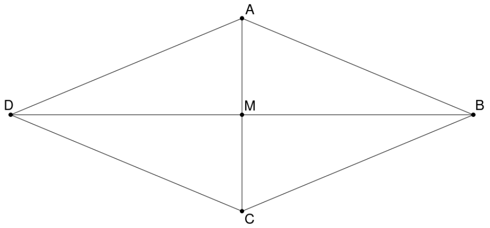
For this section, your team should give only the anwers to the problems.

1. [**20**] Find the sum of the positive integer divisors of 2^{2007}.

   Answer: $\boxed{\mathbf{2^{2008}-1.}}$ The divisors are the powers of two not exceeding 2^{2007}. So the sum is

   $$1+2+2^2+\cdots+2^{2007} = -1+2+2+2^2+\cdots+2^{2007} = -1+2^2+2^2+\cdots+2^{2007} = \cdots = -1+2^{2008}.$$

2. [**20**] The four sides of quadrilateral $ABCD$ are equal in length. Determine the perimeter of $ABCD$ given that it has area 120 and $AC = 10$.



   Answer: $\boxed{\mathbf{52.}}$ Let M be the midpoint of AC. Then triangles AMB, BMC, CMD, and DMA are all right triangles having legs 5 and h for some h. The area of $ABCD$ is 120, but also $4 \cdot (\frac{1}{2} \cdot 5 \cdot h) = 10h$, so $h = 12$. Then $AB = BC = CD = DA = \sqrt{12^2+5^2} = 13$, and the perimeter of $ABCD$ is 52.

3. [**20**] Five people are crowding into a booth against a wall at a noisy restaurant. If at most three can fit on one side, how many seating arrangements accomodate them all?

   Answer: $\boxed{\mathbf{240}.}$ Three people will sit on one side and two sit on the other, giving a factor of two. Then there are 5! ways to permute the people.

4. [**20**] Thomas and Michael are just two people in a large pool of well qualified candidates for appointment to a problem writing committee for a prestigious college math contest. It is 40 times more likely that both will serve if the size of the committee is increased from its traditional 3 members to a whopping n members. Determine n. (Each person in the pool is equally likely to be chosen.)

   Answer: $\boxed{\mathbf{16.}}$ Suppose there are k candidates. Then the probability that both serve on a 3 membered committee is $(k-2)/\binom{k}{3}$, and the odds that both serve on an n membered committee are $\binom{k-2}{n-2}/\binom{k}{n}$. The ratio of the latter to the former is

   $$\frac{\binom{k}{3}\binom{k-2}{n-2}}{(k-2)\binom{k}{n}} = \frac{k!(k-2)!1!(k-3)!n!(k-n)!}{k!(k-2)!(n-2)!(k-n)!3!(k-3)!} = \frac{n\cdot(n-1)}{3!}.$$

   Solving $n\cdot(n-1) = 240$ produces $n = 16, -15$, and we discard the latter.

5. [**20**] The curves $y = x^2(x-3)^2$ and $y = (x^2-1)(x-2)$ intersect at a number of points in the real plane. Determine the sum of the x-coordinates of these points of intersection.

   Answer: $\boxed{\mathbf{7.}}$ Because the first curve touches the x-axis at $x = 0$ and $x = 3$ while the second curve crosses the x-axis at $x = \pm 1$ and $x = 2$, there are four points of intersection. In particular, the points of intersection have x-coordinates determined by the difference of the two curves:

   $$0 = x^2(x-3)^2 - (x^2-1)(x-2) = (x^4 - 6x^3 + \cdots) - (x^3 + \cdots) = x^4 - 7x^3 + \cdots.$$

   We need only the first two coefficients to determine $x_1 + x_2 + x_3 + x_4 = -\left(\frac{-7}{1}\right) = 7$.

6. [**20**] Andrew has a fair six sided die labeled with 1 through 6 as usual. He tosses it repeatedly, and on every third roll writes down the number facing up as long as it is not the 6. He stops as soon as the last two numbers he has written down are squares or one is a prime and the other is a square. What is the probability that he stops after writing squares consecutively?

Answer: $\boxed{\mathbf{4/25}}$. We can safely ignore all of the rolls he doesn't record. The probability that he stops after writing two squares consecutively is the same as the probability that he never rolls a prime. For, as soon as the first prime is written, either it must have been preceded by a square or it will be followed by a nonnegative number of additional primes and then a square. So we want the probability that two numbers chosen uniformly with replacement from $\{1,2,3,4,5\}$ are both squares, which is $(2/5)^2$.

7. [**20**] Three positive reals x, y, and z are such that

$$\begin{aligned} x^2+2(y-1)(z-1) &= 85 \\ y^2+2(z-1)(x-1) &= 84 \\ z^2+2(x-1)(y-1) &= 89. \end{aligned}$$

Compute $x+y+z$.

Answer: $\boxed{\mathbf{18}}$. Add the three equations to obtain

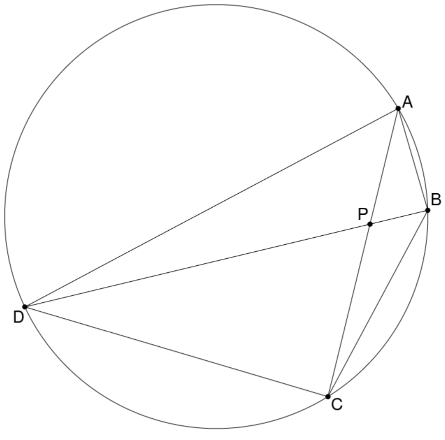
$$x^2+y^2+z^2+2xy+2yz+2zx-4x-4y-4z+6=258,$$

which rewrites as $(x+y+z-2)^2=256$. Evidently, $x+y+z=2\pm 16$. Since x, y, and z are positive, $x+y+z>0$ so $x+y+z=2+16=18$.

8. [**20**] Find the *positive* real number(s) x such that $\frac{1}{2}\left(3x^2-1\right)=\left(x^2-50x-10\right)\left(x^2+25x+5\right)$.

Answer: $\boxed{\mathbf{25+2\sqrt{159}}}$. Write $a=x^2-50x-10$ and $b=x^2+25x+5$; the given becomes $\frac{a+2b-1}{2}=ab$, so $0=2ab-a-2b+1=(a-1)(2b-1)$. Then $a-1=x^2-50x-11=0$ or $2b-1=2x^2+50x+9=0$. The former has a positive root, $x=25+2\sqrt{159}$, while the latter cannot, for obvious reasons.

9. [**20**] Cyclic quadrilateral $ABCD$ has side lengths $AB=1, BC=2, CD=3$, and $AD=4$. Determine AC/BD.



Answer: $\boxed{\mathbf{5/7}}$. Let the diagonals intersect at P. Note that triangles ABP and DCP are similar, so that $3AP=DP$ and $3BP=CP$. Additionally, triangles BCP and ADP are similar, so that $2BP=AP$. It follows that

$$\frac{AC}{BD}=\frac{AP+PC}{BP+PD}=\frac{2BP+3BP}{BP+6BP}=\frac{5}{7}.$$

10. [**20**] A positive real number x is such that

$$\sqrt[3]{1-x^3}+\sqrt[3]{1+x^3}=1.$$

Find x^2.

Answer: $\boxed{\frac{\sqrt[3]{28}}{3}}$. Cubing the given equation yields

$$1=(1-x^3)+3\sqrt[3]{(1-x^3)(1+x^3)}\left(\sqrt[3]{1-x^3}+\sqrt[3]{1+x^3}\right)+(1+x^3)=2+3\sqrt[3]{1-x^6}.$$

Then $\frac{-1}{3}=\sqrt[3]{1-x^6}$, so $\frac{-1}{27}=1-x^6$ and $x^6=\frac{28}{27}$ and $x^2=\frac{\sqrt[3]{28}}{3}$.

Adult Acorns - Gee, I'm a Tree! [200]

In this section of the team round, your team will derive some basic results concerning *tangential* quadrilaterals. Tangential quadrilaterals have an *incircle*, or a circle lying within them that is tangent to all four sides. If a quadrilateral has an incircle, then the center of this circle is the *incenter* of the quadrilateral. As you shall see, tangential quadrilaterals are related to cyclic quadrilaterals. For reference, a review of cyclic quadrilaterals is given at the end of this section.

Your answers for this section of the team test should be proofs. *Note that you may use any standard facts about cyclic quadrilaterals, such as those listed at the end of this test, without proving them. Additionally, you may cite the results of previous problems, even if you were unable to prove them.*

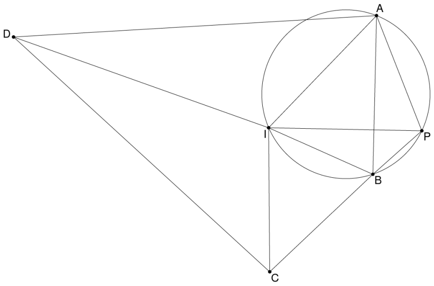
For these problems, $ABCD$ is a tangential quadrilateral having incenter I. For the first three problems, the point P is constructed such that triangle PAB is similar to triangle IDC and lies outside $ABCD$.

1. [**30**] Show that $PAIB$ is cyclic by proving that $\angle IAP$ is supplementary to $\angle PBI$.

   Solution. Note that I lies on the angle bisectors of the angles of quadrilateral $ABCD$. So writing $\angle DAB = 2\alpha, \angle ABC = 2\beta, \angle BCD = 2\gamma$, and $\angle CDA = 2\delta$, we have

$$\begin{aligned}\angle IAP+\angle PBI &= \angle IAB+\angle BAP+\angle PBA+\angle ABI\\ &= \angle IAB+\angle CDI+\angle ICD+\angle ABI\\ &= \alpha+\beta+\gamma+\delta.\end{aligned}$$

   We are done because the angles in quadrilateral $ABCD$ add up to 360°. □



2. [**40**] Show that triangle PAI is similar to triangle BIC. Then conclude that

$$PA=\frac{PI}{BC}\cdot BI.$$

   Solution. We have $\angle IBC = \angle ABI$ because I lies on the angle bisector, and $\angle ABI = \angle API$ because $PAIB$ is cyclic. Additionally,

$$\angle BCI = \angle ICD = \angle PBA = \angle PIA,$$

   by the angle bisector CI, that triangles PAB and IDC are similar, and the fact that $PAIB$ is cyclic, respectively. It follows that triangles PAI and BIC are similar. In particular, it follows that $IP/PA = BC/BI$, as required. □

3. [**25**] Deduce from the above that

$$\frac{BC}{AD} \cdot \frac{AI}{BI} \cdot \frac{DI}{CI} = 1.$$

Solution. Exchanging the roles of A and D with B and C, respectively, converts the formula from problem 2 into another formula:

$$PB = \frac{PI}{AD} \cdot AD.$$

Then one the one hand, dividing the two gives $PA/PB = (AD \cdot BI)/(BC \cdot AI)$. On the other hand, $PA/PB = DI/CI$ because triangles PAB and IDC are similar. Clearing the denominators in the equation

$$\frac{DI}{CI} = \frac{AD \cdot BI}{BC \cdot AI}$$

yields the desired form. □

4. [**25**] Show that $AB + CD = AD + BC$. Use the above to conclude that for some positive number α,

$$AB = \alpha \cdot \left(\frac{AI}{CI} + \frac{BI}{DI}\right) \qquad BC = \alpha \cdot \left(\frac{BI}{DI} + \frac{CI}{AI}\right)$$
$$CD = \alpha \cdot \left(\frac{CI}{AI} + \frac{DI}{BI}\right) \qquad DA = \alpha \cdot \left(\frac{DI}{BI} + \frac{AI}{CI}\right).$$

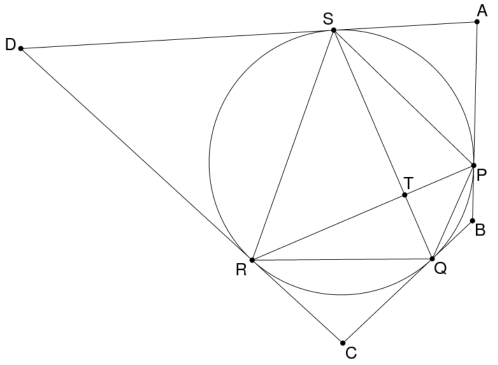
Solution. Draw in the points of tangency P, Q, R, and S, of the incircle with sides AB, BC, CD, and AD, as shown. Then we have equal tangents $AP = AS, BP = BQ, CQ = CR$, and $DR = DS$. Then

$$AB + CD = AP + BP + CR + DR = AS + (BQ + CQ) + DS = BC + AD.$$

Using the result of problem 3, we set $BC = x \cdot BI \cdot CI$ and $AD = x \cdot AI \cdot DI$ for some x, and $AB = y \cdot AI \cdot BI$ and $CD = y \cdot CI \cdot DI$ for some y. Now because $AB + CD = BC + AD$, we obtain

$$y(AI \cdot BI + CI \cdot DI) = x(BI \cdot CI + AI \cdot DI).$$

So it follows that the ratio $AB : BC : CD : DA$ is uniquely determined. One easily checks that the posed ratio satisfies the three required relations. □



5. [**40**] Show that

$$AB \cdot BC = BI^2 + \frac{AI \cdot BI \cdot CI}{DI}.$$

Solution. Returning to the original set up, Ptolemy's theorem applied to quadrilateral $PAIB$ gives $AB \cdot PI = PA \cdot BI + PB \cdot AI$. Substituting equation $PA = \frac{PI}{BC} \cdot BI$ from problem 2 and its cousin $PB = \frac{PI}{AD} \cdot AI$ allows us to write

$$AB \cdot PI = \frac{PI}{BC} \cdot BI^2 + \frac{PI}{AD} \cdot AI^2,$$

or

$$AB \cdot BC = BI^2 + \frac{BC}{AD} \cdot AI^2.$$

Substituting the formula $BC/AD = \frac{BI \cdot CI}{AI \cdot DI}$ from problem 3 finishes the problem. $\square$

6. [**40**] Let the incircle of $ABCD$ be tangent to sides AB, BC, CD, and AD at points P, Q, R, and S, respectively. Show that $ABCD$ is cyclic if and only if $PR \perp QS$.

   Solution. Let the diagonals of $PQRS$ intersect at T. Because $\overline{AP}$ and $\overline{AS}$ are tangent to ω at P and S, we may write $\alpha = \angle ASP = \angle SPA = \angle SQP$ and $\beta = \angle CQR = \angle QRC = \angle QPR$. Then $\angle PTQ = \pi - \alpha - \beta$. On the other hand, $\angle PAS = \pi - 2\alpha$ and $\angle RCQ = \pi - 2\beta$, so that $ABCD$ is cyclic if and only if

   $$\pi = \angle BAD + \angle DCB = 2\pi - 2\alpha - 2\beta,$$

   or simply

   $$\pi/2 = \pi - \alpha - \beta = \angle PTQ,$$

   as desired. $\square$

A brief review of cyclic Quadrilaterals.

The following discussion of cyclic quadrilaterals is included for reference. Any of the results given here may be cited without proof in your writeups.

A *cyclic quadrilateral* is a quadrilateral whose four vertices lie on a circle called the *circumcircle* (the circle is unique if it exists.) If a quadrilateral has a circumcircle, then the center of this circumcircle is called the *circumcenter* of the quadrilateral. For a convex quadrilateral $ABCD$, the following are equivalent:

- Quadrilateral $ABCD$ is cyclic;
- $\angle ABD = \angle ACD$ (or $\angle BCA = \angle BDA$, etc.);
- Angles $\angle ABC$ and $\angle CDA$ are *supplementary*, that is, $m\angle ABC + m\angle CDA = 180^\circ$ (or angles $\angle BCD$ and $\angle BAD$ are supplementary);

Cyclic quadrilaterals have a number of interesting properties. A cyclic quadrilateral $ABCD$ satisfies

$$AC \cdot BD = AB \cdot CD + AD \cdot BC,$$

a result known as *Ptolemy's theorem.* Another result, typically called *Power of a Point*, asserts that given a circle ω, a point P anywhere in the plane of ω, and a line ℓ through P intersecting ω at points A and B, the value of $AP \cdot BP$ is independent of ℓ; i.e., if a second line ℓ' through P intersects ω at A' and B', then $AP \cdot BP = A'P \cdot B'P$. This second theorem is proved via similar triangles. Say P lies outside of ω, that ℓ and ℓ' are as before and that A and A' lie on segments BP and $B'P$ respectively. Then triangle $AA'P$ is similar to triangle $B'BP$ because the triangles share an angle at P and we have

$$m\angle AA'P = 180^\circ - m\angle B'A'A = m\angle ABB' = m\angle PBB'.$$

The case where $A = B$ is valid and describes the tangents to ω. A similar proof works for P inside ω.