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A Successive Constraint Linear Optimization Method for Lower Bounds of Parametric Coercivity and Inf-Sup Stability Constants

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Abstract

We present an approach to the construction of lower bounds for the coercivity and inf-sup stability constants required in *a posteriori* error analysis of reduced basis approximations to affinely parametrized partial differential equations. The method, based on an Offline-Online strategy relevant in the reduced basis many-query and real-time context, reduces the Online calculation to a small Linear Program: the objective is a parametric expansion of the underlying Rayleigh quotient; the constraints reflect stability information at optimally selected parameter points. Numerical results are presented for an (coercive) elasticity problem and an (non-coercive) acoustics Helmholtz problem. *To cite this article : D.B.P. Huynh, G. Rozza, S. Sen, A.T. Patera, C. R. Acad. Sci.Paris, Ser. xxxx (2007).*

Une Méthode d'Optimisation Linéaire de Constraintes Successives pour les Bournes Inférieures des Constantes de Stabilité Paramètrique Coercive et Inf-Sup

Résumé

Nous présentons une méthode pour le calcul d'une borne inférieure de la constante de stabilité soit de coercivité soit d'inf-sup donc on a besoin pour les estimateurs d'erreurs *a posteriori* associés a l'approximation par base réduite des équations aux dérivées partielles ayant une dépendence affine en ses paramétres. La méthode — que reste fondé sur un stratagème hors-ligne/en-ligne intéressant dans les cadres de temps réel and d'évaluations nombreuses — réduit le calcul en-ligne à un problème d'optimisation linéaire peu coûteuse : l'objectif est un développment paramètrique du quotient de Rayleigh ; les constraintes portents des renseignements de stabilité sur un ensemble optimale de paramètres. Nous présentons des résultats numériques pour un problème d'élasticité (coercive) ainsi que pour un problème d'acoustique Helmholtz (non-coercive). *Pour citer cet article : D.B.P. Huynh, G. Rozza, S. Sen, A.T. Patera, C. R. Acad. Sci. Paris, Ser. xxxx (2007).*

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Version française abrégée

On construit une bourne inférieure de la constante de stabilité de coercivité ou d'inf-sup (2) : a est une forme bilinéaire, associée à une équation aux dérivees partielles sur un domain Ω, que dépend d'un paramètre $\boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^P$ telle que l'hypothèse affine (1) est satisfaite ; $X^{\mathcal{N}}(\Omega)$ est un sous-espace d'ápproximation des élémentes finis de référence de (typiquement, haute) dimension $\mathcal{N}$ auquel on associe la norm $\|\cdot\|_{X^{\mathcal{N}}}$ équivalent à la norme $H^1(\Omega)$.

On considére d'abord le cas coercive. On développe le quotient de Rayleigh (2) en forme paramètrique : $\alpha^{\mathcal{N}}(\boldsymbol{\mu}) = \min_{y \in \mathcal{Y}} \mathcal{J}(\boldsymbol{\mu}; y)$, où $\mathcal{J}(\boldsymbol{\mu}; y) = \sum_{q=1}^{Q} \Theta^q(\boldsymbol{\mu}) y_q$, et l'ensemble $\mathcal{Y}$ est donné en (3). Ensuite on introduit l'ensemble $\mathcal{Y}_{LB}$ en (4), à partir duquel on construit la borne inférieure (5) comme précisé dans la Proposition 1. La bourne inférieure (5) est en fait un problème d'optimisation linéaire en Q variables et $2Q + M_\alpha + M_+$ constraintes ; M_α et M_+ sont la nombre de constraintes engagée de stabilité et de positivité, respectivement. (Nous introduisons aussi une bourne supérieure sur la constante de coercivité qui sert dans l'étape hors-ligne comme critère d'arrêt.)

Nous poursuivons un stratagème hors-ligne/en-ligne intéressant dans les cadres d'approximations de base réduite de temps réel et d'évaluations nombreuses. Hors-ligne (prémier étape) on trouve un ensemble optimale de constraintes de stabilité : cette étape coûteuse requiert la résolution de $2Q + K_{\max}$ problémes de valeurs propres, où $K_{\max}$ est la nombre totale de constraintes de stabilité (dont on tire la M_α plus proche donné une valeur $\boldsymbol{\mu}$ d'interêt) pour arriver à une bourne inférieure de précision suffisante $\forall \boldsymbol{\mu} \in \mathcal{D}$. En-ligne (deuxiéme étape) on calcul la bourne inférieure donné une valeur $\boldsymbol{\mu} \in \mathcal{D}$ quelconque : cet étape requiert la résolution du probléme d'optimisation linéare (5) ; le coût de calcul est indépendent de $\mathcal{N}$.

Nous présentons une example d'élasticité linéare orthotropique (coercive) avec $P = 4$ paramètres pour lequel notre méthode rend $K_{\max} = 50$ pour les choix $M_\alpha = 12$ et $M_+ = 0$. En-ligne, la bourne inférière $\boldsymbol{\mu} \to \alpha_{LB}(\boldsymbol{\mu})$ est cent fois plus vite que le calcul direct $\boldsymbol{\mu} \to \alpha^{\mathcal{N}}(\boldsymbol{\mu})$ — même pour ce probléme à dimension deux de taille assez petit, $\mathcal{N} = 1296$.

Dans le cas non-coercive, d'abord on reconnait que $(\beta^{\mathcal{N}}(\boldsymbol{\mu}))^2$ peut s'exprimer comme un problème coercive équivalent, (6). Donc on peut directement faire appel a l'approche coercive, bien entendu que le problème d'optimisation linéaire maintenant s'agît de $\hat{Q} \equiv Q(Q+1)/2$ variables et $2\hat{Q} + M_\alpha + M_+$ constraintes.

Nous considérons comme example un problème d'acoustique Helmholtz : pour minimiser l'effort hors-ligne, on prend $M_\alpha = \infty$ (en-ligne, effectivement $M_\alpha = K_{\max}$) et $M_+ = 4$ qui rend $K_{\max} = 22$; pour minimiser l'effort en-ligne, on prend $M_\alpha = 4$ et $M_+ = 4$ qui rend $K_{\max} = 98$; finalement, pour bien "équilibrer" les efforts hors-ligne et en-ligne, on propose $M_\alpha = 16$ et $M_+ = 4$ qui rend $K_{\max} = 25$. Dans ce dernier cas, la bourne inférieure $\boldsymbol{\mu} \to \beta_{LB}(\boldsymbol{\mu})$ est bien plus que cent fois plus vite que le calcul direct $\boldsymbol{\mu} \to \beta^{\mathcal{N}}(\boldsymbol{\mu})$. En comparaison assez direct, on trouve que la nouvelle method est plus efficace que la technique proposée en [11] ; et peut-être même plus important, la nouvelle méthode admet un mise-en-oeuvre beaucoup plus simple et générale.

1. Introduction

We define the spatial dimension d and the dimension of our field variable d_v ; for a scalar field, $d_v = d$, whereas for a vector field, $d_v = d$. We introduce a regular spatial domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$; a typical point in Ω shall be denoted $x = (x_1, \ldots, x_d)$. Finally, we define the "exact" Hilbert space $X^{\mathrm{e}}(\Omega)$ with inner product $(w, v)_{X^{\mathrm{e}}}$ and induced norm $||w||_{X^{\mathrm{e}}} \equiv \sqrt{(w, w)_{X^{\mathrm{e}}}}$. For our class of problems, $(H_0^1(\Omega))^{d_v} \subset X^{\mathrm{e}} \subset (H^1(\Omega))^{d_v}$: $H^1(\Omega) \equiv \{v \in L^2(\Omega) | \nabla v \in (L^2(\Omega))^{d_v}\}$ with inner product $(w, v)_{H^1} \equiv \int_\Omega \nabla w \cdot \nabla v + wv$ and norm $\|w\|_{H^1} \equiv \sqrt{(w, w)_{H^1}}$; $H_0^1(\Omega) \equiv \{v \in H^1(\Omega) | v|_{\partial\Omega} = 0\}$; and $L^2(\Omega) \equiv$

$\{v \text{ measurable} | \int_\Omega v^2 \text{ finite}\}$ with inner product $(w, v)_{L^2} \equiv \int_\Omega wv$ and norm $\|w\|_{L^2} \equiv \sqrt{(w, w)_{L^2}}$. We permit any inner product $(w, v)_{X^e}$ that induces a norm equivalent to the $(H^1(\Omega))^{d_v}$ norm.

We introduce a bounded closed parameter domain $\mathcal{D} \subset \mathbb{R}^d$; we denote a parameter point in $\mathcal{D}$ as $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_P)$. We denote the parametric bilinear form associated with our partial differential equation (PDE) as $a : X^e \times X^e \times \mathcal{D} \to \mathbb{R}$. We define the inf-sup, coercivity, and continuity constants as $\beta^e(\boldsymbol{\mu}) \equiv \inf_{w \in X^e} \sup_{v \in X^e} [a(w, v; \boldsymbol{\mu})/(||w||_{X^e}||v||_{X^e})]$, $\alpha^e(\boldsymbol{\mu}) \equiv \inf_{w \in X^e}[a(w, w; \boldsymbol{\mu})/(||w||^2_{X^e})]$, and $\gamma^e(\boldsymbol{\mu}) \equiv \sup_{w \in X^e} \sup_{v \in X^e} [a(w, v; \boldsymbol{\mu})/(||w||_{X^e}||v||_{X^e})]$. We assume that a is stable, $\beta^e(\boldsymbol{\mu}) > 0, \forall \boldsymbol{\mu} \in \mathcal{D}$, and continuous, $\gamma^e(\boldsymbol{\mu})$ finite, $\forall \boldsymbol{\mu} \in \mathcal{D}$; if in addition $\alpha^e(\boldsymbol{\mu}) > 0, \forall \boldsymbol{\mu} \in \mathcal{D}$, we say that a is coercive. We assume here that a is symmetric : $a(w, v) = a(v, w), \forall w, v \in X^e$; see [8] for the non-symmetric case. We also introduce a linear bounded functional, $f : X^e \to \mathbb{R}$.

We assume that our bilinear form is "affine" in the parameter : for some finite small Q,

$$a(w, v; \boldsymbol{\mu}) \equiv \sum_{q=1}^{Q} \Theta^q(\boldsymbol{\mu}) a^q(w, v), \forall w, v \in X^e, \tag{1}$$

where the $\Theta^q, 1 \leq q \leq Q$, are continuous functions over $\mathcal{D}$, and the $a^q, 1 \leq q \leq Q$, are symmetric continuous bilinear forms over $X^e \times X^e$. The assumption (1) is crucial in Offline-Online stategies.

Our problem statement is then : given $\boldsymbol{\mu} \in \mathcal{D}$, find $u^e(\boldsymbol{\mu}) \in X^e$ such that $a(u^e(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v), \forall v \in X^e$; then evaluate the output $s^e(\boldsymbol{\mu}) = f(u^e(\boldsymbol{\mu}))$. (In general, we can consider $s(\boldsymbol{\mu}) = \ell(u^e(\boldsymbol{\mu}))$ for bounded $\ell : X^e \to \mathbb{R}$; here we restrict attention to the linear compliant case, $\ell = f$.)

We introduce a Finite Element (FE) "truth" approximation space of dimension $\mathcal{N}$, $X^{\mathcal{N}} \subset X^e$, with inherited inner product and norm $(w, v)_{X^{\mathcal{N}}} \equiv (w, v)_{X^e}$ and $\|w\|_{X^{\mathcal{N}}} \equiv \|w\|_{X^e}$. We define

$$\beta^{\mathcal{N}}(\boldsymbol{\mu}) \equiv \inf_{w \in X^{\mathcal{N}}} \sup_{v \in X^{\mathcal{N}}} \frac{a(w, v; \boldsymbol{\mu})}{||w||_{X^{\mathcal{N}}}||v||_{X^{\mathcal{N}}}}, \quad \alpha^{\mathcal{N}}(\boldsymbol{\mu}) \equiv \inf_{w \in X^{\mathcal{N}}} \frac{a(w, w; \boldsymbol{\mu})}{||w||^2_{X^{\mathcal{N}}}}, \tag{2}$$

and $\gamma^{\mathcal{N}}(\boldsymbol{\mu}) \equiv \sup_{w \in X^{\mathcal{N}}} \sup_{v \in X^{\mathcal{N}}} [a(w, v; \boldsymbol{\mu})/(||w||_{X^{\mathcal{N}}}||v||_{X^{\mathcal{N}}})]$. It immediately follows that $\alpha^e(\boldsymbol{\mu}) \leq \alpha^{\mathcal{N}}(\boldsymbol{\mu})$ and $\gamma^{\mathcal{N}}(\boldsymbol{\mu}) \leq \gamma^e(\boldsymbol{\mu})$; we *assume* that $\beta^{\mathcal{N}}(\boldsymbol{\mu}) > 0, \forall \boldsymbol{\mu} \in \mathcal{D}$.

Our "truth" FE approximation is then : given $\boldsymbol{\mu} \in \mathcal{D}$, find $u^{\mathcal{N}}(\boldsymbol{\mu}) \in X^{\mathcal{N}}$ such that $a(u^{\mathcal{N}}(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v), \forall v \in X^{\mathcal{N}}$; then evaluate the output $s^{\mathcal{N}}(\boldsymbol{\mu}) = f(u^{\mathcal{N}}(\boldsymbol{\mu}))$. We suppose that, to the desired accuracy, $u^{\mathcal{N}}(\boldsymbol{\mu})$ is indistinguishable from $u^e(\boldsymbol{\mu})$.

Our interest is in reliable real-time and many-query response $\boldsymbol{\mu} \to s^{\mathcal{N}}(\boldsymbol{\mu})$: in the real-time context such as parameter estimation the premium is on reduced *marginal* cost ; in the many-query context such as optimization the premium is on reduced *average* cost. We consider reduced basis (RB) methods [3,6,7,9] : we develop Galerkin approximations to $u^{\mathcal{N}}$ and $s^{\mathcal{N}}(\boldsymbol{\mu})$, $u_N(\boldsymbol{\mu}) \in W_N \subset X^{\mathcal{N}}$ and $s_N(\boldsymbol{\mu}) = f(u_N(\boldsymbol{\mu}))$, respectively, as well as (i) a rigorous and relatively sharp *a posteriori* error bound $\Delta^s_N(\boldsymbol{\mu})$ such that $|s^{\mathcal{N}}(\boldsymbol{\mu}) - s_N(\boldsymbol{\mu})| \leq \Delta^s_N(\boldsymbol{\mu}), \forall \boldsymbol{\mu} \in \mathcal{D}$, and (ii) an Offline–Online computational strategy such that, in the Online stage, the marginal (and hence also asymptotic average) cost to compute $\boldsymbol{\mu} \to s_N(\boldsymbol{\mu}), \Delta^s_N(\boldsymbol{\mu})$ depends only on N, the dimension of W_N, and Q — and *not* on $\mathcal{N}$.

The error bound $\Delta^s_N(\boldsymbol{\mu})$ takes the form [9] in the coercive case $\|R(\cdot; \boldsymbol{\mu})\|^2_{(X^{\mathcal{N}})'}/\alpha_{LB}(\boldsymbol{\mu})$ and in the non-coercive case $\|R(\cdot; \boldsymbol{\mu})\|^2_{(X^{\mathcal{N}})'}/\beta_{LB}(\boldsymbol{\mu})$, where $R(v; \boldsymbol{\mu}) \equiv f(v) - a(u_N(\boldsymbol{\mu}), v; \boldsymbol{\mu}), \forall v \in X^{\mathcal{N}}$, is the residual, $\| \cdot \|_{(X^{\mathcal{N}})'}$ denotes the dual norm, and $\alpha_{LB}(\boldsymbol{\mu})$ and $\beta_{LB}(\boldsymbol{\mu})$ are lower bounds for the coercivity and inf-sup constants. Our focus in this paper is on a new lower bound (Offline-Online) strategy for $\alpha_{LB}(\boldsymbol{\mu})$ and $\beta_{LB}(\boldsymbol{\mu})$. Our new approach is more efficient than our earlier coercive [7,9] and non-coercive [9,11] proposals ; also the new approach is much more easily implemented.

There are many classical techniques for the estimation of smallest eigenvalues or smallest singular values. One class of methods is based on Gershgorin's theorem and variants [2]. Within our particular context, these approaches are not optimal : generalized eigenvalue and singular value problems are difficult to treat ; the operation count will scale with $\mathcal{N}$; and finally the Gershgorin–like bounds are often not useful for elliptic PDEs. A second class of methods is based on eigenfunction/eigenvalue (e.g., Rayleigh Ritz) approximation and subsequent residual evaluation [1,4]. Unfortunately, the lower bounds are not truly rigorous : we obtain lower bounds not for the smallest eigenvalue but rather for the eigenvalue closest to the proposed approximate eigenvalue.

2. Successive Constraints Method (SCM)

We address the coercive case first. By way of preliminaries we define

$$\mathcal{Y} \equiv \left\{ y = (y_1, \ldots, y_Q) \in \mathbb{R}^Q \mid \exists\, w \in X^{\mathcal{N}} \text{s.t}\, y_q = \frac{a^q(w,w)}{||w||^2_{X^{\mathcal{N}}}}, 1 \le q \le Q \right\}. \tag{3}$$

We further define the objective function $\mathcal{J} : \mathcal{D} \times \mathbb{R}^Q \to \mathbb{R}$ as $\mathcal{J}(\boldsymbol{\mu}; y) = \sum_{q=1}^{Q} \Theta^q(\boldsymbol{\mu}) y_q$. We may then write our coercivity constant as $\alpha^{\mathcal{N}}(\boldsymbol{\mu}) = \min_{y \in \mathcal{Y}} \mathcal{J}(\boldsymbol{\mu}; y)$; we now consider relaxations.

We first define $\sigma_q^- \equiv \inf_{w \in X^{\mathcal{N}}} [a^q(w,w)/(||w||^2_{X_{\mathcal{N}}})]$ and $\sigma_q^+ \equiv \sup_{w \in X^{\mathcal{N}}} [a^q(w,w)/(||w||^2_{X_{\mathcal{N}}})]$, $1 \le q \le Q$, and let $\mathcal{B}_Q \equiv \Pi_{q=1}^{Q} [\sigma_q^-, \sigma_q^+] \in \mathbb{R}^Q$. We also introduce the two parameter sets $\Xi \equiv \{\nu_1 \in \mathcal{D}, \ldots, \nu_J \in \mathcal{D}\}$ and $\mathcal{C}_K \equiv \{\omega_1 \in \mathcal{D}, \ldots, \omega_K \in \mathcal{D}\}$. For any finite-dimensional subset of $\mathcal{D}$, E $(= \Xi$ or $\mathcal{C}_K)$, $\mathcal{P}_M(\boldsymbol{\mu}; E)$ shall denote the M points closest to $\boldsymbol{\mu}$ in E (in the Euclidean norm) ; if $card(E) \le M$ then we define $\mathcal{P}_M(\boldsymbol{\mu}; E) = E$, and if $M = 0$ we define $\mathcal{P}_M(\boldsymbol{\mu}; E) = \emptyset$.

We now define, for given $\mathcal{C}_K$ (and $M_\alpha \in \mathbb{N}$, $M_+ \in \mathbb{N}$, and Ξ),

$$\mathcal{Y}_{LB}(\boldsymbol{\mu}; \mathcal{C}_K) \equiv \Bigg\{ y \in \mathcal{B}_Q \quad | \quad \sum_{q=1}^{Q} \Theta^q(\boldsymbol{\mu}') y_q \ge \alpha^{\mathcal{N}}(\boldsymbol{\mu}'), \forall\, \boldsymbol{\mu}' \in \mathcal{P}_{M_\alpha}(\boldsymbol{\mu}; \mathcal{C}_K); \sum_{q=1}^{Q} \Theta^q(\boldsymbol{\mu}') y_q \ge 0, \forall\, \boldsymbol{\mu}' \in \mathcal{P}_{M_+}(\boldsymbol{\mu}; \Xi) \Bigg\}, \tag{4}$$

and $\mathcal{Y}_{UB}(\mathcal{C}_K) \equiv \{y^\star(\omega_k), 1 \le k \le K\}$ for $y^\star(\boldsymbol{\mu}) \equiv \arg\min_{y \in \mathcal{Y}} \mathcal{J}(\boldsymbol{\mu}; y)$. We then define

$$\alpha_{LB}(\boldsymbol{\mu}; \mathcal{C}_K) = \min_{y \in \mathcal{Y}_{LB}(\boldsymbol{\mu}; \mathcal{C}_K)} \mathcal{J}(\boldsymbol{\mu}; y) \tag{5}$$

and $\alpha_{UB}(\boldsymbol{\mu}; \mathcal{C}_K) = \min_{y \in \mathcal{Y}_{UB}(\mathcal{C}_K)} \mathcal{J}(\boldsymbol{\mu}; y)$. We then obtain
Proposition 1. For given $\mathcal{C}_K$ (and $M_\alpha \in \mathbb{N}$, $M_+ \in \mathbb{N}$, Ξ), $\alpha_{LB}(\boldsymbol{\mu}; \mathcal{C}_K) \le \alpha^{\mathcal{N}}(\boldsymbol{\mu}) \le \alpha_{UB}(\boldsymbol{\mu}; \mathcal{C}_K)$, $\forall \boldsymbol{\mu} \in \mathcal{D}$.

Proof. It is simple to recognize that $\mathcal{Y}_{UB} \subset \mathcal{Y} \subset \mathcal{Y}_{LB}$: each constraint in $\mathcal{Y}_{LB}$ is satisfied by all members of $\mathcal{Y}$; and each element of $\mathcal{Y}_{UB}$ is a member of $\mathcal{Y}$. The desired result directly follows. □
We also show in [8] that our SCM lower bound is always better than (and also more general than) our earlier "min Θ" approach [9].

We note that (4), (5) is in fact a linear optimization problem or Linear Program (LP) ; indeed (4), (5) resembles a discretized linear semi-infinite program [5]. Our LP (5) contains Q design variables and $2Q + M_\alpha + M_+$ (one-sided) inequality constraints : the operation count for the Online stage $\boldsymbol{\mu} \to \alpha_{LB}(\boldsymbol{\mu})$ is *independent* of $\mathcal{N}$.

But first we must determine $\mathcal{C}_K$ and obtain the $\alpha^{\mathcal{N}}(\omega_k), 1 \le k \le K$, by an Offline "greedy " algorithm. We first set $K = 1$ and choose $\mathcal{C}_1 = \{\omega_1\}$ "arbitrarily" ; we also specify M_α, M_+, Ξ, and a tolerance $\epsilon_\alpha \in]0,1[$. Then, given $\mathcal{C}_K$ we find $\omega_{K+1} = \arg\max_{\boldsymbol{\mu} \in \Xi} \left[\left(\alpha_{UB}(\boldsymbol{\mu}; \mathcal{C}_K) - \alpha_{LB}(\boldsymbol{\mu}; \mathcal{C}_K) \right) / \alpha_{UB}(\boldsymbol{\mu}; \mathcal{C}_K) \right]$ and update $\mathcal{C}_{K+1} = \mathcal{C}_K \bigcup \omega_{K+1}$; we repeat this process until $\max_{\boldsymbol{\mu} \in \Xi} \left[\left(\alpha_{UB}(\boldsymbol{\mu}; \mathcal{C}_{K_{\max}}) - \alpha_{LB}(\boldsymbol{\mu}; \mathcal{C}_{K_{\max}}) \right) / \alpha_{UB}(\boldsymbol{\mu}; \mathcal{C}_{K_{\max}}) \right] \le \epsilon_\alpha$. The notable offline computations are (i) $2Q$ eigenproblems to form $\mathcal{B}_Q$ and $K_{\max}$ eigenproblems to obtain $y^\star(\omega_k), \alpha^{\mathcal{N}}(\omega_k), 1 \le k \le K_{\max}$ (for a judiciously chosen inner product, the latter can be efficiently calculated by a Lanczos scheme) ; (ii) $O(\mathcal{N} Q K_{\max})$ operations to form $\mathcal{Y}_{UB}$ (we assume sparsity), and (iii)$J K_{\max}$ LPs of size $O(Q + M_\alpha + M_+)$.

As an example, we consider a linear elastic orthotropic material [10] in two dimensions (plane stress) : $d = d_v = 2$. The original domain is a rectangle $]0, L[\times]0, 1[$: we map this domain to a fixed reference domain $\Omega \equiv]0, L_{\mathrm{ref}} = 1[\times]0, 1[$; subsequently L shall appear only in the parameter-dependent coefficient functions. Our function space X^{e} is given by $\{v \in H^1(\Omega) | v|_{\Gamma^D} = 0\}^2$, where Γ^D is the left boundary of Ω ; $X^{\mathcal{N}} \subset X^{\mathrm{e}}$ is a linear finite element space of dimension $\mathcal{N} = 1296$ over a triangulation of Ω.

We consider $P = 4$ parameters : $\mu_1 \equiv E_{x_2}/E_{x_1}$, where E_{x_2} and E_{x_1} are the orthotropic Young's moduli ; $\mu_2 \equiv \nu$, the Poisson ratio ; $\mu_3 \equiv G/E_{x_1}$, where G is the tangential modulus ; and $\mu_4 \equiv L$. The parameter domain is given by $\mathcal{D} \equiv [0.05, 1.0] \times [0.1, 0.3225] \times [0.02, 0.50] \times [0.1, 15]$. Our affine development (1) then applies for $Q = 6$ and $\Theta^1(\boldsymbol{\mu}) = \frac{1}{\mu_4(1-\mu_2^2)}$, $a^1(v, w) = \int_\Omega \frac{\partial v_1}{\partial x_1}\frac{\partial w_1}{\partial x_1}$, $\Theta^2(\boldsymbol{\mu}) = \frac{\mu_2}{1-\mu_2^2}$, $a^2(v, w) = \int_\Omega \left(\frac{\partial v_2}{\partial x_2}\frac{\partial w_1}{\partial x_1} + \frac{\partial v_1}{\partial x_1}\frac{\partial w_2}{\partial x_2}\right)$, $\Theta^3(\boldsymbol{\mu}) = \mu_3\mu_4$, $a^3(v, w) = \int_\Omega \frac{\partial v_1}{\partial x_2}\frac{\partial w_1}{\partial x_2}$, $\Theta^4(\boldsymbol{\mu}) = \mu_3$, $a^4(v, w) = \int_\Omega \left(\frac{\partial v_1}{\partial x_2}\frac{\partial w_2}{\partial x_1} + \frac{\partial v_2}{\partial x_1}\frac{\partial w_1}{\partial x_2}\right)$, $\Theta_5(\boldsymbol{\mu}) = \frac{\mu_3}{\mu_4}$, $a^5(v, w) = \int_\Omega \frac{\partial v_2}{\partial x_1}\frac{\partial w_2}{\partial x_1}$, $\Theta_6(\boldsymbol{\mu}) = \frac{\mu_1\mu_4}{1-\mu_2^2}$, $a^6(v, w) = \int_\Omega \frac{\partial v_2}{\partial x_2}\frac{\partial w_2}{\partial x_2}$. We choose for our inner product $(w, v)_{X^{\mathcal{N}}} \equiv a(w, v; \boldsymbol{\mu}_{\mathrm{ref}} = \{1.0, 0.1, 0.5, 1.0\}) + \lambda_{\min}(w, v)_{L^2}$, where $\lambda_{\min}$ is the minimum of the Rayleigh quotient $a(w, w; \boldsymbol{\mu}_{\mathrm{ref}})/(w, w)_{L^2}$; this choice of inner product ensures a good spectrum for the Lanczos algorithm [8].

We apply our greedy algorithm for $\epsilon_\alpha = .75$ and Ξ a random sample of size $J = 1500$. For $M_\alpha = \infty$ (effectively $M_+ = K_{\max}$ in the Online stage), and $M_+ = 0$ we obtain $K_{\max} = 30$. However, we can significantly reduce M_α (and Online effort) with only a slight increase in $K_{\max}$ (and Offline effort) : for $M_\alpha = 12, M_+ = 0$, we obtain $K_{\max} = 50$. (Note that since $M_+ = 0$, our lower bound is a constructive proof that a is coercive over Ξ.) For this "optimal" set of parameters, we observe that our Online lower bound $\boldsymbol{\mu} \to \alpha_{LB}(\boldsymbol{\mu})$ is ≈ 100 times faster than direct computation of $\boldsymbol{\mu} \to \alpha^{\mathcal{N}}(\boldsymbol{\mu})$.

We now consider the non-coercive case. We introduce operators $T^q : X^{\mathcal{N}} \to X^{\mathcal{N}}$ as $(T^q w, v)_{X^{\mathcal{N}}} = a^q(w, v), \forall v \in X^{\mathcal{N}}, 1 \le q \le Q$, and $T^{\boldsymbol{\mu}} w \equiv \sum_{q=1}^{Q} \Theta^q(\boldsymbol{\mu}) T^q w$. It is then readily demonstrated [11] that $(\beta^{\mathcal{N}}(\boldsymbol{\mu}))^2 = \inf_{w \in X^{\mathcal{N}}}[(T^{\boldsymbol{\mu}} w, T^{\boldsymbol{\mu}} w)_{X^{\mathcal{N}}}/||w||^2_{X^{\mathcal{N}}}]$, which can be expanded as $(\beta^{\mathcal{N}}(\boldsymbol{\mu}))^2 = \inf_{w \in X^{\mathcal{N}}} \left[\sum_{q'=1}^{Q}\sum_{q''=q'}^{Q}(2 - \delta_{q'q''})\Theta^{q'}(\boldsymbol{\mu})\Theta^{q''}(\boldsymbol{\mu}) \left((T^{q'} w, T^{q''} w)_{X^{\mathcal{N}}}/||w||^2_{X^{\mathcal{N}}}\right)\right]$; here $\delta_{q'q''}$ is the Kronecker delta. We now identify $(\beta^{\mathcal{N}}(\boldsymbol{\mu}))^2 \longmapsto \hat{\alpha}^{\mathcal{N}}(\boldsymbol{\mu})$, $(2 - \delta_{q'q''})\Theta^{q'}(\boldsymbol{\mu})\Theta^{q''}(\boldsymbol{\mu}), 1 \le q' \le q'' \le Q \longmapsto \hat{\Theta}^q(\mu), 1 \le q \le \hat{Q} \equiv Q(Q+1)/2$, and $(T^{q'} w, T^{q''} w)_{X^{\mathcal{N}}}, 1 \le q' \le q'' \le Q \longmapsto \hat{a}^q(w, v), 1 \le q \le \hat{Q}$, and observe that

$$(\beta^{\mathcal{N}}(\boldsymbol{\mu}))^2 \equiv \hat{\alpha}^{\mathcal{N}}(\boldsymbol{\mu}) \equiv \inf_{w \in X^{\mathcal{N}}} \sum_{q=1}^{\hat{Q}} \hat{\Theta}^q(\boldsymbol{\mu}) \frac{\hat{a}^q(w, w)}{||w||^2_{X^{\mathcal{N}}}}; \tag{6}$$

not surprisingly, $(\beta^{\mathcal{N}}(\boldsymbol{\mu}))^2$ can be recast as an equivalent coercivity constant.

We may thus directly apply our SCM procedure to (6). The drawback is that our LP for $(\beta^{\mathcal{N}}(\boldsymbol{\mu}))^2$ now has $\approx Q^2/2$ design variables and $\approx (Q^2 + M_\alpha + M_+)$ inequality constraints. However, for Q modest, the SCM approach is still quite effective.

As an example, we consider a simple acoustics (microphone probe) Helmholtz problem [11] in two dimensions : $d = 2, d_v = 1$. The original domain is a channel probe followed by a microphone plenum of height $1/4 + L$: we map this domain to a fixed reference domain ($L_{\mathrm{ref}} = 1$) $\Omega = [-1/2, 1] \times [0, 1/4]$ $(\equiv \overline{\Omega_b}) \cup [0, 1] \times [1/4, 5/4] (\equiv \overline{\Omega_t})$; subsequently L shall appear only in the parameter–dependent coefficient functions. Our function space X^{e} is given by $\{v \in H^1(\Omega) | v|_{\Gamma^D} = 0\}$ where Γ^D is the left boundary of Ω ; $X^{\mathcal{N}} \subset X^{\mathrm{e}}$ is then a quadratic finite element space of dimension $\mathcal{N} = 4841$ over a triangulation of Ω.

We consider $P = 2$ parameters : $\mu_1 \equiv L$, and $\mu_2 = k^2$ (the reduced frequency squared) ; the parameter domain $\mathcal{D} \equiv [0.3, 0.6] \times [3.0, 6.0]$ lies between the first and second resonances. Our affine development (1) then applies for $Q = 5$, with $\Theta^1(\boldsymbol{\mu}) = 1$, $a^1(w, v) = \int_{\Omega_b} \nabla w \cdot \nabla v$, $\Theta^2(\boldsymbol{\mu}) = -\mu_2$, $a^2(w, v) = \int_{\Omega_b} wv$, $\Theta^3(\boldsymbol{\mu}) = \mu_1$, $a^3(w, v) = \int_{\Omega_t} \frac{\partial w}{\partial x_1}\frac{\partial v}{\partial x_1}$, $\Theta^4(\boldsymbol{\mu}) = 1/\mu_1$, $a^4(w, v) = \int_{\Omega_t} \frac{\partial w}{\partial x_2}\frac{\partial v}{\partial x_2}$, $\Theta^5(\boldsymbol{\mu}) = -\mu_1\, \mu_2$, $a^5(w, v) = \int_{\Omega_t} wv$; note $\hat{Q} = Q(Q+1)/2 = 15$ for this problem. For our inner product we choose $(w, v)_{X^{\mathcal{N}}} \equiv$

$a_0(w,v) + \lambda_{\min}(w,v)_{L^2}$, where $a_0(w,v) \equiv a(w,v;(0.45,0))$ and $\lambda_{\min}$ is the minimum of the Rayleigh quotient $a_0(w,w)/(w,w)_{L^2}$.

We apply our greedy algorithm for $\epsilon_\alpha = 0.75$ and Ξ a random sample of size $J = 3671$. For the case $M_\alpha = \infty$ (effectively, $M_\alpha = K_{\max}$ in the Online stage) and $M_+ = 4$ we obtain $K_{\max} = 22$: this choice minimizes the Offline effort. If we wish to minimize the Online effort we can choose $M_\alpha = 4$ (and $M_+ = 4$) — note the Online effort is independent of $K_{\max}$ — which then yields $K_{\max} = 98$: in this case, our Online lower bound $\boldsymbol{\mu} \to \beta_{LB}(\boldsymbol{\mu})$ is ≈ 160 times faster than direct computation of $\boldsymbol{\mu} \to \beta^{\mathcal{N}}(\boldsymbol{\mu})$. For a better Offline-Online balance we can consider $M_\alpha = 16$, $M_+ = 4$ which yields $K_{\max} = 25$: in this case, our Online lower bound $\boldsymbol{\mu} \to \beta_{LB}(\boldsymbol{\mu})$ is ≈ 137 times faster than direct computation of $\boldsymbol{\mu} \to \beta^{\mathcal{N}}(\boldsymbol{\mu})$. We have also applied the SCM approach to the (in fact, coercive) parameter domain considered in [11] ; the SCM performs better than the natural norm approach of [11], and is also much simpler to implement.

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