Unbalanced Random Matching Markets: The Stark Effect of Competition

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Abstract

We study competition in matching markets with random heterogeneous preferences by considering an unequal number of agents on either side. First, we show that even the slightest imbalance yields an essentially unique stable matching. Second, we give a tight description of stable outcomes, showing that matching markets are extremely competitive. Each agent on the short side of the market is matched to one of his top preferences, and each agent on the long side is either unmatched or does almost no better than being matched to a random partner. Our results suggest that any matching market is likely to have a small core, explaining why small cores are empirically ubiquitous.

Keywords: Matching markets; Random markets; Competition; Stability; Core.

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1 Introduction

Stable matching theory has been instrumental in the study and design of numerous two sided markets. In these markets, there are two disjoint sets of agents, where each agent has preferences over potential matches from the other side. Examples include entry-level labor markets, dating, and college admissions. The core of a two-sided markets is the set of stable matchings, where a matching is stable if there are no man and woman who both prefer each other over their assigned partners. Stability is a critical property in the design of centralized clearing-houses[1] and a useful equilibrium concept for predicting outcomes in decentralized markets.[2]

In this paper, we address two fundamental issues by characterizing how the core is affected by competition. First, we address the longstanding issue of multiplicity of stable matchings.[3] Previous studies have shown that the core is small only under restrictive assumptions on market structure, suggesting that the core is generally large. In contrast, empirical findings from a variety of markets suggest that matching markets have an essentially unique stable matching in practice.[4] Second, the matching literature provides almost no direct link between market characteristics and stable outcomes. For example, it is known that increasing the number of agents on one side makes agents on the other side weakly better off (Crawford (1991)), but little is known about the magnitude of this effect.

We analyze these questions by looking at randomly drawn matching markets, allowing for competition arising from an unequal number of agents on either side. Influential work by Pittel (1989b) and Roth and Peranson (1999) study the same model for balanced markets and find that the core is typically large. We show that the competition resulting from even

[1] Stable matching models have been successfully adopted in market design contexts such as school choice (Abdulkadiroglu et al. (2006); Abdulkadiroğlu et al. (2005)) and resident matching programs (Roth and Peranson (1999)). Roth (2002) shows that stability is important for the success of centralized clearing houses.

[2] See, e.g., Hitsch et al. (2010) and Banerjee et al. (2013).

[3] The potential multiplicity of stable matchings is a central issue in the literature and has led to many studies about the structure of the core (Knuth (1976)), which stable matching to implement (Schwarz and Yenmez (2011)) and strategic behavior (Dubins and Freedman (1981); Roth (1982)).

[4] A unique stable matching was reported in the National Resident Matching Program (NRMP) (Roth and Peranson (1999)), Boston school choice (Pathak and Sönmez (2008)), online dating (Hitsch et al. (2010)) and the Indian marriage market (Banerjee et al. (2013)).

the slightest imbalance yields an essentially unique stable matching. Our results, which hold for both small and large markets, suggests that any matching market is likely to have a small core, thereby providing an explanation as to why small cores are empirically ubiquitous.

Our second contribution shows that the essentially unique stable outcome can be almost fully characterized using only the distribution of preferences and the number of agents on each side. Roughly speaking, under any stable matching, each agent on the short side of the market is matched to one of his top preferences. Each agent on the long side is either unmatched or does almost no better than being matched to a random partner. Thus, we find that matching markets are extremely competitive, with even the slightest imbalance greatly benefiting the short side. We also present simulation results showing the short side's advantage is robust to small changes in the model.

Formally, we consider a matching market with uniformly random and independent complete preference lists with n agents on the short side (men) and $n+1$ or more agents on the long side (women). Let the rank of a matched agent to be her preference rank of the assigned partner. We show that with high probability: (i) the core is small in that almost all agents have the same partner in all stable matchings and all stable matchings give approximately the same average ranking for men and women, and (ii) under any stable matching men are, on average, matched to their $\log n$ most preferred wife, while matched women are on average matched to their $n/(\log n)$ most preferred man. Thus, in an unbalanced market, agents on the short side, on average, rank their partners as they would if the preferences of the long side were not taken into account by the matching process at all.[5] Matched agents on the long side, on average, rank their partners approximately the same as if they were allowed to choose their match from a limited, randomly drawn set of only $\log n$ potential partners. The results therefore show that even in a slightly imbalanced market, there is a very large advantage to being on the short side. For example, in a market with 1,000 men and 1,001 women, men are matched on average to their 7th ($\approx \log 1000$) most preferred woman, while women are matched on average to their 145th ($\approx 1000/\log 1000$) most preferred man. We further show that the benefit to the short side is amplified when the imbalance is greater.

These results imply that there is limited scope for strategic behavior in matching markets. Suppose agents report preferences to a central mechanism, which implements a matching that

[5] The corresponding mechanism would be Random Serial Dictatorship (RSD), under which men choose their partners in a random order (see e.g. Abdulkadiroğlu and Sönmez (1998)).

is stable with respect to reported preferences. Demange et al. (1987) show that agents who have unique stable partners are unable to gain from misreporting their preferences, i.e., from manipulating the mechanism. Furthermore, agents with a unique stable partners will be unaffected by profitable manipulations by other agents. Our results therefore show that even under full information, only a small fraction of agents would benefit from misreporting, and only a small fraction of agents will be affected. Thus our results may help explain the practical success of stable matching mechanisms, and suggest that agents report their preferences truthfully to stable matching mechanisms.

To gain some intuition for these results, it is useful to compare our setting with a competitive, homogeneous buyer-seller market. In a market with 100 homogeneous sellers, who have unit supply and a reservation value of zero, and 100 identical buyers, who have unit demand and value the good at one, every price between zero and one gives a core allocation. However, when there are 101 sellers, competition among sellers implies a unique clearing price of zero, since any buyer has an "outside option" of buying from the unmatched seller who will sell for any positive price.

Our results show a sharp phenomenon in matching markets similar to the one in the homogenous buyer-seller market. This is surprising since we consider markets with heterogeneous preferences and no transfers. In particular, the direct outside option argument (used above in the buyer-seller market) does not explain the strong effect of a single additional woman: while the unmatched woman is willing to match with any man, she creates a useful outside option only for a few men who rank her favorably. However, since these men must be matched to women they each rank even higher, their spouses are made worse off and in turn create an indirect outside option for other men. The economic intuition, which is key in our proof, is that these "indirect outside options" ripple throughout the entire market, creating strong outside options for all men.

Taken together, our results suggest that matching markets will generally have a small core. We show that even with uncorrelated preferences the core is small under the slightest imbalance. This result is important for real-world applications since precisely balanced markets are unlikely to arise in practice. It is generally thought that correlation in preferences reduces the size of the core,[6] and we present simulations supporting this. We also simulate

[6] See for example Roth and Peranson (1999).

markets with varying correlation structures, market sizes, and list lengths, and do not find a setting (apart from the balanced random market) that has a large core.

Our proof requires developing some technical tools that may be of independent interest. We build on the work of McVitie and Wilson (1971) and Immorlica and Mahdian (2005) to construct an algorithm that progresses from the men-optimal stable matching to the women-optimal stable matching through a series of rejection chains. These rejection chains unveil whether a woman can be made better off without inducing some man to prefer the unmatched woman and block the match. Immorlica and Mahdian (2005) and Kojima and Pathak (2009) analyze rejection chains in markets where there are many unmatched agents. When there are many unmatched agents, rejection chains are likely to be short, allowing each chain to be analyzed independently. When the imbalance is small, understanding the ripple effect relies on the joint analysis of how the different rejection chains are interconnected. We modify the McVitie-Wilson algorithm to account for the interdependence between different rejection chains. A run of the modified algorithm on random preferences can be modeled by a tractable stochastic process, the analysis of which reveals that different chains are likely to be highly connected.

1.1 Related literature

Most relevant to our work are Pittel (1989a) and Knuth et al. (1990), which extensively analyze balanced random matching markets. They characterize the set of stable matchings for a random matching market with n men and n women, showing that the men's average rank of wives ranges from $\log n$ to $n/\log n$ for different stable matchings and that the fraction of agents with multiple stable partners approaches 1 as n grows large. Our results show that the addition of a single woman makes the core collapse, leaving only the most favorable outcomes for men.

Several papers study the size of the core to understand incentives to misreport preferences. They analyze random matching markets where one side has short, randomly drawn preference lists.[7] In this case Immorlica and Mahdian (2005) show that women cannot manipulate in

[7]The random, short list assumption is motivated by the limited number of interviews as well as the NRMP restriction that allows medical students to submit a rank ordered list of up to only 30 programs (http://www.nrmp.org/match-process/create-and-certify-rol-applicants/) and by the limited number of in-

a one-to-one marriage market, and Kojima and Pathak (2009) show that schools cannot manipulate in many-to-one matching markets. Our results differ in two ways. First, these papers are limited to studying manipulation and the size of the core, while we characterize outcomes and show that competition benefits the short side of the market. Second, their analysis relies on the specific market structure generated by short, randomly drawn preference lists. Because preference lists are short in their setting, many schools are not ranked by *any* student, and many students remain unmatched. Thus, though they assume an equal number of seats and students, their model behaves like a highly unbalanced market.[8] The short list assumption was necessary for the analysis in these papers, leading them to conclude that the core is likely to be small only under restrictive assumptions. In contrast, our results suggest that matching markets very generally have a small core, driven by a strong effect of competition.

Coles and Shorrer (2012) and Lee (2011) study manipulation in asymptotically large balanced matching markets, making different assumptions about the utility functions of agents. Coles and Shorrer (2012) define agents' utilities to be equal to the rank of their spouse (varying between 0 and n), which grows linearly with the number of agents in the market. They show that women are likely to gain substantially from manipulating men-proposing deferred acceptance by truncating their preferences. Lee (2011) allows for correlation in preferences but assumes that utilities are kept bounded as the market grows large. He shows that in a large enough market, most agents cannot gain much utility from manipulating men-proposing deferred acceptance. The difference between the two conclusions can be reconciled as a result of the different utility parameterizations. In a balanced market, agents are likely to have stable partners of rank ranging from $\log n$ to $n/\log n$, and this difference in rank can be small or large in terms of utilities.[9] In contrast, in the unbalanced market, agents are likely to have a unique stable partner.

The literature on matching markets with transferable utilities provides theoretical pre-

terviews.

[8]In fact, their proof requires that preference lists are short enough for agents to have a significant probability of remaining unmatched. Unless being unmatched is an attractive option, it implies that agents are not submitting long enough lists.

[9]Both $\log n/n$ and $1/\log n$ converge to 0, meaning that both asymptotically give the top percentile, although the latter converges very slowly. For example, $\log n/n = 1\%$ for $n \approx 600$, but $1/\log n = 1\%$ for $n \approx 2.68 \times 10^{43}$.

dictions on who matches to whom based on market characteristics. For instance, related to this paper is the phenomenon that increasing the relative number of agents of one type ("buyers") is well-known to benefit agents of the other type ("sellers") (Shapley and Shubik, 1971; Becker, 1973).[10] Our paper provides the first quantification of this effect in matching markets without transfers.

Another situation in which the set of stable matchings is small is in matching markets with highly correlated preferences. When all men have the same preferences over women, there is a unique stable matching. Holzman and Samet (2013) generalize this observation, showing that if the distance between any two men's preference lists is small, the set of stable matchings is small. Azevedo and Leshno (2012) look at large many-to-one markets with a constant number of schools and an increasing number of students; they find these generically converge to a unique stable matching in a continuum model. The core of these markets can be equivalently described as the core of a one-to-one matching market between students and seats in schools, where all seats in a school have identical preferences over students.

1.2 Organization of the paper

Section 2 presents our model and results. Section 3 provides intuition for the results, outlines our proof and presents the new matching algorithm that is the basis for our proof. Section 4 presents simulation results, which show that the same features occur in small markets, and checks the robustness of our results. Section 5 gives some final remarks and discusses the limitations of our model.

Appendix A proves the correctness of our new matching algorithm. The proof of our main results is in Appendix B. In Appendix C, we discuss how our results may be extended to many-to-one random matching markets.

[10] Rao (1993); Abramitzky et al. (2009) exploit random variation in female-male balance to show that outcomes in the marriage market favor the short side.

2 Model and results

2.1 Random matching markets

In a two-sided matching market, there is a set of men $\mathcal{M} = \{1, \ldots, n\}$ and a set of women $\mathcal{W} = \{1, \ldots, n+k\}$. Each man m has a complete strict preference list $\succ_m$ over women, and each woman w has a complete strict preference list $\succ_w$ over the set of men. A *matching* is a mapping μ from $\mathcal{M} \cup \mathcal{W}$ to itself such that for every $m \in \mathcal{M}$, $\mu(m) \in \mathcal{W} \cup \{m\}$, and for every $w \in \mathcal{W}$, $\mu(w) \in \mathcal{M} \cup \{w\}$, and for every $m, w \in \mathcal{M} \cup \mathcal{W}$, $\mu(m) = w$ implies $\mu(w) = m$.

A matching μ is *unstable* if there are a man m and a woman w such that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. A matching is *stable* if it is not unstable. We say that m is stable for w (and vice versa) if there is a stable matching in which m is matched to w. It is well-known that the core of a matching market is the set of stable matchings.

Our focus in this paper is to characterize the stable matchings of randomly generated matching markets. A *random matching market* is generated by drawing a complete preference list for each man and each woman independently and uniformly at random. That is, for each man m, we draw a complete ranking $\succ_m$ from a uniform distribution over the $|\mathcal{W}|!$ possible rankings. Section 4 provides simulation results for more general distributions over preferences, allowing for correlated preferences.

A stable matching always exists, and can be found using the Deferred Acceptance (DA) algorithm by Gale and Shapley (1962). The men-proposing DA (MPDA) algorithm repeatedly selects an unassigned man m who in turn proposes to his most preferred woman who has not yet rejected him. If m has been rejected by all women, the algorithm assigns m to be unmatched. If a woman has more than one proposal, she rejects all but her most preferred one, leaving the rejected men unassigned. When every man is assigned to either a woman or being unmatched, the algorithm terminates and outputs the matching.

Gale and Shapley (1962) show that men-proposing DA finds the *men-optimal stable matching* (MOSM), in which every man is matched to his most preferred stable woman. The MOSM matches every woman to her least preferred stable man. Likewise, the women proposing DA produces the women-optimal stable matching (WOSM) with symmetric properties.

We are interested in the size of the core, as well as how matched agents rank their assigned partners. Denote the rank of woman w in the preference list $\succ_m$ of man m by $\mathrm{Rank}_m(w) \equiv |w' : w' \succeq_m w|$, where m's most preferred woman has a rank of 1. Symmetrically denote the rank of m in the preference list of w by $\mathrm{Rank}_w(m)$.

Definition 2.1. *Given a matching μ, the* men's average rank of wives *is given by*

$$R_{\text{MEN}}(\mu) = \frac{1}{|\mathcal{M} \backslash \bar{\mathcal{M}}|} \sum_{m \in \mathcal{M} \backslash \bar{\mathcal{M}}} \mathrm{Rank}_m(\mu(m)) \, ,$$

where $\bar{\mathcal{M}}$ is the set of men who are unmatched under μ.

Similarly, the women's average rank of husbands *is given by*

$$R_{\text{WOMEN}}(\mu) = \frac{1}{|\mathcal{W} \backslash \bar{\mathcal{W}}|} \sum_{m \in \mathcal{W} \backslash \bar{\mathcal{W}}} \mathrm{Rank}_w(\mu(w)) \, ,$$

where $\bar{\mathcal{W}}$ is the set of women who are unmatched under μ.

We use two metrics for the size of the core. First, we measure the fraction of agents that have multiple stable partners. Second, we adopt the difference between men's average rankings under MOSM and WOSM as a measure for how distinct these two extreme stable matchings are.

2.2 Previous results

Previous literature has analyzed balanced random matching markets, which have an equal number of men and women. We start by citing a key result on the structure of stable matchings in balanced markets:

Theorem [Pittel (1989a)]. *In a random matching market with n men and n women, the fraction of agents that have multiple stable partners converges to 1 as $n \to \infty$. Furthermore,*

$$\frac{R_{\text{MEN}}(\text{MOSM})}{\log n} \xrightarrow{p} 1 \, ,$$

$$\frac{R_{\text{MEN}}(\text{WOSM})}{n / \log n} \xrightarrow{p} 1$$

where $\xrightarrow{p}$ denotes convergence in probability.

This result shows that in a balanced market, the core is large under both measures as not only do most agents have multiple stable partners, but also men's average rankings under the extreme stable matchings are significantly different. We next show that this does not extend to unbalanced markets. In the next section, we show that in an unbalanced market, the core is small under both measures. In the subsequent subsection, we further show that an inequality in the numbers of men and women is highly favorable for the short side of the market.

2.3 The size of the core in unbalanced markets

In our main result, we show that an unequal number of men and women leads to a small core, even if the difference is as small as one. We show that almost all men have a unique stable partner in a typical realization of the market, and that each side receives a similar average rank under the MOSM and WOSM. We omit quantifications for the sake of readability and give a more detailed version of the theorem in Appendix B.

Theorem 1. *Fix $\varepsilon > 0$. Consider a sequence of random matching markets, indexed by n, with n men and $n + k$ women, for arbitrary $k = k(n) \geq 1$. With high probability,*[11] *we have that*

(i) the fraction of men and fraction of women who have multiple stable partners tend to zero as $n \to \infty$, and

(ii) the men's average rank of spouses is almost the same under all stable matchings,[12] *as is the women's average rank of spouses:*

$$R_{\text{MEN}}(\text{WOSM}) \leq (1 + \varepsilon) R_{\text{MEN}}(\text{MOSM}) \,,$$
$$R_{\text{WOMEN}}(\text{WOSM}) \geq (1 - \varepsilon) R_{\text{WOMEN}}(\text{MOSM}) \,.$$

Theorem 1 shows that there is an essentially unique stable matching in unbalanced markets. For centralized unbalanced markets, this result implies that which side proposes in DA would make little difference. For decentralized markets, this implies that stability gives an almost unique prediction. In Section 2.6, we show that Theorem 1 further implies that there is limited scope for manipulation when the market is unbalanced. In Appendix C, we sketch how to extend our results to unbalanced many-to-one markets.

[11] Given a sequence of events $\{\mathcal{E}_n\}$, we say that this sequence occurs *with high probability* (whp) if $\lim_{n\to\infty} \mathbb{P}(\mathcal{E}_n) = 1$.

[12] For any stable matching μ it is well known that $R_{\text{MEN}}(\text{MOSM}) \leq R_{\text{MEN}}(\mu) \leq R_{\text{MEN}}(\text{WOSM})$.

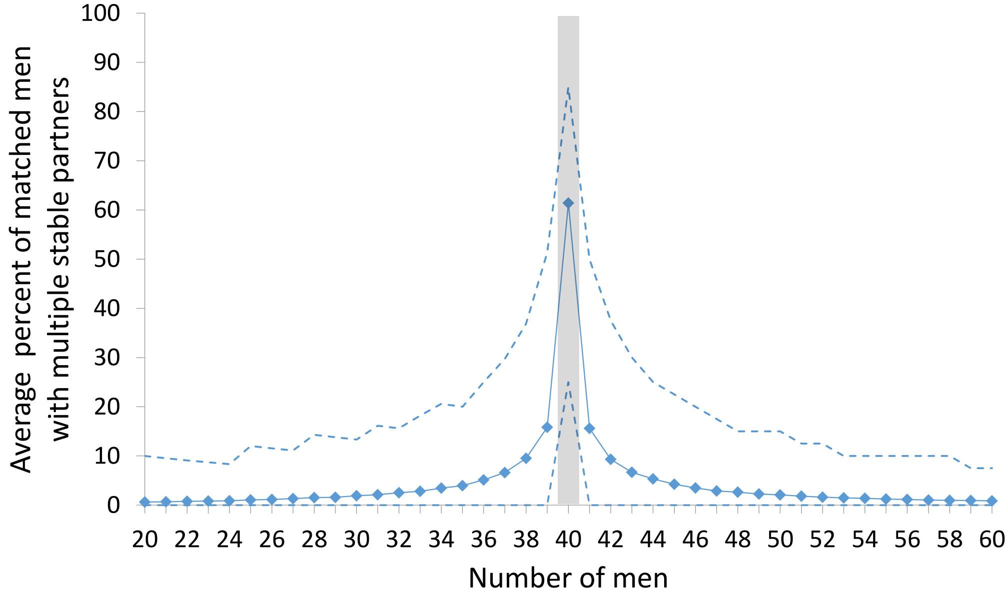


Figure 1: Percent of men with multiple stable partners, in random markets with 40 women and a varying number of men. The main line indicates the average over 10,000 realizations. The dotted lines indicate the top and bottom 2.5th percentiles.

Figures 1 and 2 illustrate the results and show that the same features hold in small unbalanced markets. Figure 1 reports the fraction of men who have multiple stable partners in random markets with 40 women and 20 to 60 men. Figure 2 plots the men's average rank of wives under MOSM and WOSM. Observe that, even in such small markets, the large core of the balanced market (40 men and 40 women) is a knife-edge case, such that the core is small under both measures of core size for any market that is unbalanced even slightly.

Given that the smallest imbalance leads to a small core, even when preferences are heterogenous and uncorrelated, we expect that general matching markets will have a small core. Real markets are likely to have correlated preferences, but we expect correlation in preferences to reduce the size of the core. When there are multiple stable matchings there is a subset of men who all become strictly better off when moving from one stable matching to another. However, such a joint improvement is unlikely when preferences are correlated. In Section 4 we use simulations to examine the effect of correlation on the size of the core, and find that the core tends to shrink as preferences become more correlated. In addition, after simulating various distributions, we have yet to find a distribution that yields a large core outside of a balanced market. This leads us to believe that any real life matching market will have an almost unique stable matching.

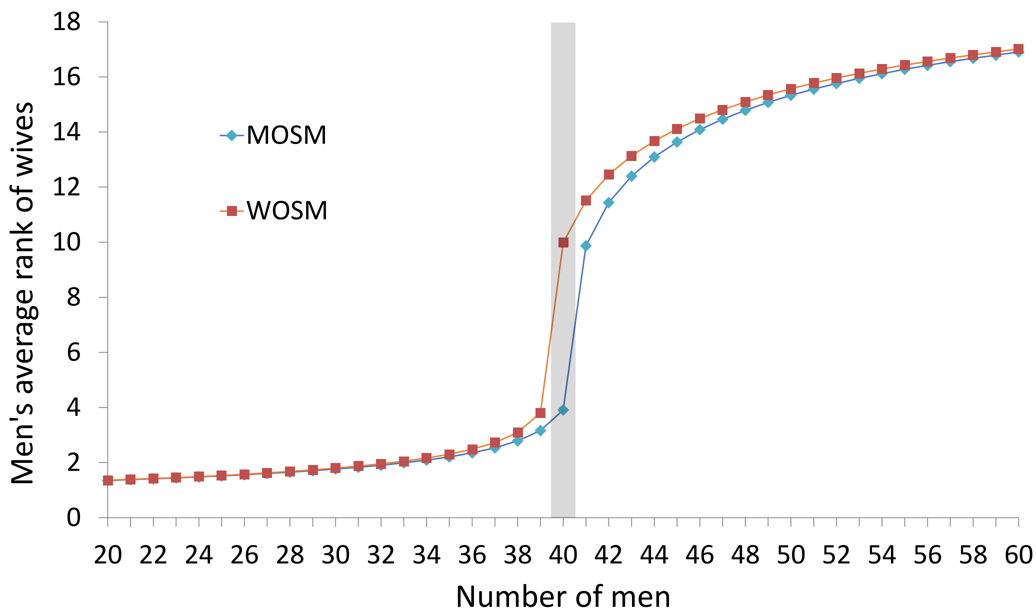


Figure 2: Men's average rank of wives under MOSM and WOSM in random markets with 40 women and varying number of men. The lines indicate the average over 10,000 realizations.

2.4 Characterization of stable outcomes

The following theorem characterizes the stable outcomes of unbalanced markets. Theorem 2 shows that under any stable matching, agents on the short side are matched to one of their top choices, while agents on the long side either remain unmatched or are assigned to a partner that is ranked not much better than a random partner. The theorem states the result for a general imbalanced market, and we give simplified expressions for special cases of interest in the next subsection. A more detailed version of the theorem is given in Appendix B.

Theorem 2. *Consider a sequence of random matching markets, indexed by n, with n men and $n + k$ women, for arbitrary $k = k(n) \geq 1$. With high probability, the following hold for* *every stable matching μ:*

$$R_{\text{MEN}}(\mu) \le (1+\varepsilon)\left(\tfrac{n+k}{n}\right)\log\left(\tfrac{n+k}{k}\right),$$

$$R_{\text{WOMEN}}(\mu) \ge n\Big/\left[1+(1+\varepsilon)\left(\tfrac{n+k}{n}\right)\log\left(\tfrac{n+k}{k}\right)\right].$$

For comparison, consider the assignments generated by the men's random serial dictatorship (RSD) mechanism. In RSD, men are ordered at random, and each man chooses his favorite woman that has yet to be chosen, such that the mechanism completely ignores women's preferences. The men's average rank of wives under RSD is approximately $(1+\varepsilon)\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)$.[13] Thus, under any stable matching, men's average rank would be almost the same as under RSD. Women's average rank under any stable matching is better than getting a random husband by a factor of at most only $\Theta(\log\frac{n}{k})$.[14] Thus, roughly speaking, in any stable matching, the short side "chooses" while the long side is "chosen."

Figure 2 illustrates the advantage of the short side. It plots men's average rank in markets with 40 women and 20 to 60 men. When men are on the short side (there are fewer than 40 men), they are matched to one of their top preferences on average. When men are on the long side, they are either unmatched or rank their spouse barely better than a random match. Figure 3 includes a plot of men's average rank under RSD. In Section 4, we provide simulation results indicating that Theorem 2 gives a good approximation in finite markets. We further conduct simulations that demonstrate the advantage of the short side when preferences are correlated.

2.5 Special cases for unbalanced matching markets

To highlight two particular cases of interest, we present the following two immediate corollaries. We first focus on markets with the minimal imbalance, where there is only one extra woman.

Corollary 2.2. *Consider a sequence of random matching markets with n men and $n+1$ women. With high probability, in every stable matching, the men's average rank of wives is*

[13]Following an analysis very similar to the proof of Lemma B.4(i), we can show that the men's average rank under RSD is, with high probability, at least $(1-\epsilon)\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)$ and at most $(1+\epsilon)\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)$.

[14]For any two functions f and g, we write $f(n) = \Theta(g(n))$ if there exist constants $a \le b$ such that $ag(n) \le f(n) \le bg(n)$ for sufficiently large n.

no more than $1.01 \log n$*, the women's average rank of husbands is at least* $\frac{n}{1.01 \log n}$*, and the fractions of men and women who have multiple stable partners converge to* 0 *as* $n \to \infty$*.*

The next case of interest is a random matching market with a large imbalance between the sizes of the two sides of the market. We look at $k = \lambda n$ for fixed λ, i.e., a matching market with linearly more women than men.

Corollary 2.3. *Let* $\lambda > 0$ *be any positive constant. Consider a sequence of random matching markets with* $|\mathcal{M}| = n$, $|\mathcal{W}| = (1+\lambda)n$*. Define the constant* $\kappa = 1.01(1+\lambda)\log(1+1/\lambda)$*. We have that with high probability, in every stable matching, the average rank of wives is at most* κ*, the average rank of husbands is at least* $n/(1+\kappa)$*, and the fractions of men and women who have multiple stable partners converge to* 0 *as* $n \to \infty$*.*

When there is a substantial imbalance in the market, the allocation is largely driven by men's preferences. For example, in a large market with 5% extra women, men will be matched, on average, to their 4th most preferred woman. The average rank of women under WOSM is only a factor $(1+\kappa)/2$ better than being assigned to a random man. Thus, the benefit of being on the short side becomes more extreme when there is more imbalance.[15]

2.6 Implications for strategic behavior

In this section, we consider the implications of our results for strategic agents in matching markets. A *matching mechanism* is a function that takes reported preferences of men and women and produces a matching of men to women. A *stable matching mechanism* is a matching mechanism which produces a matching that is stable with respect to reported preferences. A matching mechanism induces a direct revelation game, in which each agent reports a preference ranking and receives utility from being assigned to his/her assigned partner.

A mechanism is said to be strategy-proof for men (women) if it is a dominant strategy for every man (woman) to report preferences truthfully. The MPDA is strategy-proof for men, but some women may benefit from misreporting their preferences. Roth (1982) shows that

[15] As we would expect, $\kappa = \kappa(\lambda)$ is monotone decreasing in λ. At the extreme, we have $\lim_{\lambda \to \infty} \kappa(\lambda) = 1.01$, such that when there are many women for each man, $R_{\text{MEN}} \le 1.01$ (meaning that men typically get their top choice) and $R_{\text{WOMEN}} \ge n/2.01$ (meaning that women typically get an almost random match)).

no stable matching mechanism is strategy-proof for both sides of the market. Demange et al. (1987) show that the scope of strategic behavior is limited - in a stable matching mechanism, a woman can never achieve a man she prefers over her husband in the WOSM. In particular, a woman cannot manipulate a stable matching mechanism if she has a unique stable partner (who is her match under both the MOSM and WOSM). In a random unbalanced matching market, most women will have a unique stable partner and therefore cannot gain from misreporting their preferences.

Consider the following direct revelation game induced by a stable matching mechanism. Each man m independently draws utilities over matching with each woman w $u_m(w) \sim F$. Symmetrically, each woman w draws utilities over matching with each man m $u_w(m) \sim G$. We assume that F, G are continuous probability distributions with finite support $[0, \bar{u}]$. Each agent privately learns his/her own preferences, submits a ranking to the matching mechanism, and receives the utility of begin matched to his/her assigned partner. We say that an agent reports truthfully if he/she submits a ranking list of spouses in the order of his/her utility of being matched to them.

Theorem 2.4. *Consider a stable matching mechanism, and let $\varepsilon > 0$ and $k > 0$. There exists n_0 such that for any unbalanced market with $n > n_0$ men and $n + k$ women, it is an ε-Bayes-Nash equilibrium for all agents to report truthfully.*

Note that Theorem 2.4 applies to *any* stable matching mechanism, regardless of the stable matching it selects. While the theorem is stated for large n, the simulations in Section 4 show that there is little scope for manipulation even in small markets.

Proof. From Theorem 1, we know that the expected fraction of women with multiple stable partners converges to 0 as n tends to infinity. All women are ex ante symmetric; then since preferences are drawn independently and uniformly at random, the women are still symmetric after we reveal the preference list of one woman. Therefore, the interim probability that a woman has multiple stable partners, conditional on her realized preferences, is equal to the expected fraction of woman with multiple stable partners, and thus tends to 0 as n tends to infinity. Choosing n_0 such that this probability is less than $\varepsilon/\bar{u}$ guarantees that any woman can gain at most ε by misreporting her true preferences. The argument for men is identical. □

Generally, a woman w can manipulate a stable matching mechanism by reporting the partner that the mechanism would have assigned her to be unacceptable. This forces the mechanism to match w to a different stable partner, if one exists, or leaves her unmatched, if she has a unique stable partner. Our analysis in Appendix B bounds the probability that another stable partner exists, giving us the result above. We believe that by following the analysis in Appendix B more delicately, one can produce tighter bounds on the gains from manipulation and extend the above results to utilities with unbounded support.

3 Proof idea and algorithm

This section presents the intuition for Theorems 1 and 2. We first provide a rough calculation to illustrate the stark advantage of the short side. Our proof follows a different and more constructive approach, which uses a new matching algorithm to trace the structure of the core. We present the main ideas of the proof, as well as the new matching algorithm.

3.1 Intuition for the advantage of the short side

Rough intuition for the large advantage of the short side can be gained through the Rural Hospital Theorem (Roth, 1986). Suppose there are $n+1$ women. By the Rural Hospital Theorem, the same woman $\bar{w}$ is unmatched in all stable matchings; thus, every stable matching must also induce a stable matching for the balanced market (after dropping the unmatched woman $\bar{w}$). A stable matching of the balanced market without $\bar{w}$ remains stable after we add $\bar{w}$ if no man prefers $\bar{w}$ over his current match. Therefore, the stable matchings in the unbalanced market are the stable matchings for the balanced market in which men's average rank is low enough such that no man prefers $\bar{w}$ over his match.

3.2 Sketch of the proof

We prove our results by calculating both the MOSM and WOSM through a series of proposals by men, through the use of Algorithm 2. Since men receive a low average rank, the algorithm runs quickly and provides us with a tractable stochastic process, which we analyze to determine the average rank of men and women.

We first calculate the MOSM using MPDA. Algorithm 2 first calculates the MOSM and proceeds to determine for every woman whether she has a more preferred stable husband until it finds the WOSM. The algorithm works as follows: consider a woman w who is matched to man m in a stable matching. A stable matching that gives w a more preferred partner must give m a less preferred partner. In order to check whether w has a better stable husband than m, the algorithm triggers a rejection chain by rejecting m: m proposes to his most preferred woman to whom he has not already proposed, possibly displacing another man who proposes in a similar way. This chain, which we call a *phase*, can end in one of two ways: (a) an *improvement phase* ends with w accepting a proposal from a man she prefers over m, and (b) a *terminal phase* ends with a proposal to an unmatched woman. An improvement phase finds a new stable matching that matches w to a stable partner she prefers over m. A terminal phase implies that m is w's most preferred stable partner. The algorithm terminates when the rejection chain from each woman is a terminal phase.

Analyzing the run of the algorithm for markets of n men and $(1+\lambda)n$ women is simpler. In this case, an arbitrary phase beginning with any w is likely to be terminal: the probability that a man in the chain will propose to an unmatched woman is roughly $\frac{\lambda}{1+\lambda}$, while the probability that he will propose to w is of order $\frac{1}{n}$. Thus, improvement phases are rare, and most women will be matched to the same man under the MOSM and WOSM.

The analysis for markets of n men and $n+1$ women is more involved and requires us to use links between different rejection chains. Denote by S the set of women for whom the algorithm has already found their best stable partners. After finding the MOSM, we can initialize S to contain all unmatched women (by the Rural Hospital Theorem, they are unmatched in all stable matchings). If a chain includes a sub-chain that begins and ends with the same woman w', we call that sub-chain an Internal Improvement Cycle (IIC). An IIC is equivalent to a separate improvement phase for w', which can be cut out from the original chain. After removing all IICs, the chain of every terminal phase includes each woman only once. Every woman in a terminal phase can be added to S, as divorcing their husbands will result in a sub-chain that is also terminal. When a new chain reaches a woman $w' \in S$, the phase must also be terminal, since reaching $w' \in S$ implies that the new chain merges with a previous terminal chain. The set S thus allows us to track rejection chains, and how they merge together.

Consider starting the first phase in the algorithm, which starts with the woman w who received the most proposals in the MPDA algorithm. The probability that a proposal will be to w is approximately the same as the probability it will be to the unmatched woman; both are $\sim \frac{1}{n}$. However, the phase is likely to be terminal because the unmatched woman accepts all proposals, while w previously received at least $\log(n)$ proposals and will only accept a man whom she prefers to all of them. The total length of the chain is likely to be of order $\Theta(n)$, and after we exclude IICs, a chain of length $\sim \sqrt{n}$ is left. Therefore, by the end of the first phase, S is likely to be of size $\sim \sqrt{n}$. Thus, S grows quickly, and once S is large, the algorithm will have almost reached the WOSM. Almost every subsequent phase will be a terminal phase, and almost all women are already matched to their most preferred stable husband.

To calculate the average rank, we calculate the number of proposals made by men under MPDA, using methods similar to Pittel (1989a). We use a variant of MPDA in which one man proposes at a time, eventually also producing the MOSM, and show that the number of proposals by men is roughly equal to the solution of the coupon collector's problem. The average rank of wives is calculated from the number of proposals each woman receives.

In Appendix C, we discuss how to extend our proof to many-to-one matching markets. A many-to-one market can be decomposed to a one-to-one market where a school is represented by several seats. By running Algorithm 2 on the decomposed market, we can find the two extreme stable matchings. When the capacity of each school is small relative to the size of the market, rejection chains behave similarly to how they behave in the one-to-one case, and the same proof approach can be used. We sketch the required alterations to our proof in Appendix C.

3.3 Matching algorithm

This section presents Algorithm 2, which calculates the WOSM from the MOSM through a series of proposals by men. This algorithm can be used to quickly calculate the WOSM when women are on the long side. All algorithms in this section assume that there are strictly more women than men (that is, $|\mathcal{W}| > |\mathcal{M}|$). For further details, refer to Appendix A.

For completeness, we first present the MPDA algorithm that outputs the MOSM.

Algorithm 1. Men-proposing deferred acceptance (MPDA)

1. *Every unmatched man proposes to his most preferred woman who has not already rejected him. If no new proposal is made, output the current matching.*

2. *If a women has multiple proposals, she tentatively keeps her most preferred man and rejects the rest. Go to step 1.*

We now present Algorithm 2, which calculates the WOSM. It tracks rejection chains by using S, the set of women whose most preferred stable match has already been found. In each phase, the rejection chain is an ordered set $V = (v_1, v_2, .., v_J)$. If a proposal is accepted by a woman in V, an improvement phase or IIC is implemented. When a proposal is accepted by a woman in S, the phase is terminal, and all the women in V are added to S. The algorithm keeps track of $\nu(w)$, the current number of proposals received by women w, and $R(m)$, the set of women who have rejected m so far. μ records the current proposals, and $\tilde{\mu}$ records the current candidate for the women-optimal stable matching. We use the notation $x \leftarrow y$ for the operation of copying the value of variable y to variable x.

Algorithm 2. MOSM to WOSM

- Input: *A matching market with n men and $n + k$ women.*

- Initialization: *Run the men-proposing deferred acceptance to get the men-optimal stable matching μ, and set $R(m)$ and $\nu(w)$ accordingly. Set S to be the set of women unmatched under μ, bar$\mathcal{W}$. Index all women in $\mathcal{W}$ according to some order. Set $\hat{w}$ to be the woman in $\mathcal{W}$ who has received the most proposals so far. Set t to be the total number of proposals made so far.*

- New phase:

  1. *Set $\tilde{\mu} \leftarrow \mu$. Set $V = (v_1) = (\hat{w})$.*
  2. Divorce: *Set $m \leftarrow \mu(\hat{w})$ and have $\hat{w}$ reject m (add $\hat{w}$ to $R(m)$).*
  3. Proposal: *Man m proposes to w, his most preferred woman has not yet rejected him.*[16] *Increment $\nu(w)$ and proposal number t by one each.*
  4. w's Decision:

[16] See footnote 26.

(a) *If $w \notin V$ prefers $\mu(w)$ to m, or if $w \in V$ and w prefers $\tilde{\mu}(w)$ to m, then reject m (add w to $R(m)$) and return to step 3.*

(b) *If $w \notin S \cup V$ and w prefers m to $\mu(w)$, then w rejects her current partner. Set $m' \leftarrow \mu(w)$, $\mu(w) \leftarrow m$. Add w to $R(m')$, append w to the end of V. Set $m \leftarrow m'$ and return to step 3.*

(c) New stable matching: *If $w \in V$ and w prefers m to $\tilde{\mu}(w)$, then we have found a stable matching. If $w = \hat{w} = v_1$, set $\mu(\hat{w}) \leftarrow m$. Select $\hat{w} \in \mathcal{W} \backslash S$ and start a new phase from step 1.*

If $w = v_\ell$ for $\ell > 1$, record her current husband as $m' \leftarrow \mu(w)$. Call the set of all proposals made since and including the proposal of m' to w an internal improvement cycle *(IIC). Update $\tilde{\mu}$ for the women in the loop by setting $\tilde{\mu}(v_j) \leftarrow \mu(v_j)$ for $j = \ell+1, ..., J$; also set $\tilde{\mu}(w) \leftarrow m$ and $\mu(w) \leftarrow m$. Remove $v_\ell, .., v_J$ from V, set the proposer $m \leftarrow m'$, decrement $\nu(w)$, decrement t, and return to step 3, in which m will propose to w.*

(d) End of terminal phase: *If $w \in S$ and w prefers m to $\mu(w)$, then restore $\mu \leftarrow \tilde{\mu}$ and add all the women in V to S. If $S = \mathcal{W}$, terminate and output $\tilde{\mu}$. Otherwise, set $\hat{w}$ to be the woman in $\mathcal{W} \backslash S$ with the smallest index and begin a new phase by returning to 1.*

In Appendix A, we prove that Algorithm 2 outputs the women-optimal stable matching (see Proposition A.2). We can calculate the difference between the sum of men's rank of wives under WOSM and MOSM from the algorithm. Since all terminal phases are rolled back, this difference equals the number of proposals in improvement phases and IICs during Algorithm 2.

4 Computational experiments

This section presents simulation results that complement our theoretical results. We first present simulation results for small markets and test the asymptotic fit of our theoretical findings. We then investigate the effects of correlation in preferences and test the robustness of our results. Last, we present simulation results for unbalanced many-to-one matching markets.

4.1 Numerical results for our model

To complement our asymptotic theoretical results we conduct computational experiments for varying market sizes. The simulation results show that our findings tend to hold for small markets as well as large markets. Furthermore, we find that the asymptotic results give a good approximation for small markets.

The first computational experiment illustrates the sharp effect of imbalance in small markets. We consider markets with 40 women and a varying number of men, from 20 to 60 men. For each market size we simulate 10,000 realizations by drawing uniformly random complete preferences independently for each agent. For each realization we compute the MOSM and WOSM. Figure 1 (Section 2) shows the fraction of men who have multiple stable partners (averaged across realizations with 95% confidence interval); it shows the fraction is small in all unbalanced markets.

Figure 3 shows men's average rank of wives[17] under the MOSM and WOSM (averaged across realizations). The results for the balanced market (40 men and 40 women) replicate the previous analysis by Pittel (1989a) and Roth and Peranson (1999), but the figure shows that this balanced market is not typical. In any unbalanced market, mens' average rank under MOSM and WOSM is similar. When there are fewer men than women (that is, less than 40 men) men are matched to one of their top choices under any stable matching. For comparison we also plot mens' average rank under the Random Serial Dictatorship (RSD) mechanism,[18] and find that in an unbalanced market with fewer men, RSD gives almost the same average rank for men as any stable matching. When there are more men than women in the market, the average rank of matched men is not much better than that resulting from a random assignment (note that assigning a man to a random woman leads to an average rank of 20.5).

Tables 1 and 2 present simulation results for markets with varying size and imbalance. Table 1 reports men's average rank of wives under the MOSM and WOSM, showing the sharp effect of imbalance across different market sizes. Under both WOSM and MOSM the men's average rank of wives is high when there are strictly fewer women (columns $-10, -5, -1$) and low when there strictly fewer men (columns $1, 2, 3, 5, 10$). In addition, Table 1 reports

[17] Women's average rank of husbands is symmetrically given by switching the number of men and women.

[18] In the RSD mechanism men are ordered at random, and each man chooses his favorite woman that has yet to be chosen (thus RSD ignores womens' preferences).

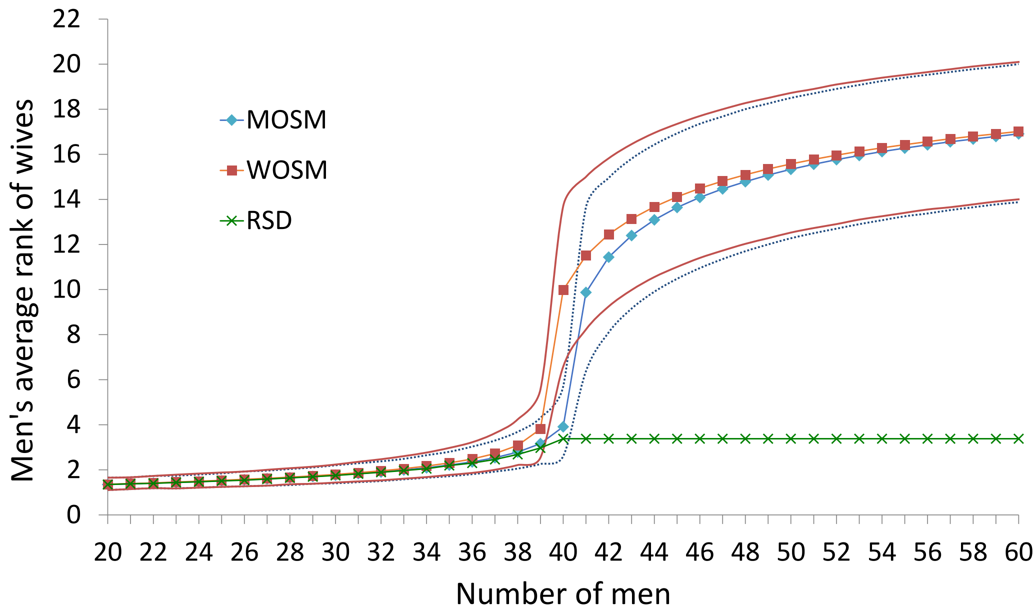


Figure 3: Men's average rank of wives under MOSM and WOSM in random markets with 40 women and a varying number of men. The main lines indicate the average over matched mens' average rank of wives in all 10,000 realizations. The dotted lines indicate the top and bottom 2.5th percentile of the 10,000 realizations. The labeled RSD gives the men's average rank under the Random Serial Dictatorship mechanism.

theoretical estimates of the average rank based on the asymptotic results in section 2. The estimate from our theorem is a tight asymptotic bound for men's average rank of wives when men are on the short side, while it is only a lower bound when men are the long side (see Appendix D for a details). Observe that when men are on the short side of the market (the right side of the table) our estimate gives a surprisingly good approximation for both the MOSM and WOSM. When men are the long side (the left side of the table) our estimate is only about $10 - 25\%$ below the true values. Notice also that the values for the MOSM and the WOSM are quite close to each other in all markets. Table 2 presents the percentage of men who have multiple stable partners. This percentage is high in balanced markets but is low for all unbalanced markets with even the slightest imbalance. We infer that imbalance leads to a small core.

Using the matching algorithm from Section 3 we were able to run simulations of rather large unbalanced matching markets. Table 3 provides numerical results of 1,000 draws of

$\lvert\mathcal{M}\rvert$ \ $\lvert\mathcal{W}\rvert - \lvert\mathcal{M}\rvert$		-10	-5	-1	0	1	2	3	5	10
100	MOSM	29.5	27.2	20.3	5.0	4.1	3.7	3.4	3.0	2.6
	WOSM	30.1	28.2	23.6	20.3	4.9	4.1	3.6	3.2	2.6
	EST	25.3	22.9	17.5		4.7	4.0	3.6	3.2	2.6
200	MOSM	53.6	48.1	35.3	5.7	4.8	4.3	4.1	3.7	3.1
	WOSM	54.7	49.9	41.0	35.5	5.7	4.7	4.4	3.8	3.2
	EST	45.7	40.8	31.5		5.3	4.7	4.3	3.8	3.2
500	MOSM	115.8	102.6	75.9	6.7	5.7	5.3	5.0	4.5	3.9
	WOSM	118.0	106.3	86.6	76.2	6.7	5.7	5.3	4.7	4.0
	EST	98.2	87.6	69.0		6.2	5.5	5.2	4.7	4.0
1000	MOSM	203.8	181.4	136.2	7.4	6.4	6.0	5.7	5.2	4.6
	WOSM	207.5	187.6	155.1	137.3	7.4	6.5	6.0	5.4	4.7
	EST	175.2	157.3	126.2		6.9	6.2	5.8	5.3	4.7
2000	MOSM	364.5	324.2	249.6	8.1	7.1	6.7	6.3	5.9	5.3
	WOSM	370.8	334.4	280.7	249.1	8.1	7.1	6.6	6.1	5.4
	EST	314.6	284.7	232.3		7.6	6.9	6.5	6.0	5.3
5000	MOSM	793.1	713.5	560.0	9.1	8.1	7.6	7.3	6.8	6.2
	WOSM	804.7	732.8	622.5	560.2	9.1	8.1	7.6	7.0	6.3
	EST	690.5	631.1	525.2		8.5	7.8	7.4	6.9	6.2

Table 1: Men's average rank of wives in MOSM and WOSM for different market sizes. The numbers for each market size are averages over 1,000 realizations. A man's most preferred wife has rank 1, and larger rank indicates a less preferred wife. EST is the theoretical estimate of the men's average rank based on Theorem 2 (see Appendix D for details).

matching markets of different sizes.

4.2 Size of the core under correlated preferences

This section presents simulation results to examine the effects of correlation in preferences on the size of the core. We simulate a large class of distributions and find a large core only in balanced markets with small correlation in preferences. In general our simulations suggest that the core becomes smaller as preferences become more correlated, although when the fraction of agents with multiple stable partners is small, this relationship may not be monotone.

We present results on preferences generated from the following random utility model,

$\lvert\mathcal{M}\rvert$ \ $\lvert\mathcal{W}\rvert - \lvert\mathcal{M}\rvert$	-10	-5	-1	0	1	2	3	5	10
100	2.1	4.1	15.1	75.3	15.4	9.5	6.5	4.5	2.3
200	2.2	3.8	14.6	83.6	14.6	8.0	6.2	4.1	2.1
500	2.0	3.8	12.6	91.0	13.1	7.1	5.5	3.6	2.0
1000	1.9	3.5	12.3	94.5	12.2	7.2	5.1	3.4	2.0
2000	1.8	3.1	11.1	96.7	11.1	6.1	4.8	2.9	1.7
5000	1.5	2.7	10.1	98.4	10.2	6.0	4.3	2.8	1.5

Table 2: Percentage of men who have multiple stable partners for different market sizes. The numbers for each market size are averages over 1,000 realizations.

$\lvert\mathcal{M}\rvert$	$\lvert\mathcal{W}\rvert - \lvert\mathcal{M}\rvert = +1$			$\lvert\mathcal{W}\rvert - \lvert\mathcal{M}\rvert = +10$		
	Men's avg rank under MOSM	Men's avg rank under WOSM	% Men w. mul. stable partners	Men's avg rank under MOSM	Men's avg rank under WOSM	% Men w. mul. stable partners
10	1.98 (0.45)	2.29 (0.60)	13.84 (18.82)	1.31 (0.20)	1.33 (0.21)	1.19 (5.13)
100	4.09 (0.72)	4.89 (1.08)	15.16 (12.98)	2.55 (0.26)	2.61 (0.27)	2.30 (3.15)
1,000	6.47 (0.79)	7.44 (1.28)	11.90 (10.17)	4.59 (0.30)	4.69 (0.31)	1.95 (2.03)
10,000	8.80 (0.79)	9.80 (1.30)	9.45 (8.30)	6.88 (0.30)	6.98 (0.32)	1.46 (1.47)
100,000	11.11 (0.83)	12.09 (1.31)	7.66 (6.60)	9.16 (0.31)	9.26 (0.32)	1.08 (1.02)
1,000,000	13.40 (0.80)	14.41 (1.27)	6.62 (6.04)	11.46 (0.30)	11.56 (0.32)	0.85 (0.80)

Table 3: Men's average rank in different market sizes with a small imbalance. The numbers for each market size are averages over 1,000 realizations. Standard deviation are given in parentheses.

adapted from Hitsch et al. (2010). The utility of agent i for matching with agent j is given by

$$u_i(j) = \beta x_j^A - \gamma^+(|x_i^D - x_j^D|_+)^2 - \gamma^-(|x_j^D - x_i^D|_+)^2 + \varepsilon_{ij}, \tag{1}$$

where ε_{ij} is an idiosyncratic term that is i.i.d. with a logistic distribution, agent i has characteristics x_i^A, x_i^D, and $|a - b|_+ = \max(a - b, 0)$. x_i^A is the vertical quality of agent i. x_i^D is the location of agent i and γ^+, γ^- determines agent's preferences over distance to their spouse, where distances from above are allowed to be different from distance below.

We draw characteristics $x_i^A, x_i^D \sim U[0,1]$ for each agent i.i.d. as well as ε_{ij} to generate utilities $u_i(\cdot)$ which induce i's complete preference list over spouses. We use the same coefficients $\beta, \gamma^+, \gamma^-$ for both men and women.

When $\beta = \gamma^+ = \gamma^- = 0$ this model reduces to our i.i.d. preferences model. As β increases (keeping γ^+ and γ^- fixed), preferences of agents on the same side become more correlated, and as $\beta \to \infty$, all men will have the same preferences over women (and symmetrically, all women will have the same preferences over men). The coefficients γ^+, γ^- generate correlation between $u_m(w)$ and $u_w(m)$. When $\gamma^+ > 0, \gamma^- > 0$ both m and w will get higher utility if $|x_m^D - x_w^D|^2$ is low, generating positive cross sides correlation between $u_m(w)$ and $u_w(m)$.[19] When $\gamma^+ > 0, \gamma^- < 0$ the term $|x_m^D - x_w^D|^2$ will appear with opposite signs in the utility of the matched couple, generating negative cross side correlation between $u_m(w)$ and $u_w(m)$.

Figure 4 presents the size of the core for different levels of imbalance and a range of coefficient values. For each market we simulate 2,000 realizations and report the average percent of agents with multiple stable partners.[20] We consider markets with 40 women and either 40, 41, 45 or 60 men, $\beta \in [0, 20]$ and $\gamma^+ \in [-20, 20]$ and allow both positive cross sides correlation by taking $\gamma^+ = \gamma^-$ and negative cross sides correlation by taking $\gamma^+ = -\gamma^-$.

The plot shows that correlation tends to reduce the size of the core.[21] The only markets that have a large core are balanced markets with low levels of correlation in preferences. We interpret these results as complementary to our theoretical results, giving additional suggestive evidence that general preference distributions are likely to generate a small core.

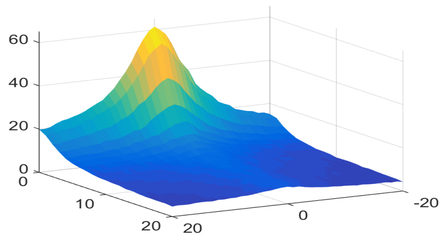
4.3 Robustness of the short side advantage

We conduct simulations to test the robustness of the advantage of the short side of the market. Theorem 2 shows that if there are strictly less men than women, and preferences are complete and uncorrelated, men will have a large advantage: men will match to one

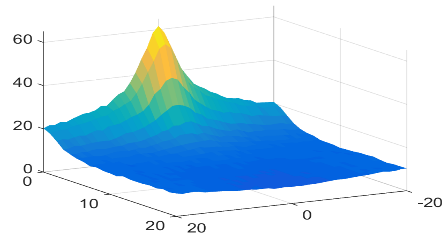
[19] See Yariv and Lee (2014) for analysis of fully aligned random preferences.

[20] We also measured the size of the core using the difference/ratio between the men's rank of wives under MOSM and under WOSM. These measures produce very similar results and are therefore omitted.

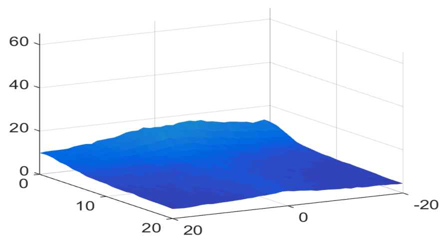
[21] It is possible to construct specific examples such that an increase in correlation increases the size of the core. For example, if there are 41 men and 40 women with i.i.d. preferences we can create a balanced market (and hence a large core) by making all women rank the same man at the bottom of their list. However, in all the simulations we ran, we did not find a natural setting, in which more correlation significantly increases the size of the core.



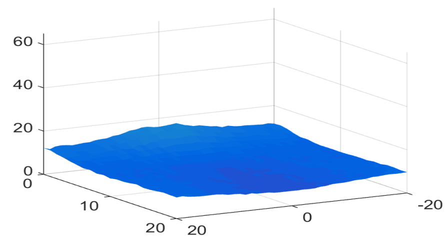
(a) Positive, 40 men



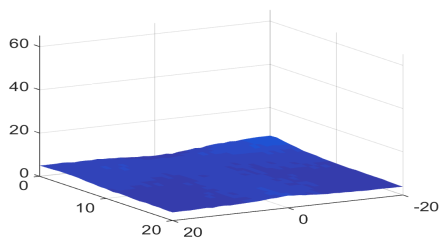
(b) Negative, 40 men



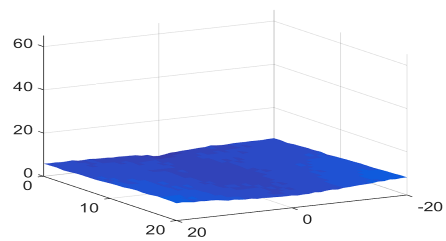
(c) Positive, 41 men



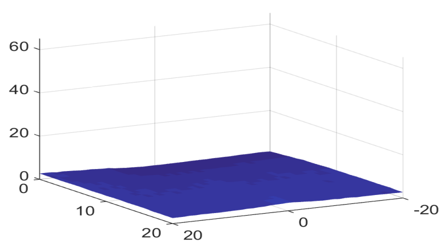
(d) Negative, 41 men



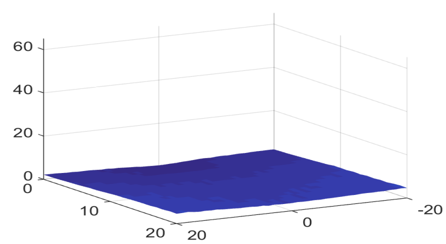
(e) Positive, 45 men



(f) Negative, 45 men



(g) Positive, 60 men



(h) Negative, 60 men

Figure 4: The percent of men with multiple stable partners for correlated preferences, generated as per Eq. (1). In each plot the z-axis (vertical and color) gives the percentage of agents with multiple stable partners, the x-axis is the value of the coefficient γ^+ (ranging from -20 to 20), and the y-axis is the value of coefficient β (ranging from 0 to 20). The number of men is 40, 41, 45 or 60, as labeled for each plot. In the plots labeled 'Positive' we use $\gamma^- = \gamma^+$, and in the plots labeled 'Negative' we use $\gamma^- = -\gamma^+$. In all markets there are 40 women.

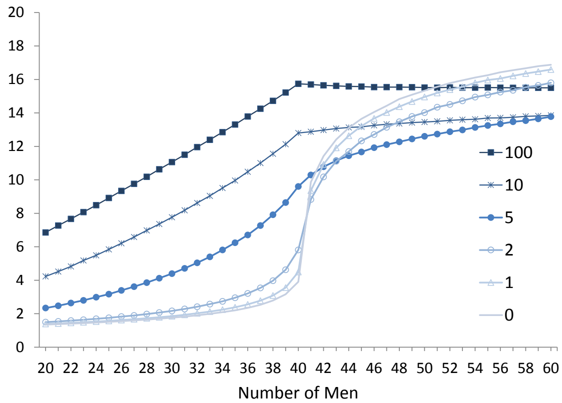
of their top choices, while women will match to a husband that is not much better than a random man. But when preferences are correlated, comparing the number of men and women may not be meaningful. For example, if all men have the same preferences, the top ranked woman must be matched to her top choice, regardless of the number of men and women. We therefore used simulations to investigate the sensitivity of the advantage of the short side to correlation in preferences.

We generate different correlation structures using the utility model from Section 4.2 and examine how the men's average rank varies with the imbalance. In every panel we give the men's average rank under different values of $\beta, \gamma^+, \gamma^-$ for markets with 40 women and from 20 to 60 men. Each panel contains as a reference point the graph for $\beta = \gamma^+ = \gamma^- = 0$ which replicates the strong advantage to the short side in Figure 3. We only plot the average rank under the MOSM, as the graph for the average rank under WOSM is almost identical (it differs only for balanced markets).

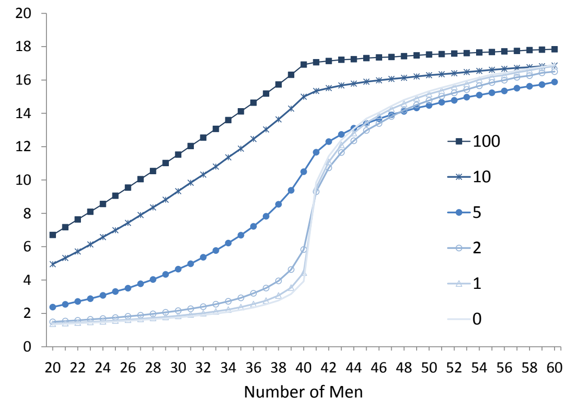
Panel 5a shows the results when taking $\gamma^+ = \gamma^- = \beta$, and each line is labeled by the common value for all coefficients. A higher value of $\gamma^+, \gamma^-, \beta$ makes preferences more correlated. The advantage of the short side gradually decreases as correlation increases, but is still evident even when there is a sizeable correlation. Panel 5b shows results with $\gamma^+ = \gamma^- = 0$ and different values of β as indicated. A higher value of β makes preferences of agents on the same side more correlated. Similarly, we find that when β is very large all men have almost the same preferences over women, and therefore men's average rank is $(\min(|\mathcal{M}|, |\mathcal{W}|) + 1)/2$.

Panel 5c shows results for varying values of $\gamma^+ = -\gamma^-$, holding $\beta = 0$. If $\gamma^+ \cdot \gamma^- < 0$ agents m, w have opposing preferences over $|x_m^D - x_w^D|^2$ as one prefers a larger distance and one prefers a smaller distance. This negative cross-side correlation increases men's average rank both when there are less than 40 men and when there are more than 40 men. Panel 5d shows results for varying values of $\gamma^+ = \gamma^-$, holding $\beta = 0$. If $\gamma^+ = \gamma^-$ both agents prefer a close partner, generating alignment between the preferences of men and women. As preferences become more correlated the men's average rank increases in markets with less than 40 men, but it decreases in markets with more than 40 men since matched men are likely to be favorably ranked by their spouses, and therefore likely to rank their spouse favorably.

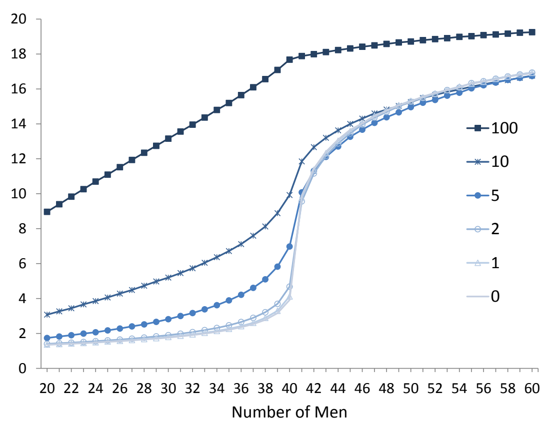
Put together, we find that for low levels of correlation, men tend to receive a lower average rank when there are fewer men. This advantage is continuously attenuated as agents' preferences become more correlated.



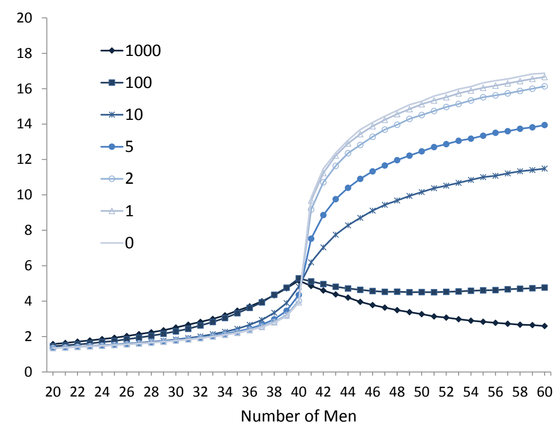
(a) Equal Coefficients



(b) Absolute Correlation



(c) Negative Correlation



(d) Positive Correlation

Figure 5: Men's average rank of wives under MOSM for correlated preferences, generated as per Eq. (1), with 40 women and varying number of men. Panel 5a plots the average rank when $\gamma^- = \gamma^+ = \beta$. Different lines correspond to different values of β ranging from 0 to 100. Similarly, panel 5b plots the average rank when $\gamma^- = \gamma^+ = 0$ and β ranging from 0 to 100. Panel 5c plots the average rank when $\beta = 0$ and $\gamma^- = -\gamma^+$ for γ^+ ranging from 0 to 100. Panel 5d plots the average rank when $\beta = 0$ and $\gamma^- = \gamma^+$ for γ^+ ranging from 0 to 1000.

We further simulate markets in which agents find some potential matches to be unacceptable. Utilities (and thus preferences) for each man m are drawn from $u_m(w) \sim U[0,1]$

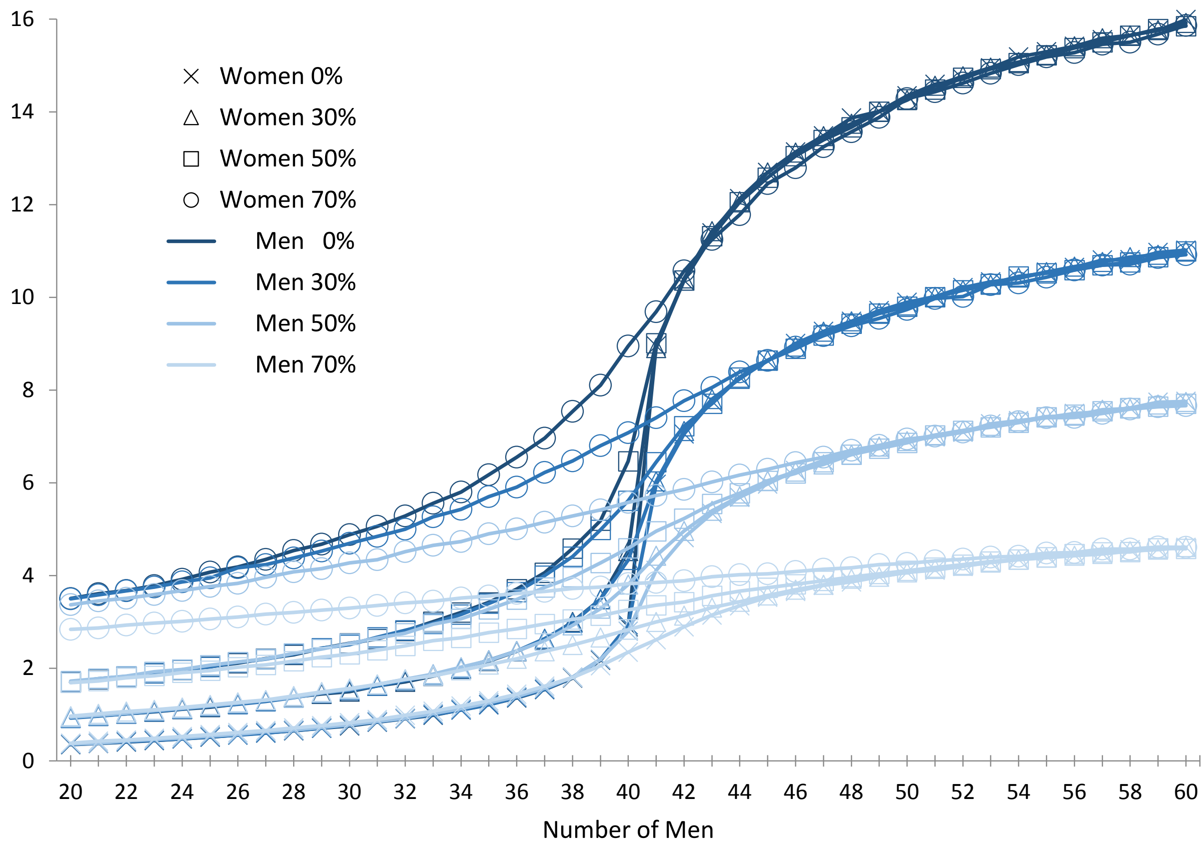


Figure 6: Men's average rank of wives for various levels of imbalance and selectivity in random markets with 40 women and different number of men. Each of the 16 lines indicates a different level of selectivity for men and women, marked by color and shape. The shape indicates the percent of men a woman finds unacceptable (on average), and the color indicates the percent of women that a man finds unacceptable (on average).

independently, setting w to be unacceptable if $u_m(w) < \bar{u}^M$. Preferences for women were drawn analogously with a threshold $\bar{u}^W$. We considered $(\bar{u}^M, \bar{u}^W) \in \{0, .3, .5, .7\}^2$, to allow either 0%, 30%, 50% or 70% of potential acceptable spouses. Figure 6 plots the men's average rank under the MOSM for markets with 40 women and from 20 to 60 men for each $(\bar{u}^M, \bar{u}^W)$.[22] The graph does not report the percent of matched agents, but we find this to be always more than 80% of $\min(|\mathcal{M}|, |\mathcal{W}|)$. The selectivity level of men $\bar{u}^M$ is indicated by

[22] We do not plot the men's average rank under WOSM as it is almost identical for all unbalanced markets.

the color of the line and the selectivity level of women $\bar{u}^W$ is indicated by the shape of the markers. Notice that only the selectivity level of agents on the long side matters, and that the benefit of the short side is still apparent and is continuously attenuated as the long side becomes more selective.

4.4 Many-to-one markets

We run computational experiments to investigate the effect of imbalance in many-to-one matching markets, in which each college has relatively few seats. In our simulations, each student has an independent uniformly random complete preference list over colleges, and each college has responsive preferences (Roth (1985)) with an independent uniformly random complete preference list over individual students. We find that unbalanced many to one markets are similar to unbalanced one to one markets, and investigate how college capacity influences the average ranks on the two sides of the market. See also Section C for a discussion about extending our theoretical results to such markets.

Denote by $\mathcal{S}$ the set of students, by $\mathcal{C}$ the set of colleges, and by q the number of seats in each college. A market is unbalanced if the number of students differs from the total number of seats, i.e., $|\mathcal{S}| \neq |\mathcal{C}| \times q$. For each market size we drew 1,000 realizations. For each realization we compute the extreme stable matchings, i.e., the student optimal stable matching (SOSM) and the college optimal stable matching (COSM). The students' average rank of their colleges (under both SOSM and COSM) is defined as before. The colleges' average rank of students is computed by averaging the rank of student in all filled seats. We report the average of each quantity as well as the percent of students with multiple stable matches.

Tables 4, 5 and 6 report simulation results for markets with an overall number of seats of $q \times |\mathcal{C}| = 100, 200, 500, 1000$, with a varying number of seats per college ($q = 2, 5$) and a varying number of students $|\mathcal{S}|$ (varying from $q|\mathcal{C}| - 10$ to $q|\mathcal{C}| + 10$). Again, we find that there is a big difference between the SOSM and COSM when the market is balanced, but in unbalanced markets the SOSM and COSM give almost the same average rank to both colleges and students. The percent of students with multiple stable matches is high when the market is balanced, but it is low under imbalance.

If we increase q while holding $q \times |\mathcal{C}|$ and $|\mathcal{S}|$ fixed, we find lower multiplicity, and higher college's average rank especially when colleges are on the short side. Note that an increase in q effectively makes preferences (over seats) more correlated, and hence it is not surprising that our findings here resemble the findings reported in Sections 4.2 and 4.3. The average rank of students is not directly comparable, since the length of the students preference list changes.

$\lvert\mathcal{C}\rvert \times q$ \ $\lvert\mathcal{S}\rvert - \lvert\mathcal{C}\rvert \times q$		-10	-5	-2	-1	0	1	2	5	10
100 (50×2)	SOSM	1.76	2.05	2.41	2.67	3.08	10.41	11.98	14.36	16.30
	COSM	1.79	2.13	2.64	3.08	10.54	12.01	12.99	14.89	16.61
200 (100×2)	SOSM	2.12	2.44	2.84	3.06	3.56	17.79	20.44	24.68	28.12
	COSM	2.16	2.53	3.08	3.49	17.9	20.61	22.27	25.62	28.71
500 (250×2)	SOSM	2.60	2.95	3.36	3.64	4.12	38.02	43.33	51.58	59.11
	COSM	2.65	3.05	3.62	4.10	38.2	43.48	46.95	53.42	60.21
1000 (500×2)	SOSM	2.98	3.34	3.76	3.99	4.54	69.09	78.01	91.75	103.75
	COSM	3.03	3.44	4.02	4.49	68.52	78.15	83.8	94.64	105.21
100 (20×5)	SOSM	1.30	1.46	1.63	1.75	1.90	4.46	5.09	6.02	6.82
	COSM	1.32	1.49	1.73	1.88	4.50	5.12	5.49	6.24	6.94
200 (40×5)	SOSM	1.50	1.66	1.84	1.96	2.14	7.47	8.51	10.19	11.60
	COSM	1.51	1.70	1.95	2.12	7.56	8.55	9.27	10.56	11.83
500 (100×5)	SOSM	1.76	1.93	1.12	2.27	2.45	15.65	17.75	20.98	23.86
	COSM	1.78	1.98	2.24	2.44	15.65	17.84	19.08	21.74	24.34
1000 (200×5)	SOSM	1.95	2.12	2.33	2.46	2.67	27.67	31.42	36.79	41.61
	COSM	1.97	2.18	2.46	2.65	27.99	31.43	33.78	38.03	42.31

Table 4: Students' average rank of colleges in the SOSM and COSM, for different markets. A student's most preferred college has rank 1, and larger rank indicates a less preferred college.

5 Discussion

Our results show that matching markets are extremely competitive. In an unbalanced matching market with preferences drawn uniformly at random, the short side essentially "chooses" while the long side is "chosen". The effect of competition is sharp; the addition of even a

$\lvert\mathcal{S}\rvert - \lvert\mathcal{C}\rvert \times q$ / $\lvert\mathcal{C}\rvert \times q$		-10	-5	-2	-1	0	1	2	5	10
100 (50×2)	SOSM	33.4	31.5	28.9	27.1	24.5	7.12	5.95	4.69	3.88
	COSM	32.9	30.6	26.8	23.9	7.2	6.01	5.42	4.50	3.79
200 (100×2)	SOSM	61.3	56.8	50.9	48.3	42.8	8.47	7.16	5.67	4.77
	COSM	60.4	55.1	47.4	42.9	8.5	7.16	6.50	5.43	4.65
500 (250×2)	SOSM	134.7	123.1	110.3	103.3	92.4	10.00	8.63	7.03	5.97
	COSM	132.6	119.1	102.9	92.7	10.1	8.61	7.89	6.76	5.85
1000 (500×2)	SOSM	242.3	220.4	199.1	188.3	168.1	11.07	9.64	8.03	7.01
	COSM	238.7	213.9	187.0	169.4	11.2	9.64	8.92	7.77	6.87
100 (20×5)	SOSM	38.1	36.9	34.8	33.4	31.5	13.36	11.29	8.97	7.46
	COSM	37.8	36.3	33.2	31.2	13.4	11.34	10.32	8.60	7.32
200 (40×5)	SOSM	72.2	68.1	63.3	60.5	56.4	16.14	13.76	11.03	9.27
	COSM	71.4	66.5	60.2	56.4	16.1	13.80	12.49	10.57	9.06
500 (100×5)	SOSM	162.6	152.1	140.8	132.7	123.7	19.38	16.77	13.85	11.87
	COSM	160.7	148.4	133.9	123.7	19.5	16.73	15.48	13.32	11.61
1000 (200×5)	SOSM	300.0	278.6	256.8	244.5	229.3	22.11	19.09	16.04	13.93
	COSM	296.7	272.3	244.7	228.4	21.9	19.17	17.66	15.48	13.68

Table 5: Colleges' average rank of students in the SOSM and COSM, for different markets. A college's most preferred student has rank 1, and larger rank indicates a less preferred student.

single extra man to a balanced market tips the market in favor of women. Furthermore, the core is small. Our results apply to actual mechanisms that are used in practice to implement stable outcomes in numerous centralized matching markets.

Competition and its consequences have been at the heart of economic theory since Edgeworth (1881), and a sharp effect of competition has been demonstrated in Bertrand models and homogeneous buyer-seller markets.[23] We find that competition has a stark effect in matching markets as well, despite the heterogeneity of preferences and the lack of monetary transfers. In fact, we find that this is true in the extreme case of fully heterogeneous preferences. To investigate the robustness of our findings, we examine numerous simulations of many markets. We find that our results on the advantage of the short side also hold in small

[23]See the famous example left-glove right-glove by Shapley and Shubik (1969).

$\lvert\mathcal{C}\rvert \times q$ \ $\lvert\mathcal{S}\rvert - \lvert\mathcal{C}\rvert \times q$	-10	-5	-2	-1	0	1	2	5	10
100 (50×2)	1.8	3.45	7.81	12.43	70.66	14.14	8.42	3.93	2.07
200 (100×2)	1.84	3.35	7.34	11.4	79.84	14.09	8.58	3.97	2.25
500 (250×2)	1.69	3.39	6.89	10.45	89.04	12.75	7.93	3.66	2
1000 (500×2)	1.58	3.03	6.2	10.24	93.26	11.75	7.03	3.16	1.96
100 (20×5)	1.19	1.95	4.97	6.8	57.54	13.83	7.91	3.92	1.75
200 (40×5)	1.28	2.6	5.19	7.12	71.41	13.13	8.5	3.88	2.14
500 (100×5)	1.21	2.56	5.02	6.84	84.06	12.48	7.22	3.7	2.1
1000 (200×5)	1.14	2.33	4.82	6.67	90.35	12.04	7.12	3.4	1.79

Table 6: Percentage of students who have multiple stable matches.

markets and are robust to a small amount of correlation in preferences. Importantly, we find the core is small in all our simulated markets (except the balanced random market), which vary in terms of list length, market size, and degree of correlation.[24]

Under more general preference distributions that allow for correlations, identifying the "short side" may require knowing more than just the number of men and women in the market. As an example, consider a tiered market with 40 women in two tiers and 30 men; half of the women are "top," while the other half are "middle." Preferences are drawn uniformly at random, except that every man prefers any top woman over any middle woman. It can be shown that every stable matching in this market can be found by first selecting a stable matching in the market that comprises the 30 men and the 20 top women, and then finding a stable matching in the market with the remaining 10 unmatched men and the 20 middle women. Our findings imply that the 20 top women will get to "choose" men, and the men are either chosen by a top woman or get to choose a middle woman. Our results can be extended by the argument above to characterize the stable matching of any (multileveled) tiered market. For any tiered market with unbalanced tiers, the core will be small, suggesting

[24]Previous papers showing a small core also implicitly rely on competition, albeit under "stronger" assumptions that roughly correspond to highly unbalanced markets in our model. While Kojima and Pathak (2009) and Immorlica and Mahdian (2005) analyze a balanced matching market, they essentially create a highly unbalanced market with women on the long side by assuming that the men can rank only a small number of women.

again that general matching markets are likely to have a small core.

Various markets such as residency matching and college admissions can be modeled as many-to-one matching markets. In Appendix C, we discuss how to extend our analysis to this setting, assuming that colleges have responsive preferences and that each college is small relative to the size of the market. We argue that the sharp effect of competition also holds in these markets: whenever there is even a slight difference between the number of seats and the number of students in the market, the core is small, and the short side will "choose".

Our results have implications for welfare in matching markets. When the imbalance is small, say n men and $n+1$ women, most agents are matched to partners they rank in their top fractiles (note that women are matched to a man at their $1 - 1/\log n$-th fractile on average). However, this interpretation is valid only asymptotically; for almost any realistic market size, one should view the average rank of $\frac{n}{\log n}$ as much worse than an average rank of $\log n$. When there is a large imbalance, say n men and $(1+\lambda)n$ women, women will not, on average, match to their top percentile even asymptotically. This paper is a first step in the analysis of welfare in matching markets; for additional insights on this topic, see subsequent work by Yariv and Lee (2014) and Che and Tercieux (2014), who build on our results.

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A Matching algorithm

The goal of this section is to present Algorithm 2 which is the basis of our analysis. This algorithm allows us to calculate the WOSM by a process of successive proposals by men. It first finds the men optimal stable matching using the men-proposing DA algorithm (Algorithm 1 or MPDA), and then progresses to the women optimal stable matching through a series of divorces of matched women followed by men proposals. At the end of this section we show how the run of the algorithm on a random matching market is equivalent to a randomized algorithm. In Appendix B we analyze the randomized algorithm to prove Theorem 1. Throughout our analysis and this section we assume that there are strictly more women than men, that is $|\mathcal{W}| > |\mathcal{M}|$.

Before presenting Algorithm 2 we first give a simplified version. The following algorithm (adapted from McVitie and Wilson (1971) and Immorlica and Mahdian (2005)) produces the WOSM from the MOSM by finding each woman's most preferred stable match. It maintains a set S of women whose most preferred stable match has been found. Denote the set of women who are unmatched under the MOSM by $\bar{\mathcal{W}}$.[25] We start with $S = \bar{\mathcal{W}}$, as by the rural hospital theorem these women are unmatched under any stable matching.

We use the notation $x \leftarrow y$ for the operation of copying the value of variable y to variable x. We do not explicitly declare variables. Instead we think of variables as being declared implicitly when they are first assigned a value in the algorithm.

Algorithm 3. MOSM to WOSM (simplified)

- Input: *A matching market with n men and $n + k$ women.*
- Initialization: *Run the men-proposing deferred acceptance to get the men-optimal stable matching μ. Set $S = \bar{\mathcal{W}}$ to be the set of women unmatched under μ. Select any $\hat{w} \in \mathcal{W} \backslash S$.*
- New phase:

  1. *Set $\tilde{\mu} \leftarrow \mu$.*
  2. Divorce: *Set $m \leftarrow \mu(\hat{w})$ and have $\hat{w}$ reject m.*
  3. Proposal: *Man m proposes to his most preferred woman w to whom he has not yet proposed.*[26]
  4. w's Decision:

     (a) *If $w \neq \hat{w}$ prefers her current match $\mu(w)$ to m, or if $w = \hat{w}$ and she prefers $\tilde{\mu}(\hat{w})$ to m, she rejects m. Go to step 3.*

     (b) *If $w \notin \{\hat{w}\} \cup \bar{\mathcal{W}}$, and w prefers m to $\mu(w)$, then w rejects her current partner and accepts m (More precisely, we use a temporary variable m' as follows: $m' \leftarrow \mu(w)$, $\mu(w) \leftarrow m$ and $m \leftarrow m'$). Go to step 3.*

     (c) New stable matching: *If $w = \hat{w}$ and $\hat{w}$ prefers m to all her previous proposals, w accepts m and a new stable matching is found. Select $\hat{w} \in \mathcal{W} \backslash S$ and start a new phase from step 1.*

[25] Since there are more women than men $\bar{\mathcal{W}} \neq \phi$.

[26] Since there are more women than men, such a woman always exists. To account for a man who prefers to be unmatched, add the "empty woman" $\phi \in \bar{\mathcal{W}}$ to represent being unmatched. A man who chooses to be unmatched over any woman he has yet to propose simply chooses $w = \phi \in \bar{\mathcal{W}}$, and the algorithm continues directly to step 4(d).

(d) End of terminal phase: *If* $w \in \bar{\mathcal{W}}$*, restore* $\mu \leftarrow \tilde{\mu}$*, erase rejections accordingly, and add* $\hat{w}$ *to* S*. Select* $\hat{w} \in \mathcal{W} \backslash S$ *if the set is not empty, otherwise terminate and output* μ*. Start a new phase from step 1.*

It will be convenient to use the following terminology. A **phase** is a sequence of proposals made by the algorithm between visits to step 1. An **improvement phase** is a phase that terminates in step 4(c) (a new stable matching is found). A **terminal phase** is a phase that terminates in step 4(d) (there is no better stable husband for $\hat{w}$). We refer to the sequence of women who reject their husbands in a phase as the **rejection chain**.

In each phase the algorithm tries to find a more preferred husband for $\hat{w}$, which requires divorcing $m = \tilde{\mu}(\hat{w})$ and assigning him to another woman. In improvement phases the algorithm finds a more preferred stable husband for $\hat{w}$. In terminal phases the algorithm finds that $\hat{w}$ cannot be assigned a man she prefers over m without creating a blocking pair, and therefore m is $\hat{w}$'s most preferred stable parter.

Proposition A.1. *Algorithm 3 outputs the women optimal stable matching.*

Proof. The algorithm terminates as each man can make only a finite number of proposals, and there is at most one terminal phase (that is rolled back) per woman. Consider a phase which begins with a stable $\tilde{\mu}$. Immorlica and Mahdian (2005) show that if the phase ends in step 4(c), the matching $\tilde{\mu}$ at the end of the phase is stable as well. By induction, every $\tilde{\mu}$ is a stable matching. Immorlica and Mahdian (2005) also show that if a phase ends in step 4(d) then $\tilde{\mu}(\hat{w})$ is $\hat{w}$'s most preferred stable man. Any subsequent matching $\tilde{\mu}$ is a stable matching in which $\hat{w}$ is weakly better off, and therefore $\tilde{\mu}$ also matches $\hat{w}$ to her most preferred stable man. Finally, any woman in $\bar{\mathcal{W}}$ is unmatched under the WOSM by the rural hospital theorem (Roth (1986)). Thus the algorithm terminates with $\tilde{\mu}$ being the WOSM which matches all women to their most preferred stable husband. □

We follow to refine the algorithm. Since proposals made during terminal phases are rolled back, we can end a phase (and roll back to $\tilde{\mu}$) as soon as we learn that the phase is a terminal phase. During the run of Algorithm 3 each woman in S is matched to her most preferred stable husband. Therefore we can terminate the phase (and roll back) if a woman in S accepts a proposal, as this can only happen in terminal phases. Furthermore, suppose that in a terminal phase the rejection chain includes each woman at most once. We show that every woman in the rejection chain is matched under $\tilde{\mu}$ to her most preferred stable husband, and can therefore be added to S. When the rejection chain includes a woman more than once there are improvement cycles in the chain. We can identify these improvement cycles and implement them as an **Internal Improvement Cycle** (IIC). Specifically, whenever a woman in the chain receives a new proposal we check if she prefers the proposing man over her best stable partner we found so far. If she prefers the proposing man, the part of the rejection chain between this proposal and her best stable partner so far forms an improvement cycle. We implement the IIC by recording the stable matching we found in $\tilde{\mu}$ and removing the cycle from the rejection chain. By removing these cycles from the rejection chain we are left a rejection chain that includes each woman at most once, and can be added to S when the phase is terminal. See Algorithm 2 step 4(c) below for a precise definition of IICs.

Applying these modifications to Algorithm 3 gives us Algorithm 2. It keeps track of the women in the current rejection chain as an ordered set $V = (v_1, v_2, .., v_J)$, and adds all of them to S if the phase is terminal. It also ends a phase as terminal when a woman in S accepts a proposal. The algorithm keeps track of $\nu(w)$ - the current number of proposals received by women w, and $R(m)$ - the set of women who rejected m so far. μ keeps track of the current proposals and $\tilde{\mu}$ records the women-optimal stable matching we found so far.

Algorithm 2. MOSM to WOSM

- Input: *A matching market with n men and $n + k$ women.*
- Initialization: *Run the men-proposing deferred acceptance to get the men-optimal stable matching μ and set $R(m)$ and $\nu(w)$ accordingly. Set $S = \bar{\mathcal{W}}$ to be the set of women unmatched under μ. Index all women in $\mathcal{W}$ according to some order. Set $\hat{w}$ to be the woman in $\mathcal{W} \backslash S$ who received maximum number of proposals so far. Set t to be the total number of proposals made so far.*
- New phase:
    1. *Set $\tilde{\mu} \leftarrow \mu$. Set $V = (v_1) = (\hat{w})$.*
    2. Divorce: *Set $m \leftarrow \mu(\hat{w})$ and have $\hat{w}$ reject m (add $\hat{w}$ to $R(m)$).*

3. Proposal: *Man m proposes to w, his most preferred woman who hasn't rejected him yet.*[27] *Increment $\nu(w)$ and proposal number t by one each.*

4. w's Decision:

(a) *If $w \notin V$ prefers $\mu(w)$ to m, or if $w \in V$ and w prefers $\tilde{\mu}(w)$ to m, then reject m (add w to $R(m)$) and return to step 3.*

(b) *If $w \notin S \cup V$ and w prefers m to $\mu(w)$ then w rejects her current partner. Do $m' \leftarrow \mu(w)$, $\mu(w) \leftarrow m$. Add w to $R(m')$, append w to the end of V. Set $m \leftarrow m'$ and return to step 3.*

(c) New stable matching: *If $w \in V$ and w prefers m to $\tilde{\mu}(w)$ then we found a stable matching. If $w = \hat{w} = v_1$ do $\mu(\hat{w}) \leftarrow m$. Select $\hat{w} \in \mathcal{W} \backslash S$ and start a new phase from step 1.*

If $w = v_\ell$ for $\ell > 1$, record her current husband $m' \leftarrow \mu(w)$. Call all proposals made since, and including, m' proposed w an internal improvement cycle *(IIC). Update $\tilde{\mu}$ for the women in the loop by setting $\tilde{\mu}(v_j) \leftarrow \mu(v_j)$ for $j = \ell + 1, ..., J$ and set $\tilde{\mu}(w) \leftarrow m$ and $\mu(w) \leftarrow m$. Remove $v_\ell, .., v_J$ from V, set the proposer $m \leftarrow m'$, decrement $\nu(w)$, decrement t, and return to step 3 in which m will apply to w.*

(d) End of terminal phase: *If $w \in S$ and w prefers m to $\mu(w)$ then restore $\mu \leftarrow \tilde{\mu}$ and add all the women in V to S. If $S = \mathcal{W}$, terminate and output $\tilde{\mu}$. Otherwise, set $\hat{w}$ to be the woman with smallest index in $\mathcal{W} \backslash S$ and begin a new phase by returning to 1.*

In step 4(c) we found a new stable matching. If the rejection chain cycles back to the original woman, we have an improvement phase. If the rejection chain cycles back to a woman w_ℓ in the middle of the chain we implement the IIC (Implementing the IIC is equivalent to an improvement phase that begins with w_ℓ). Update the best stable matching $\tilde{\mu}$ for all the women in the cycle, and make it the current assignment. Then take m' and make him propose again to w_ℓ, as we changed $\mu(w_\ell)$. Decrement t and $\nu(w_\ell)$ in order not to count this proposal twice.

Proposition A.2. *Algorithm 2 outputs the women optimal stable matching.*

[27]See footnote 26.

Proof. Consider the run of Algorithm 2 for a given sequence of selections of $\hat{w}$. We will find an sequence of selections that such that Algorithm 2 gives the same output, and on this sequence Algorithm 2 is equivalent to Algorithm 3. Since Algorithm 3 outputs the WOSM for every sequence this will prove that Algorithm 2 finds the WOSM as well. After the initialization step both algorithms are in an identical state. To produce the required sequence of selections of $\hat{w}$ we follow both algorithms in parallel, noting that Algorithm 2 skips some of the phases that Algorithm 3 performs. We start from the given sequence of selections, and modify it so that the two algorithms will hold identical $\mu, \tilde{\mu}$ at the end of every phase.

Both algorithms perform the same actions when Algorithm 3 performs steps 4(a), 4(b) and step 4(c) when $w = \hat{w}$. Assume that the algorithms are identical up to proposal $t-1$ and suppose that in proposal t Algorithm 2 performs step 4(c) with $w \neq \hat{w}$, that is, we found an IIC with $w = v_\ell$. We change the sequence of selections of $\hat{w}$ so that v_ℓ is chosen to be $\hat{w}$ just before the current phase. That makes the previous phase an improvement phase for $\hat{w} = v_\ell$ in which the cycle of the IIC is implemented. At the end of the (updated) previous phase both algorithms are identical. In the the current phase both algorithms will continue to be identical until woman v_ℓ receives a proposal, which will happen in step t. In that step the two algorithms will also be identical since $v_\ell \notin V \cup S$ at that time. Thus under the revised ordering the two algorithms are identical up to step t.

Assume that the algorithms are identical up to proposal $t-1$ and suppose that in proposal t Algorithm 2 performs step 4(d), that is, a woman $w \in S$ accepted the proposal. First we show that Algorithm 3 also finds a terminal phase, and thus at the end of the phase both algorithms revert to the same $\tilde{\mu}$. Since $w \in S$ we either have that $w \in \bar{\mathcal{W}}$ or that w was added to S in a previous phase. If $w \in \bar{\mathcal{W}}$ both algorithms will declare a terminal phase and revert to $\tilde{\mu}$. Otherwise, w was added to S in some previous phase as a part of a rejection chain that ended in a woman already in S. Recursively build a rejection chain $C = \{w_1, w_2, ..., w_q\} \subset S$ where $w_1 = w$, $w_q \in \bar{\mathcal{W}}$ such that in the terminal phase in which woman w_j was added to S woman w_j rejected her husband m_j, triggering a series of proposals by m_j that resulted in m_j making a proposal which is accepted by w_{j+1}. At the time of proposal t under Algorithm 2 we have that $\mu(w_j) = m_j$, since at the end of that terminal phase each woman in the rejection chain is reverted to her match under $\tilde{\mu}$, and this man must be the man she rejected during the phase (recall that a woman appears in a rejection chain at most once). Under the induction assumption this is true also under Algorithm 3.

Consider the continuation of the run of Algorithm 3. When $w = w_1$ rejects her husband m_1, he will make proposals in the same order as he did in the terminal phase in which w_1 was added to S under Algorithm 2. All the women which m_1 prefers over w_2 rejected him back then, and since throughout the algorithm, women can only improve to whom they are matched to, they will reject him again in the current phase. Therefore, m_1 will end up proposing to w_2. Since at that point $\mu(w_2) = m_2$ the proposal will be accepted by w_2, making m_2 the new proposer. By induction, m_j will make an accepted proposal to w_{j+1}, until we reach a proposal to $w_q \in \bar{\mathcal{W}}$. At that step Algorithm 3 declares a terminal phase and rolls back to $\tilde{\mu}$, at which point the phase ends with the same $\mu, \tilde{\mu}$ as Algorithm 2.

Next, any phase that is skipped by Algorithm 2 starts with a $\hat{w}$ that is already in S. We have shown before that when a woman in S rejects her husband the resulting rejection chain ends in S, and therefore all such phases are terminal phases and skipping them does not change μ or $\tilde{\mu}$.

To complete the proof observe that once we move a woman to be selected as $\hat{w}$ we never change the selected order before and including her. By the end of the run we therefore found a sequence of selections such that the two algorithms hold identical $\mu, \tilde{\mu}$ at the end of every phase, and that every phase that Algorithm 2 skips is a terminal phase. Thus when the Algorithm 2 terminates, it outputs the same matching as Algorithm 3, which is the WOSM by Proposition A.1. □

The following lemma shows how Algorithm 2 allows us to compare the WOSM and MOSM.

Lemma A.3. *The difference between the sum of mens' rank of wives under WOSM and the sum of mens' rank of wives under MOSM is equal to the number of proposals in improvement phases and IICs during Algorithm 2.*

Proof. Note that at the end of each terminal phase (Step 4(d)) we roll back all proposals made and return to $\tilde{\mu}$, the matching from the previous phase. Therefor, we can consider only improvement phases and IIC, in which each proposal increases the rank of the proposing man by one. □

A.1 Randomized algorithm

As we are interested in the behavior of Algorithm 2 on a random matching market, we transform the deterministic algorithm on random input into a randomized algorithm which will be easier to analyze. The randomized, or coin flipping, version of the algorithm does not receive preferences as input, but draws them through the process of the algorithm.[28] This is often called the *principle of deferred decisions*.

The algorithm reads the next woman in the preference of a man in step 3 and whether a woman prefers a man over her current proposal in step 4. Since the algorithm ends a phase immediately when a woman $w \in S$ accepts a proposal, no man applies twice to the same woman during the algorithm, and therefore the algorithm never reads previously revealed preferences. When preferences are drawn independently and uniformly at random the distribution of both can be calculated from $R(m)$, the set of women who rejected m, and $\nu(w)$, the number of proposals woman w received so far. In step 3 the randomized algorithm selects the woman w uniformly at random from $\mathcal{W}\backslash R(m)$. In step 4 the probability that w prefers m over her current match can be given directly from $\nu(w)$ for $w \notin S$ or bounded for $w \in S$.[29] Table 7 describes the probabilities for the possible decisions of w.

Step	**Event**	**Probability**
4(b)	$w \notin S \cup \{\hat{w}\}$ prefers m to $\mu(w)$	$\frac{1}{\nu(w)+1}$
4(c)	$\hat{w}$ prefers m to $\tilde{\mu}(\hat{w})$	$\frac{1}{\nu(\hat{w})+1}$
4(d)	$w \in S\backslash\bar{\mathcal{W}}$ prefers m to $\mu(w)$	at least $\frac{1}{\nu(w)+1}$

Table 7: Probabilities in a run of Algorithm 2 on a random matching market.

Note that the event Step 4(a) is the complement of the union of the events in Table 7.

[28] The initialization step of the randomized version of Algorithm 2 calls the randomized version of Algorithm 1.

[29] The probability that a woman $w \in S$ accepts a man m can be calculated from the number of proposals she received during improvement phases or MPDA and the number of proposals she received during terminal phases. Since the bound on the acceptance probability we calculate from $\nu(w)$ is sufficient for our analysis we omit the additional counters from the algorithm.

B Proof

We will prove the following quantitative version of the main theorem:

Theorem 3. *Fix any $\epsilon > 0$. Consider a sequence of random matching markets, indexed by n, with n men and $n+k$ women, for arbitrary $1 \le k = k(n)$. There exists $n_0 < \infty$ such that for all $n > n_0$, with probability at least $1 - \exp\big\{-(\log n)^{0.4}\big\}$, we have*

(i) In every stable matching μ:

$$R_{\text{MEN}}(\mu) \le (1+\epsilon)\big((n+k)/n\big)\log\big((n+k)/k\big)$$

$$R_{\text{WOMEN}}(\mu) \ge n\big/\big[1 + (1+\epsilon)\big((n+k)/n\big)\log\big((n+k)/k\big)\big]\,.$$

(ii) Less than $n/(\log n)^{0.5}$ men, and less than $n/(\log n)^{0.5}$ women have multiple stable partners.

(iii) The men are almost as well off under the WOSM as under the MOSM:

$$\frac{R_{\text{MEN}}(\text{WOSM})}{R_{\text{MEN}}(\text{MOSM})} \le 1 + (\log n)^{-0.4}\,.$$

(iv) The women are almost as badly off under the WOSM as under the MOSM:

$$\frac{R_{\text{WOMEN}}(\text{WOSM})}{R_{\text{WOMEN}}(\text{MOSM})} \ge 1 - (\log n)^{-0.4}\,.$$

Remark 1. *In our proof of Theorem 3 (ii), we actually bound the number of different stable partners. We show that the sum over all men (or women) of the number of different stable partners of each man is no more than $n + n/\sqrt{\log n}$. Thus, in addition to the bound stated in Theorem 3 (ii), we rule out the possibility that there are a few agents who have a large number of different stable partners.*

Definition B.1. *Given a sequence of events $\{\mathcal{E}_n\}$, we say that this sequence occurs* with very high probability (wvhp) *if*

$$\lim_{n\to\infty} \frac{1 - \mathbb{P}(\mathcal{E}_n)}{\exp\big\{-(\log n)^{0.4}\big\}} = 0\,.$$

Clearly, it suffices to show that (i)-(iv) in Theorem 3 hold wvhp.

To prove Theorem 3, we analyze the number of proposals in Algorithm 1 followed by Algorithm 2, which will provide us the average rank of wives in the women optimal stable match. We partition the run of Algorithm 2 leading to the WOSM into three parts (Parts II through IV below).

1. **Part I is the run of DA (Algorithm 1)**, which by an analysis similar to that in (Pittel (1989a)), takes no more than $3n\log(n/k)$ proposals wvhp.

2. **Part II are the proposals in Algorithm 2 that take place before the end of first terminal phase.** We show that wvhp,

   - Part II takes no more than $(n/k)(\log n)^{0.45} \leq n(\log(n/k))^{0.45}$ proposals.
   - When part II ends the set S contains at least $n^{(1-\varepsilon)/2}$ elements.

3. **Part III are the proposals in Algorithm 2 after Part II that take place until $|S| \geq n^{0.7}$. Thus, this part ends at the end of a terminal phase when $|S|$ exceeds $n^{0.7}$ for the first time.** We show that, wvhp, part III requires $O(n^{0.47})$ phases, and $o(n)$ proposals.[30]

4. **Finally, Part IV includes the remaining proposals from the end of part III until Algorithm 2 terminates or $50n\log n$ total proposals have occurred (including proposals made in Parts I and II), whichever occurs earlier.** Because the set S is large, most phases are terminal phases containing no IICs, and most acceptances lead to eventual inclusion in S. We show that, wvhp, part IV ends with termination of the algorithm, and that the number of proposals in improvement phases and IICs is $o(n)$. But the increase in sum of men's rank of wives from the MOSM to the WOSM is exactly the number of proposals in improvement phases and IICs, yielding the result.

The definition of Part III based on when $|S|$ exceeds $n^{0.7}$ is for technical reasons, with the exponent of 0.7 being a choice for which our analysis goes through. Throughout this section,

[30]For any two functions $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}_{>0}$ we write $f = o(g)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ and $f = O(g)$ if there exist a constant a such that $f(n) \leq ag(n)$ for sufficiently large n.

we consider the preferences on both sides of the market as being revealed sequentially as the algorithm proceeds, as discussed in Appendix A.1.

Lemma B.2. *Consider a man m, who is proposing at step 3 of Algorithm 2. Consider a subset of women $\mathcal{A} \subseteq \mathcal{W}\backslash R(m)$. Let $\nu(\mathcal{A}) = \frac{1}{|\mathcal{A}|}\sum_{\tilde{w}\in\mathcal{A}} \nu(\tilde{w})$ be the average number of proposals received by women in $\mathcal{A}$. The man m proposes to some woman w in the current step. Conditional on $w \in \mathcal{A}$ and all preferences revealed so far, the probability that m is the most preferred man who proposed w so far, is at least $\frac{1}{\nu(\mathcal{A})+1}$.*

Proof. For any woman $\tilde{w} \notin R(m)$ the probability that m is the most preferred man who applied to w so far is $\frac{1}{\nu(\tilde{w})+1}$. Conditional on $w \in \mathcal{A}$, the probability that m is the most preferred man who applied to w so far is at least

$$\frac{1}{|\mathcal{A}|}\sum_{\tilde{w}\in\mathcal{A}} \frac{1}{\nu(\tilde{w})+1} \geq \frac{1}{\nu(\mathcal{A})+1}$$

by Jensen's inequality. □

The following lemma will be convenient and its proof is trivial:

Lemma B.3. *If all men have lists of length $|\mathcal{W}|$ and $|\mathcal{W}| > |\mathcal{M}|$, then no man ever reaches the end of his list in Algorithm 1 or Algorithm 2.*

B.1 Part I

For the analysis in this section we consider the following equivalent version of deferred acceptance:

Algorithm 4. *Index the men $\mathcal{M}$. Initialize $S_{\mathcal{M}} = \phi$, $\bar{\mathcal{W}} = \mathcal{W}$, $\nu(w) = 0$ $\forall w \in \mathcal{W}$ and $R(m) = \phi$ $\forall m \in \mathcal{M}$.*

1. *If $\mathcal{M}\backslash S_{\mathcal{M}}$ is empty then terminate. Else, let m be the man with the smallest index in $\mathcal{M}\backslash S_{\mathcal{M}}$. Add m to $S_{\mathcal{M}}$.*

2. *Man m proposes to his most preferred woman w whom he has not yet applied to (increment $\nu(w)$). If he is at the end of his list, go to Step 1.*

3. *Decision of w:*

- *If $w \in \bar{\mathcal{W}}$, i.e, w is unmatched then she accepts m. Remove w from $\bar{\mathcal{W}}$. Go to Step 1.*
- *If w is currently matched, she accepts the better of her current match and m and rejects the other. Set m to be the rejected man, add w to $R(m)$ and continue at Step 2.*

Note that the output of Algorithm 4 is the same as the output of Algorithm 1, i.e., it is the man optimal stable match (we have just reordered the proposals). The output of Algorithm 4 is given as an input to Algorithm 2. Again, we think of preferences as being revealed as the algorithm proceeds, with the women only revealing preferences among the set of men who have proposed them so far.

The next lemma establishes upper bounds on the average and maximum rank of men over women and a lower bound on the women's average rank over men. The upper bound for the worst possible rank of men is due to Pittel (1989a) who obtained this bound in a balanced market (by adding more women to the market men are only becoming better off).

Lemma B.4. *Fix any $\epsilon > 0$. Let μ be the men-optimal stable matching. The following hold wvhp:*

(i) the men's average rank of wives in μ is at most $(1+\epsilon)\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)$ and is at least $(1-\epsilon)\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)$,

(ii) $\max_{m\in\mathcal{M}} \mathrm{Rank}_m(\mu(m)) \le 3(\log n)^2$,

(iii) the women's average rank of husbands in μ is at least $n\big/\left[1+(1+\epsilon)\left(\frac{n+k}{n}\right)\log\left(\frac{n+k}{k}\right)\right]$.

Proof. We first prove the upper bound in (i), then (ii), then the lower bound in (i), and finally (iii).

Tracking Algorithm 4 like in Pittel (1989a), we claim that, wvhp, the sum of men's rank of wives is at most $(1+\epsilon)(n+k)\log((n+k)/k)$ for small enough $\epsilon > 0$. This claim immediately implies the stated bound (i) on men's average rank of wives. To prove the claim, we use the fact that the number of proposals is stochastically dominated by the number of draws in the coupon collector's problem, when n distinct coupons must be drawn from $n+k$ coupons. This latter quantity is a sum of Geometric$((n+k-i+1)/(n+k))$ random variables for

$i = 1, 2, \ldots, n$. The mean is

$$\sum_{i=1}^{n} \frac{n+k}{n+k-i+1} = (n+k)\Big(\frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{n+k}\Big)$$
$$= (n+k)\Big(\log((n+k)/k) + O(1/k)\Big)$$
$$= (n+k)\log((n+k)/k)\Big(1 + O(1/(k\log((n+k)/k)))\Big)\,.$$

A short analytical exercise[31] shows that $1/(k\log((n+k)/k))$ is monotone decreasing in k, and is thus maximized at $k = 1$. It follows that $1/(k\log((n+k)/k)) \le 1/\log(n+1) \le 1/(\log n)$, which establishes that the error term $O(1/(k\log((n+k)/k))) = O(1/\log n)$ vanishes in the limit. Now routine arguments (e.g., Durrett (2010)) can be used to show that, in fact, this sum exceeds $(1+\epsilon)(n+k)\log((n+k)/k)$ with probability $\exp(-\Theta(n))$. This establishes the upper bound in (i).

Intuitively, the upper bound in (ii) should hold since it holds for a balanced market (see Pittel (1989a)), and adding more women should presumably only make the bound tighter. We show that this is indeed the case. Our proof works as follows: we first show that for any man m, the probability that $\mathrm{Rank}_m(\mu(m)) > 3(\log n)^2$ is bounded above by $1/n^{1.2}$. We then use a union bound over the men to establish (ii).

Fix a man m and consider Algorithm 4, where one additional man is processed at a time. From McVitie and Wilson (1971), we know that the final outcome is the MOSM, and this does not depend on the order in which the men are processed. Therefore, we can assume that m is processed last.

Let $t = 1, 2, \ldots$ be the index of the proposals. From the upper bound in (i) proved above, we know that with probability $1 - \exp(-\Theta(n))$, the MOSM is found before

$$t = T_* = (1+\epsilon)(n+k)\log((n+k)/k)$$
$$\le (1+\epsilon)(n\log(1+n/k) + n\lim_{k\to\infty}(k/n)\log((n+k)/k))$$
$$\le n(1+\epsilon)(\log(1+n/k)+1)$$
$$\le 1.1n\log n\,,$$

[31] Define $f(k) = k\log((n+k)/k)$, where we think of n as fixed, and $k \in (0,\infty)$ as varying. We obtain $f'(k) = \log(1+n/k) - 1 + 1/(1+n/k)$. It suffices to show that $f'(k) > 0$ for all $k > 0$. To show this, we define $g : (0,\infty) \to \mathbb{R}$ as $g(x) = \log(1+x) - 1 + 1/(1+x)$. Now $\lim_{x\to 0} g(x) = 0$ and $g'(x) = x/(1+x)^2 > 0$ for all $x > 0$, leading to $g(x) > 0$ for all $x > 0$. Hence, $f'(k) > 0$ for all $k > 0$, as required.

for small enough ϵ and large enough n. Let $\hat{\mathcal{E}}$ be the event that this bound holds. Then we know that

$$\mathbb{P}(\hat{\mathcal{E}}^c) \leq \exp(-\Theta(n)) \, .$$

We track[32] Algorithm 4 until it terminates or index t exceeds T_*. If the index t exceeds T_*, i.e., we observe $\hat{\mathcal{E}}^c$, we declare failure and stop. Hence, we can use $t \leq T_*$ in what follows. Consider the i-th proposal by man m, and suppose $i \leq 3(\log n)^2$. Then there are at least $n + k - 3(\log n)^2$ women that m has not yet proposed to, and these women have together received no more than T_* proposals in total so far. Using Lemma B.2, the probability of the proposal being accepted is at least

$$1/(T_*/(n + k - 3(\log n)^2) + 1) \geq 1/(1.1 n \log n/(n - 3(\log n)^2) + 1) \geq 1/(1.2 \log n)$$

for large enough n. If the proposal is accepted by a woman $\hat{w}$, each subsequent proposal in the chain is, independently, at least as likely to go to an unmatched woman as it is to go to $\hat{w}$. Hence, the subsequent rejection chain has a probability at least $1/2$ of terminating in an unmatched woman before there is another proposal to woman $\hat{w}$. As such, the probability that the i-th proposal will be the last proposal made by m is at least $(1/2) \cdot 1/(1.2 \log n) = 1/(2.4 \log n)$. It follows that the probability that m has to make more than $3(\log n)^2$ proposals before the MOSM is reached or failure occurs is no more than

$$\Big(1 - \frac{1}{2.4 \log n}\Big)^{3(\log n)^2} \leq \Big(\exp\Big\{-\frac{1}{2.4 \log n}\Big\}\Big)^{3(\log n)^2} \leq \exp(-1.25 \log n) = 1/n^{1.25} \, .$$

Combined with the probability of failure, using a union bound, the overall probability that man m makes more than $3(\log n)^2$ proposals is bounded above by $1/n^{1.25} + \mathbb{P}(\hat{\mathcal{E}}^c) \leq 1/n^{1.25} + \exp(-\Theta(n)) \leq 1/n^{1.2}$, for large enough n. Since the same bound applies to any man, we can use a union bound to find that

$$\mathbb{P}(\text{Any man makes more than } 3(\log n)^2 \text{ proposals}) \leq n \cdot 1/n^{1.2} = 1/n^{0.2} \, .$$

We conclude that wvhp, no man makes more than $3(\log n)^2$ proposals, establishing (ii).

[32]As usual, we reveal information about each preference list as it is needed (cf. Appendix A.1): for men we reveal the next entry in the preference list just before a new proposal; for women, we reveal whether a new proposal is the best one so far, only when the proposal is made.

We now establish the lower bound in (i), i.e., that the sum of men's rank of wives is at least $(1-\epsilon)(n+k)\log((n+k)/k)$. The proof is similar to that of the upper bound in (i). From (ii), we have that wvhp, no man makes more than $3(\log n)^2$ proposals. It follows that for each proposal that occurs during the search for the i-th unmatched woman, the probability that an unmatched woman is found is at most

$$p_i = \frac{n+k-i+1}{n+k-\min(3(\log n)^2, i-1)}\,.$$

It follows that the number of proposals needed to find the i-th woman stochastically dominates Geometric(p_i), conditional on what has happened. It follows that the mean total number of proposals is at least

$$\begin{aligned}\sum_{i=1}^{n} \frac{n+k-3(\log n)^2}{n+k-i+1} &= \big(n+k-3(\log n)^2\big)\Big(\frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{n+k}\Big) \\ &= (n+k)\log((n+k)/k)(1+O(1/(\log n))\,,\end{aligned}$$

where we bound the error term as above and using $3(\log n)^2/(n+k) = O((\log n)^2/n) = O(1/(\log n))$. Again, routine arguments (e.g., Durrett (2010)) can be used to show that, in fact, a sum of independent Geometric(p_i) random variables for $i = 1, 2, \ldots, n$ is less than $(1-\epsilon)(n+k)\log((n+k)/k)$ with probability $\exp(-\Theta(n))$. This establishes the lower bound on men's average rank of wives.

Now consider the women's ranks of husbands. For a woman w, who has received $\nu(w)$ proposals in Part I, the rank of her husband is a random variable that depends only on $\nu(w)$, and not anything else revealed so far. We have

$$\mathbb{E}[\mathrm{Rank}_w(\mu(w))] = \frac{n+1}{\nu(w)+1}\,.$$

Define

$$M \equiv \mathbb{E}\left[\sum_{w\in\mathcal{W}\setminus\bar{\mathcal{W}}} \mathrm{Rank}_w(\mu(w))\right] = (n+1)\sum_{w\in\mathcal{W}\setminus\bar{\mathcal{W}}} \frac{1}{\nu(w)+1}\,.$$

Using Azuma's inequality (see Durrett (2010)), we have

$$\mathbb{P}\Big((1/n)\sum_{w\in\mathcal{W}\setminus\bar{\mathcal{W}}} \mathrm{Rank}_w(\mu(w)) \le \frac{M}{n} - \Delta\Big) \le \exp\left\{-\frac{n\Delta^2}{2n^2}\right\}\,,$$

since $\text{Rank}_w(\mu(w)) \in [0, n]$. Plugging in $\Delta = n^{3/4}$ yields

$$\mathbb{P}\Big((1/n) \sum_{w \in \mathcal{W} \setminus \bar{\mathcal{W}}} \text{Rank}_w(\mu(w)) \leq \frac{M}{n} - n^{3/4}\Big) \leq \exp\left\{-\frac{n^{1/2}}{2}\right\} . \tag{2}$$

Using Jensen's inequality in the definition of M, we have

$$\begin{aligned} M &\geq (n+1)n \cdot \frac{1}{1 + (1/n)\sum_{w \in \mathcal{W}\setminus\bar{\mathcal{W}}} \nu(w)} \\ &\geq \frac{n^2}{1 + (1+\epsilon/2)\left(\frac{n+k}{n}\right) \log\left(\frac{n+k}{k}\right)} \quad \text{wvhp} , \end{aligned}$$

where we used (i) with ϵ replaced by $\epsilon/2$, i.e., $(1/n)\sum_{w \in \mathcal{W}\setminus\bar{\mathcal{W}}} \nu(w) \leq (1+\epsilon/2)\left(\frac{n+k}{n}\right) \log\left(\frac{n+k}{k}\right)$ wvhp. Using $\left(\frac{n+k}{n}\right) \log\left(\frac{n+k}{k}\right) \leq 1.1 \log n$ (see the bound on T_* in the proof of (ii)), i.e., the $n^{3/4}$ is negligible in comparison to M, we can deduce that

$$M - n^{3/4} \geq \frac{n^2}{1 + (1+\epsilon)\left(\frac{n+k}{n}\right) \log\left(\frac{n+k}{k}\right)} \quad \text{wvhp} , \tag{3}$$

for large enough n (notice that we used the $\epsilon/2$ slack).

Combining Eqs. (2) and (3), we obtain (iii). □

Lemma B.5. *Suppose $k \leq n^{0.1}$. Then, wvhp, there are fewer than $n^{0.99}$ women who each receive less than $(1/2)\log n$ proposals.*

Proof. Now consider a woman w'. For each proposal, it goes to w' with probability at least $1/(n+k)$, unless the proposing man has already proposed w'. Suppose w' receives fewer than $(\log n)/2 \leq (2/3)(1-2\epsilon)\log(n+k)/k$ proposals, where, for instance, we can define $\epsilon = 0.01$. Since, wvhp, each man makes at most $3(\log n)^2$ proposals (Lemma B.4(ii)), wvhp there are at most $(3/2)(\log n)^3$ proposals by men who have already proposed w'. Using Lemma B.4(i), wvhp, there are at least $(1-\epsilon)(n+k)\log((n+k)/k)$ proposals in total. It follows that, wvhp, there are at least $(1-\epsilon)(n+k)\log((n+k)/k) - (3/2)(\log n)^3 \geq (1-2\epsilon)(n+k)\log((n+k)/k)$ proposals by men who have not yet proposed w'. Using Fact E.1 (i), the probability that fewer than $(2/3)(1-2\epsilon)\log((n+k)/k)$ of these proposals go to w' is

$$\begin{aligned} &\mathbb{P}\Big(\text{Binomial}((1-2\epsilon)(n+k)\log((n+k)/k), 1/(n+k)) < (2/3)(1-2\epsilon)\log(n+k)/k)\Big) \\ &\leq 2\exp\{-(1-2\epsilon)\log((n+k)/k)/27\} \leq ((n+k)/k)^{-1/28} \leq n^{-0.02} , \end{aligned}$$

for $k \leq n^{0.1}$. It follows that the expected number of women who receive fewer than $(1/2)\log n$ proposals is no more than $(n+k)n^{-0.02} \leq 2n^{0.98}$. By Markov's inequality, the number of women who receive fewer than $(1/2)\log n$ proposals is no more than $n^{0.99}$, with probability at least $1 - 2n^{0.98}/n^{0.99} = 1 - o(\exp\{-(\log n)^{0.4}\})$. □

B.2 Part II

Lemma B.7 below shows that by the end of Part II, the number of proposals by each man is "small", the number of proposals received by each woman is "small", and that the set S is "large". Since S will be large at the end of this part, Part III will terminate "quickly". The next lemma says that wvhp, there will not be too many proposals in Part II (this bound will be assumed in the proof of Lemma B.7).

Lemma B.6. *Part II completes in no more than* $\frac{n+k}{k}(\log n)^{0.45} \leq (n+1)(\log n)^{0.45}$ *proposals wvhp.*

Proof. For each proposal (Step 3) in Part II, the probability of Step 4(d), which will end Part II, is the probability that the man m proposes to an unmatched woman $\frac{k}{|\mathcal{W}\setminus R(m)|} \geq \frac{k}{n+k}$. Therefore the probability that the number of proposals in part II exceeds $((n+k)/k)(\log n)^{0.45}$ is at most $\left(1 - \frac{k}{n+k}\right)^{((n+k)/k)(\log n)^{0.45}} \leq \exp(-(\log n)^{0.45}) = o(\exp\{-(\log n)^{0.4}\})$, leading to the first bound. Noticing that $(n+k)/k = 1 + n/k \geq 1 + n$, we obtain the bound of $(n+1)(\log n)^{0.45}$. □

Lemma B.7. *Fix any* $\varepsilon > 0$. *At the end of Part II, the following hold wvhp:*

(i) No man has applied to a lot of women:

$$\max_{m \in \mathcal{M}} |R(m)| < n^{\varepsilon}. \tag{4}$$

(ii) The set S *is large:* $|S| \geq n^{(1-\varepsilon)/2}$.

(iii) No woman received many proposals

$$\max_{w \in \mathcal{W}} \nu(w) < n^{\varepsilon}. \tag{5}$$

Proof. Using Lemma B.4, we know that wvhp, there have been no more than $3n\log(n/k)$ proposals and no man has proposed more than $3(\log n)^2$ women in Part I. Assume that these two conditions hold for the rest of the proof.

We begin with (i). We say that a man m starts a **run of proposals** when m is rejected by a woman at step 4(b) or is divorced from $\hat{w}$ at step 2. We say that a *failure* occurs if a man starts more than $(\log n)^2$ runs or if the length of any run exceeds $(\log n)^3$ proposals. We associate a failure with a particular proposal t, when for the first time, a man starts his $(\log n)^2+1$-th run, or the proposal is the $(\log n)^3+1$-th proposal in the current run.

Consider the number of runs of a given man m. Man m starts at most one run at step 2. The other runs start when the proposing man $m' \neq m$ proposes to the women m is currently matched to and m' is accepted. At any proposal the probability that m' proposes to any particular woman is no more than the probability that he proposes to $\bar{\mathcal{W}}$. Now if the latter happens, Part II ends. Therefore, it follows that the number of runs man m has in part II is stochastically dominated[33] by $1+\text{Geometric}(1/2)$. Hence, the probability that a man has more than $(\log n)^2$ runs is bounded by $\left(\frac{1}{2}\right)^{(\log n)^2-1} \leq 1/n^2$, showing that man m has fewer than $(\log n)^2$ runs in Part II with probability at least $1-1/n^2$. It follows from a union bound over all men $m \in \mathcal{M}$ that wvhp failure due to number of runs does not occur.

Assume failure did not occur before or at the beginning of a run of man m. The number of proposals man m accumulates until either the run ends or a failure occurs is bounded by

$$(\log n)^2 \cdot (\log n)^3 \leq n/2$$

for sufficiently large n. In each proposal in the run before failure, man m proposes to a uniformly random woman in $\mathcal{W}\backslash R(m)$. Since there were at most $4n\log n$ proposals so far, we have that

$$\nu\left(\mathcal{W}\backslash R(m)\right) \leq \frac{4n\log n}{n/2} = 8\log n\,.$$

From Lemma B.2, we have that the probability of acceptance at each proposal is at least $\frac{1}{\nu(\mathcal{W}\backslash R(m))+1} \geq \frac{1}{8\log n+1}$. Therefore the probability of man m making $(\log n)^3$ proposals with-

[33] A real values random variable with cumulative distribution F_1 is said to be stochastically dominated by another r.v. with cumulative distribution F_2 if $F_2(x) \leq F_1(x)$ for all $x \in \mathbb{R}$.

out being accepted is bounded by[34]

$$\left(\frac{1}{8\log n+1}\right)^{(\log n)^3} \le \frac{1}{n^3}\,.$$

Thus, the run has length no more than $(\log n)^3$ with probability at least $1-1/n^3$. Now the number of runs is bounded by n^2, so we conclude that wvhp failure due to number of runs does not occur. Finally, assuming no failure,

$$|R(m)| \le (\log n)^2 \cdot (\log n)^3 < n^{\varepsilon}$$

establishing (i).

We now prove (ii). If $k \ge n^{(1-\varepsilon)/2}$, the set S is already large enough at the beginning of Part II, and there is nothing to prove. Suppose $k < n^{(1-\varepsilon)/2}$. Consider the evolution of $|V|$ during Part II. We first provide some intuition. Part II contains about $n/k = \omega(n^{1/2})$ proposals before it ends. We start with $|V| = 0$, and $|V|$ initially builds up without any new stable matches found. We can estimate the size of $|V|$ when Step 4(c) (new stable match) occurs as follows[35]: Suppose we reach $|V| \sim N$. Consider the next accepted proposal. Ignoring factors of $\log n$, the probability that the woman who accepts is in V is $\sim |V|/n \sim N/n$. Thus, for one of the next N accepted proposals to include a woman in V, we need $N \cdot N/n \sim 1$, i.e., $N \sim \sqrt{n}$. Thus, when $|V|$ reaches a size of about $\sqrt{n}$, then an IIC forms over the next $\sim \sqrt{n}$ proposals, reducing the size of $|V|$. This occurs repeatedly, with $|V|$ converging to an 'equilibrium' distribution with mean of order $\sqrt{n}$, and this distribution has a light tail. Thus, when the phase ends, we expect $|V| \sim \sqrt{n}$.

We now formalize this intuition. Whenever $|V| < n^{(1-\varepsilon)/2}$, for the next proposal, the probability that:

- The proposal goes to a woman in V is less than $2/n^{(1+\varepsilon)/2}$. Such a proposal is necessary to creating an IIC.
- The proposal goes to a woman in $S = \bar{\mathcal{W}}$, is at most $2k/(n+k) \le 2k/n \le 2/(n^{(1+\varepsilon)/2})$. Such a proposal would terminate the phase.

[34] Again this inequality holds for large enough n. We omit the explicit mention of the condition "for large enough n" when such inequalities appear subsequently in this section.

[35] This analysis is analogous to that of the birthday paradox.

- The proposal goes to a woman in $\mathcal{W}\backslash(S \cup V)$, who accepts it, is at least $1/(5 \log n)$, using the fact that there have been no more than $4n \log n$ proposals so far and Lemma B.2.

Suppose we start with any $|V| < n^{(1-\varepsilon)/2}$, for instance we have $|V| = 0$ at the start of the phase, we claim that with probability at least $1 - 3k/\sqrt{n}$, we reach $|V| = n^{(1-\varepsilon)/2}$ (call this an 'escape') before there is a proposal to S and before $\sqrt{n}$ proposals occur. We prove this claim as follows: There is a proposal to S among the next $\sqrt{n}$ proposals with probability no more than $2k/\sqrt{n}$. Suppose that $|V|$ stays less than $n^{(1-\varepsilon)/2}$. Then there are $n^{\varepsilon/4}$ or more proposals to women in $|V|$ among $\sqrt{n}$ total proposals with probability no more than $2^{-n^{\varepsilon/4}}$ using Fact E.1 (ii) on Binomial$(\sqrt{\mathrm{n}}, 2/\mathrm{n}^{(1+\varepsilon)/2})$. Also, the probability that there are less than $n^{1/2-\varepsilon/8}$ proposals accepted by women in $\mathcal{W}\backslash(V \cup S)$ (these women are added to V) is at most $2^{-n^{1/2-\varepsilon/8}}$, using Fact E.1 (ii) on Binomial$(\sqrt{\mathrm{n}}, 1/(5 \log \mathrm{n}))$, since at each proposal, such a woman is added with probability at least $(1/(5 \log n))$. But if there are

- less than $n^{\varepsilon/4}$ proposals to V, each such proposal reducing $|V|$ by at most $n^{1/2-\varepsilon/2}$,
- no proposal to S, and
- at least $n^{1/2-\varepsilon/8}$ women added to V,

then we must reach $|V| = n^{(1-\varepsilon)/2}$. Thus, the overall probability of *not* reaching $|V| = n^{(1-\varepsilon)/2}$ before there is a proposal to S and before $\sqrt{n}$ proposals occur is at most $2k/\sqrt{n} + 2^{-n^{\varepsilon/2}} + 2^{-n^{1/2-\varepsilon/4}} \leq 3k/\sqrt{n}$. In particular, the probability of a failed escape is at most $3k/\sqrt{n}$.

We now bound the number of times $|V|$ reduces from a value larger than $n^{(1-\varepsilon)/2}$ to a value smaller than $n^{(1-\varepsilon)/2}$. Suppose $|V| \geq n^{(1-\varepsilon)/2}$. The probability that a proposal goes to S is at least $k/(n+k) \geq k/(2n)$. The probability that a proposal goes to one of the first $n^{(1-\varepsilon)/2}$ women in $|V|$ is at most $2n^{(1-\varepsilon)/2}/n \leq 2/n^{(1+\varepsilon)/2}$. Thus, the number of times the latter occurs is stochastically dominated by Geometric$(k/(4n^{(1-\varepsilon)/2})) - 1$. Thus, the total number of escapes needed to ensure $|V| \geq n^{(1-\varepsilon)/2}$, including the one at the start of the phase, is stochastically dominated by Geometric$(k/(4n^{(1-\varepsilon)/2}))$, which exceeds $n^{1/2-\varepsilon/4}/k$ with probability at most

$$\left(1 - k/(4n^{(1-\varepsilon)/2})\right)^{n^{1/2-\varepsilon/4}/k} \leq \exp(-n^{\varepsilon/4}/4)\,.$$

Assuming no more than $n^{1/2-\varepsilon/4}/k$ escapes are needed, one of these escapes fails with probability at most $(n^{1/2-\varepsilon/4}/k)\cdot(3k/\sqrt{n}) = 3n^{-\varepsilon/4}$. Thus, the overall probability of $|V| < n^{(1-\varepsilon)/2}$ when the phase ends is bounded by $\exp(-n^{\varepsilon/4}/4)+n^{-\varepsilon/4} = o(\exp\{-(\log n)^{0.4}\})$. Thus, wvhp, $|V| \geq n^{(1-\varepsilon)/2}$ for all phases in Part II, including the terminal phase. This establishes (ii).

Finally, we establish (iii). Again we assume in our proof that Parts I and II end in no more than $4n\log n$ proposals in total, and that (i) holds (if not, we abandon our attempt to establish (iii), but this does not happen wvhp). Fix a woman w. For each proposal, the probability that *she* receives the proposal is no more than $2/n$, using (i). Thus, the total number of proposals she receives is no more than Binomial(2n log n, 2/n) which is less than n^{ε}, except with probability $2^{-n^{\varepsilon}/(4\log n)}$ by Chernoff bound (see Fact E.1 in Appendix E). Union bound over the women gives us that (iii) holds wvhp. □

Lemma B.8. *Wvhp, the number of accepted proposals in Part II is no more than* $n/(2\sqrt{\log n})$ *and the improvement in sum of women's ranks of husbands during Part II is no more than* $n^2/(2(\log n)^{3/2})$.

Proof. If $k \geq n^{0.1}$, then we already know that wvhp the number of proposals is no more than $n^{0.95}$ using Lemma B.6.

If $k < n^{0.1}$, then we know from Lemma B.5, that, wvhp, fewer than $n^{0.99}$ women each received fewer than $\log n/2$ proposals in Part I. Further, from Lemma B.7, wvhp, no man has proposed to more than n^{ε} women in Parts I and II. It follows that for each proposal in Part II, it goes to a woman who has already received $\log n/2$ or more proposals with probability at least $1-n^{-0.01}/2$. Hence, the probability that the proposal is accepted is at most $2.5/\log n$. But the total number of proposals in Part II, wvhp, is less than $(n+1)(\log n)^{0.45}$ from Lemma B.6. It follows using Fact E.1 that, wvhp, fewer than $3(n+1)/(\log n)^{0.55} \leq n/(10\sqrt{\log n})$ proposals are accepted in Part II.

We now bound the improvement in the sum of women's ranks of husbands. Using Markov's inequality, there are, wvhp, at most $n^{0.995}$ proposals to women who have received fewer than $\log n/2$ proposals so far. The maximum possible improvement in rank from these proposals is $(n+k)n^{0.995} \leq 2n^{1.995}$. The number of proposals accepted by women who have received at least $\log n/2$ proposals so far is, wvhp, at most $n/(10\sqrt{\log n})$, as we showed above. For such a proposal accepted by a woman w' who has received $\nu(w') \geq (\log n)/2$ previous proposals, the expected improvement in rank is

$$\frac{n-\nu(w')}{\nu(w')+1} - \frac{n-\nu(w')-1}{\nu(w')+2} \le n/((\log n)/2)^2 = 4n/\log n\,.$$

Further the improvement in rank is in the interval $[1, n-1]$. Thus, the total improvement in rank is stochastically dominated by a sum of independent X_i, for $i = 1, 2, \ldots, n/(10\sqrt{\log n})$, with $\mathbb{E}[X_i] \le 4n/\log n$ and $1 \le X_i \le n-1$. It follows using Azuma's inequality (see, e.g., Durrett (2010)) that this sum exceeds $n/(10\sqrt{\log n}) \cdot 4n/\log n + n^{1.6} \le n^2/(2.1(\log n)^{3/2})$ with probability at most

$$2\exp\left\{-\frac{(n^{1.6})^2}{2\cdot n/(10\sqrt{\log n})\cdot n^2}\right\} = \exp\{-n^{0.1}\} = o(\exp\{-(\log n)^{0.4}\})\,.$$

Thus, the total improvement in the sum of women's ranks of husbands is, wvhp, no more than $2n^{1.995} + n^2/(2.1(\log n)^{3/2}) \le n^2/(2(\log n)^{3/2})$. □

B.3 Part III

Let S_{II} be the set S at the end of part II.

The next lemma provides upper bounds (that are achieved wvhp) on the number of proposals each man makes and the number of proposals each woman receives throughout Parts III and IV.

Let $\mathcal{E}_t$ be the event that until proposal t, no man has applied to more than $n^{0.6}$ women in total or to more than $n^{3\varepsilon}$ women in S_{II}, and no woman has received $n^{2\varepsilon}$ or more proposals. Let $\mathcal{E}_\infty$ be the event that these same conditions hold when Part IV ends.

Lemma B.9. *The event $\mathcal{E}_\infty$ occurs wvhp.*

Proof. By Lemma B.7, we know that at the end of Part II, no man has made more than n^{ε} proposals, that $|S_{\mathrm{II}}| \ge n^{(1-\varepsilon)/2}$, and that no woman has received more than n^{ε} proposals, wvhp. We assume that all these conditions hold.

Fix a man m. We argue that if m makes a successful proposal to a woman in $S_{\mathrm{II}} \cup \backslash \bar{\mathcal{W}}$, then he makes no further proposals in Algorithm 2: If m makes a successful proposal to a woman in S, this ends the phase making the phase a terminal one, man m goes back to the woman to whom he was matched at the beginning of the phase, and this woman becomes a member of S. Thus, if a m makes a successful proposal to a woman in S, he makes no further proposals. In particular, if m makes a successful proposal to a woman in $S_{\rm II}\backslash\bar{\mathcal{W}}$, then he makes no further proposals.

Suppose man m is proposing in proposal t and that $\mathcal{E}_t$ holds. Then m has not yet applied to at least $3n^{(1-\varepsilon)/2}/4$ women in $S_{\rm II}$. Hence the probability of applying to a woman in $S_{\rm II}$ is at least $n^{(1-\varepsilon)/2}/2$. Further, since no woman has received $n^{2\varepsilon}$ or more proposals, the probability of the proposal being accepted is at least $1/n^{2\varepsilon}$. Hence, the probability of the proposal going to a woman in $S_{\rm II}$ and being accepted is at least $n^{-3\varepsilon-1/2}$. Hence, the man makes fewer than $n^{0.6}/2$ proposals in Part IV, and proposes to fewer than $n^{3\varepsilon}/2$ additional women in $S_{\rm II}$, except with probability $\exp(-n^{\varepsilon}/2)$. Using a union bound over the men, wvhp, no man has applied to more than $n^{0.6}$ women in total or to more than $n^{3\varepsilon}$ women in $S_{\rm II}$ until the end of Part IV.

Fix a woman w. Each time a proposal occurs, since no man has proposed to more than $n^{0.6}$ women (assuming $\mathcal{E}_t$ holds), the probability of the proposal going to w is less than $2/n$. Since there are at most $50n\log n$ proposals in total, the number of proposals received by w in Part III is more than n^{ε} with probability less than $\mathbb{P}(\text{Binomial}(50n\log n, 2/n) \geq n^{\varepsilon}) \leq 2^{-n^{\varepsilon}}$, using Chernoff bounds (see Fact E.1(ii) in Appendix E). Using a union bound over the women, wvhp, no woman has received more than n^{ε} proposals until the end of Part IV.

The result follows combining the analyses in the two paragraphs above. □

We now focus on Part III. We show (Lemma B.10) that for every phase in Part III, whp:

- the phase is a terminal phase, and
- that $|V|$ at the end of the phase is at least $n^{0.25}$.

For each such phase, $|S|$ increases by at least $n^{0.25}$. In addition, we show that phases are short, with the expected length of a phase being $O(n^{1/2+3\varepsilon})$. We infer that, wvhp, we reach $|S| \geq n^{0.7}$, i.e., the end of Part III, in $o(n^{0.47})$ phases, containing $o(n)$ proposals. Lemma B.11 below formalizes this.

Lemma B.10. *Assume $|S_{\rm II}| \geq n^{(1-\varepsilon)/2}$, cf. Lemma B.7. Consider a phase during Part III. Suppose $\mathbb{I}(\mathcal{E}_t) = 1$ at the start of the phase. Then, whp, either $\mathbb{I}(\mathcal{E}_{t'}) = 0$ at the end of the phase, or we have:*

- *The phase is a terminal phase.*

- *At the end of the phase is at least* $|V| \geq n^{0.25}$.

Proof. Assume $\mathbb{I}(\mathcal{E}_\tau) = 1$ throughout the phase (otherwise there is nothing to prove). Since we are considering a phase during Part III, we know that $|S| < n^{0.7}$. Also, $|S| \geq |S_{\mathrm{II}}| \geq n^{(1-\varepsilon)/2}$ by assumption. For each proposal, there is a probability of at least $|S|n^{-2\varepsilon}/(2n) \geq n^{-1/2-3\varepsilon}$ and at most $2|S|/n \leq 2n^{-0.3}$, that the proposal is to a woman in S and is accepted. It follows that, whp, the phase is a terminal phase, and the number of proposals in the phase is in $[n^{0.28}, n^{0.52}]$. It is easy to see that with probability at least $(1 - 2n^{0.28}/n)^{n^{0.28}} = 1 - o(1)$, all of the first $n^{0.28}$ proposals in the phase are to distinct women, meaning that there are no IICs. For each proposal, the probability of acceptance is at least $1/(1 + n^{2\varepsilon})$, since no woman has received $n^{2\varepsilon}$ proposals, so whp, there are at least $n^{0.25}$ accepted proposals among the first $n^{0.28}$ proposals, using Fact E.1 (ii) on $\mathrm{Binomial}(\mathrm{n}^{0.26}, 1/(1 + \mathrm{n}^{2\varepsilon}))$. Now, consider the first $n^{0.25}$ women in V. These women receive no further proposals during the phase with a probability at least $(1 - 2n^{0.25}/n)^{n^{0.52}} = 1 - o(1)$. Hence, whp, these women are part of V at the end of the phase, establishing $|V| \geq n^{0.25}$ at the end of the phase as needed. □

Lemma B.11. *Wvhp, Part III contains less than* $n^{0.99}$ *proposals.*

Proof. We first show that the next $n^{0.47}$ phases after the end of Part II complete in fewer than $n^{0.99}$ proposals. Since, wvhp, $|S_{\mathrm{II}}| \geq n^{(1-\varepsilon)/2}$, if $\mathbb{I}(\mathcal{E}_t) = 1$ then for proposal t the probability of ending the phase (due to acceptance by a woman in S) is at least $n^{-1/2-3\varepsilon}$. It follows that either $\mathbb{I}(\mathcal{E}_\infty) = 0$ or wvhp, the next $n^{0.47}$ phases after the end of Part II complete in no more than $n^{0.47+1/2+4\varepsilon} \leq n^{0.99}$ proposals, using Fact E.1 (ii) on $\mathbb{P}(\mathrm{Binomial}(\mathrm{n}^{0.97+4\varepsilon}, \mathrm{n}^{-1/2-3\varepsilon}) \geq \mathrm{n}^{0.47})$.

Now we show that wvhp, Part III contains fewer than $n^{0.47}$ phases. Suppose this is not the case, then, by our definition of Part III, at most $n^{0.45}$ of these phases increase $|S|$ by $n^{0.25}$ or more. But using Lemma B.10, either $\mathbb{I}(\mathcal{E}_\infty) = 0$, or this occurs with probability at most

$$\mathbb{P}\big(\mathrm{Binomial}(\mathrm{n}^{0.47}, 1 - \varepsilon) \leq \mathrm{n}^{0.45}\big) \leq \mathbb{P}\big(\mathrm{Binomial}(\mathrm{n}^{0.47}, 1/2) \leq \mathrm{n}^{0.47}/4\big) \leq 2\exp(-\mathrm{n}^{0.47}/24)\,,$$

using Fact E.1 (i). In other words, either $\mathbb{I}(\mathcal{E}_\infty) = 0$ or, wvhp, Part III contains fewer than $n^{0.47}$ phases.

But Lemma B.9 tells us that $\mathbb{I}(\mathcal{E}_\infty) = 1$ wvhp. Combining the above, we deduce that wvhp, Part III contains fewer than $n^{0.47}$ phases and fewer than $n^{0.99}$ proposals. □

B.4 Part IV

Lemma B.12. *Suppose we are at Step 3 (time t) of Algorithm 2 during Part IV, we have $\mathbb{I}(\mathcal{E}_t) = 1$, and man m is proposing. Then, for large enough n, the probability that:*

(i) Man m proposes to S and is accepted is at least $n^{-0.31}$.

(ii) Man m proposes to $\mathcal{W}\backslash(R(m) \cup \hat{w})$ and is accepted is at least $0.9n/t$.

Proof. Note that $|S| \geq n^{0.7}$, whereas by definition of $\mathcal{E}_t$ the man has proposed to no more than $n^{0.6}$ women and no woman has received more than $n^{2\varepsilon}$ proposals. (i) follows from Lemma B.2.

Proof of (ii): Since m has not applied to more than $n^{0.6}$ women so far, we know that $|\mathcal{W}\backslash(R(m) \cup \hat{w})| \geq 0.95n$. Also, the total number of proposals so far is $t-1$. Using Lemma B.2, the probability of applying to $\mathcal{W}\backslash(R(m) \cup \hat{w})$ and being accepted is at least

$$\frac{1}{1+(t-1)/(0.95n)} \geq \frac{0.9n}{t},$$

for large enough n, using $t > n$ since Part I itself requires at least n proposals. □

Lemma B.13. *Wvhp, Part IV ends due to termination of the algorithm.*

Proof. Suppose Part IV does not end with termination (if not we are done) and that $\mathcal{E}_\infty$ occurs (Lemma B.9 guarantees this wvhp). Reveal each proposal sequentially.

For $t \leq 40n \log n$, call proposal t a 'seemingly-good' proposal when acceptance by $w' \in \mathcal{W}\backslash(R(m) \cup \hat{w})$ occurs. Denote the set of seemingly good proposals by $\mathcal{A}$. We use Lemma B.12. For each proposal t, there is a probability at least $0.9n/t$ of it being a seemingly-good proposal, conditioned on the history so far. Define independent $X_t \sim \text{Bernoulli}(0.9n/t)$ for $t = t_0, t_0+1 \ldots, 40n \log n$, where t_0 is the first proposal in Part III. Then we can set up a coupling so that proposal $t \in \mathcal{A}$ whenever $X_t = 1$. Now

$$\sum_{t=4n\log n}^{40n\log n} 1/t \geq (0.99)\ln(10) \geq 2.27\,,$$

$$\Rightarrow \qquad \sum_{t=4n\log n}^{40n\log n} \mathbb{E}[X_t] \qquad \geq 2n$$

Using Fact E.1, we deduce that

$$\sum_{t=4n\log n}^{40n\log n} X_t \geq 7n/4$$

wvhp, implying

$$|\mathcal{A}| \geq \sum_{t=t_0}^{40n\log n} X_t \geq 7n/4 \tag{6}$$

wvhp, since we know that $t_0 \leq 4n\log n$ wvhp using Lemmas B.4 and B.6.

We call a seemingly-good proposal $t \leq 40n\log n$ a 'good' proposal if the following conditions are satisfied:

- During the current phase, there is no proposal to a woman in V. In particular, there are no IICs.
- The phase is a terminal phase that ends during Part IV.

We denote the set of good proposals by $\mathcal{G} \subseteq \mathcal{A}$. We now argue that

$$|S_{\mathrm{II}}| \geq |\mathcal{G}|\,, \tag{7}$$

where S_{II} is the set S at the end of Part IV. If $w' \notin (S \cup \bar{w})$, then w' becomes part of S at the end of the phase if the proposal is a good proposal. For each terminal phase, there is exactly one good proposal to a woman in $S \cup \bar{w}$, which we think of as accounting for $\hat{w}$, which also becomes a part of S. Thus, for every good proposal, one woman joins S during Part IV, establishing Eq. (7).

Now consider any phase in Part IV that starts before the $40n\log n$-th proposal. Call such a phase an *early* phase. Using Lemma B.12, the phase contains more than $n^{0.32}$ proposals with probability at most $(1-n^{-0.31})^{n^{0.32}} \leq \exp(-n^{0.01}) \leq 1/n^2$. But the total number of early phases is no more than $40n\log n$. It follows that using a union bound that, wvhp, there is no early phase that contains more $n^{0.32}$ proposals.

Now, the probability of a phase containing fewer than $n^{0.32}$ proposals, and containing a proposal to a woman in V is at most $n^{0.32} \cdot 2n^{0.32}/n \leq n^{-0.35}$, since $|V| \leq n^{0.32}$ throughout such a phase. Further, there are at most $40n\log n \leq n^{1.01}$ early phases. It follows, using Fact E.1 (ii) on $\mathrm{Binomial}(\mathrm{n}^{1.01}, \mathrm{n}^{-0.35})$, that the number of early phases containing a proposal to a woman in V is, wvhp, no more than $n^{0.67}$. It follows that, wvhp, no more than $n^{0.67} \cdot n^{0.32} =$

$n^{0.99}$ proposals occur in early phases containing a proposal to V. But all proposals in $\mathcal{A}\backslash\mathcal{G}$ must occur in such phases. We deduce that, wvhp,

$$|\mathcal{A}\backslash\mathcal{G}| \leq n^{0.99}\,. \tag{8}$$

Combining Eqs. (6) and (8), we deduce that $|\mathcal{G}| \geq 3n/2 \geq |\mathcal{W}|$ wvhp. Plugging in Eq. (7), we obtain $|S_{\mathrm{II}}| \geq |\mathcal{W}|$ at the end of Part IV wvhp, which we interpret[36] as "With wvhp, our assumption that Part IV does not end with termination was incorrect. In other words, Part IV ends with termination wvhp". □

Lemma B.14. *The number of proposals in improvement phases and in IICs in Part IV is no more than $n^{0.99}$ wvhp.*

Proof. In the proof of Lemma B.13, we in fact showed that whp in Part IV, the number of proposals in phases that include a proposal to a woman in V is no more than $n^{0.99}$. (Actually, we showed this bound for 'early' phases, and also showed that wvhp, the algorithm terminates with an early phase, so that all phases in Part IV are early phases). But improvement phases and phases containing IICs must include a proposal to a woman in V. The result follows. □

We now establish two claims that follow from elementary calculus.

Claim B.15. *For large enough n and any $k \geq 1$ we have $k\log(1+n/k) \geq (\log n)/2$.*

Proof. We divide possible values of k into three ranges, and establish the bound for each range.

First, suppose $k \leq 10\log n$. Then $\log(1+n/(10\log n)) \geq \log\sqrt{n} \geq \log n/2$ for large enough n. It follows that $k\log(1+n/k) \geq \log n/2$ since $k \geq 1$.

Next, suppose $k \in (10\log n, 10n]$. Now $\log(1+n/k) \geq \log(1+n/(10n)) = \log 1.1 \geq 0.09$. The bound follows by multiplying with $k \geq 10\log n$.

Finally, consider $k > 10n$. Now, $n/k \leq 1/10$ leading to $\log(1+n/k) \geq n/k - (n/k)^2/2 \geq 0.95n/k$. It follows that $k\log(1+n/k) \geq 0.95n \geq \log n/2$ for large enough n. □

Claim B.16. *For any $n \geq 1$ and any $k \geq 1$, we have $(1+1/n)\log(1+n) \geq (1+k/n)\log(1+n/k) \geq 1$.*

[36] Recall our initial assumption that Part IV does not end with termination. Finding that $|S| \geq n$ under this assumption simply means that Part IV did, in fact, end with termination.

Proof. Consider $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = (1 + 1/x) \log(1 + x)$. Then

$$f'(x) = \frac{x - \log(1 + x)}{x^2} > 0 \quad \forall x > 0\,,$$

using $\log(1 + x) < x$ for all $x > 0$. It follows that for any $x > 0$, we have

$$f(x) \geq \lim_{x \to 0} f(x) = 1,$$

using $\lim_{x\to 0} \log(1+x)/x = 1$. The lower bound in the claim follows by plugging in $x = n/k$. The upper bound follows by plugging in $k = 1$, since this maximizes n/k for fixed n. □

Finally, using these lemmas and claims we give the proof of Theorem 3.

Proof of Theorem 3. Using Lemma A.3, we can calculate the sum of men's rank of wives by summing up the rank under the MOSM and the number of proposals made during improvement phases and IICs during the run of Algorithm 2. By Lemma B.6, B.11 and Lemma B.14, the total number of proposals that occur in improvement phases and IICs (in Parts II-IV) of Algorithm 2 is, wvhp, no more than $(1 + n/k)(\log(n))^{0.45} + 2n^{0.99}$. Using Lemma A.3, we get that

$$\begin{aligned}
&\text{Sum of men's ranks of wives(WOSM)} - \text{Sum of men's ranks of wives(MOSM)} \\
&\leq (1 + n/k)(\log(n))^{0.45} + 2n^{0.99} \qquad (9)
\end{aligned}$$

But

$$\begin{aligned}
&\text{Sum of men's ranks of wives(MOSM)} \\
&\geq 0.99(n + k) \log((n + k)/k) \\
&\geq \max(0.49(1 + n/k) \log n, 0.99n) \qquad (10)
\end{aligned}$$

wvhp, from Lemma B.4 (i), along with Claims B.15 and B.16. We deduce from Eqs. (9) and (10) that

$$\frac{R_{\text{MEN}}(\text{WOSM}) - R_{\text{MEN}}(\text{MOSM})}{R_{\text{MEN}}(\text{MOSM})} \leq (\log n)^{-0.55}/0.49 + 2n^{-0.01}/0.99 \leq (\log n)^{-0.4}$$

wvhp, immediately implying Theorem 3 (iii).

The only agents whose partner changes in going from the MOSM to the WOSM are the ones who make or receive accepted proposals during improvement phases and IICs. But this number on each side of the market is, wvhp, no more than $n/(2\sqrt{\log n})$ in Part II (Lemma B.8), and no more than $n^{0.99}$ each in Part III (Lemma B.11) and Part IV (Lemma B.14, leading to a bound of $n/(2\sqrt{\log n}) + 2n^{0.99} \leq n/\sqrt{\log n}$ on the total number of agents with multiple stable partners on each side of the market, establishing Theorem 3 (ii).

By Lemma B.8, wvhp, the improvement in the sum of women's ranks of husbands in Part II is at most $n^2/(2(\log n)^{3/2})$. In Parts III and IV, wvhp there are at most $2n^{0.99}$ women who obtain better husbands (since each such woman must have received a proposal during an improvement phase or IIC; see above), and the rank improves by less than n for each of these women, so the improvement in sum of ranks is less than $2n^{1.99}$. It follows that the total improvement in sum of ranks is, wvhp, less than $n^2/(\log n)^{3/2}$. On the other hand, using Lemma B.4 (iii) and the upper bound in Claim B.16, we obtain that wvhp

$$\text{Sum of women's ranks of husbands(MOSM)} \geq n^2/(2\log n)\,.$$

Theorem 3 (iv) follows.

Lemma B.4 (i) and (iii) with $\epsilon' = \epsilon/2$, combined with Theorem 3 (iii) and (iv) (established above) yields Theorem 3 (i).

□

C Many-to-one matching markets

This section discusses the extension of our results to many-to-one matching markets, in which colleges are matched to more than one student. We consider many-to-one markets where colleges have a small capacity relative to the size of the market, each student has an independent, uniformly random complete preference list over colleges, and each college has responsive preferences (Roth (1985)) and an independent, uniformly random complete preference list over individual students. In Section 4.4, we present computational experiments demonstrating that imbalance in such markets again leads to a small core and allows the short side to approximately "choose." We follow to describe how our theoretical results can be extended.

Assume that each college has a constant number of seats q. Students are on the short side of the market if there are fewer students than seats, and, symmetrically, colleges are the short side of the market if there are more students than seats. We denote the extreme stable matchings by SOSM (the student-optimal stable matching) and COSM (the college-optimal stable matching). The students' average rank of their colleges is defined as before. We define the colleges' rank of students to be the average rank of students assigned to the college.

We argue that the bounds on average rank stated in Theorem 2 will deteriorate by a factor that depends on q,[37] whereas Theorem 1 will hold as stated. Thus, with high probability, the core will be small and the short side will "choose".

Our proof can be extended as follows. First, using results from Roth and Sotomayor (1989), one can decompose the many-to-one market into a one-to-one market as follows. For each seat in a college, create an "agent" that ranks students according to that colleges preferences. Students rank all seats according to their preferences for the corresponding schools, and all students rank seats within a college in the same order (i.e., there is a "top" seat and a "bottom" seat in each college). A matching is stable in the original market if and only if there is a corresponding stable matching in the decomposed market. With this decomposition, all our results from Appendix A extend to the many-to-one case, allowing us to use our algorithms to calculate the extreme stable matchings via a sequence of proposals by agents on the short side.

Next, the stochastic analysis can be extended to the many-to-one case as follows. Suppose there are n colleges each with q seats each. First consider the case in which there are fewer than qn students. The first part of the analysis is student-proposing DA, and we can bound the number of proposals in this stage by considering q repetitions of the coupon collector's problem. The next steps in our proof follow with slight modifications, using the fact that rejection chains have the same structure as in the one-to-one case. Whenever a seat rejects a student, the student matched to the bottom seat in the college gets rejected, and that bottom student in turn applies to a randomly drawn college that ranks the student uniformly at random. A college will accept the student if the applying student is ranked higher than the q-th best student currently at the college and will reject that q-th best student if it accepts the applying student. A phase (chain), initiated by a college c rejecting a student, can terminate either with an application to a school that has not filled its seats, or with a successful application to college c. Therefore, a phase consists of a series of proposals to random colleges that in case of acceptance, always reject their lowest ranked student. The

[37] That is, there is a similar upper bound on the average rank of partners for the short side of the market, and there is a similar lower bound on the average rank for the long side of the market.

main difference is that in order to calculate the acceptance probability, we need to calculate the probability that a (randomly ranked) proposing student is better than the q-th best proposal the college received (instead of being better than the best proposal the woman received in the one-to-one case). Bounds on this probability will be affected by at most a constant factor.

Similar arguments allow us to extend our proof to the case when there are more students than seats. The first part of the analysis is college-proposing DA, and we can again bound the number of proposals in this stage by considering the coupon collector's problem. The rest of our analysis, controlling rejection chains, is almost unchanged. Whenever a student in a college rejects a seat, the seat accepts the student matched to the next lower seat at the college and so on, until the bottom seat in the college is rejected. This seat proposes to a randomly drawn student who ranks the college uniformly at random and accepts if she ranks the college higher than her current match. A phase (chain) initiated by a student s rejecting a seat can either terminate with an offer to an unmatched student, or with a successful offer to student s. Overall, the analysis in this case will be almost identical to our original proof.

We note that these results do not imply that colleges cannot gain from manipulation in unbalanced matching markets, as a college can potentially manipulate even if there is a unique stable matching. Kojima and Pathak (2009) show that a college can manipulate only if a rejection of one of the students assigned to that college will trigger an rejection chain that cycles back to the college. We therefore conjecture that when the imbalance is larger than a college's capacity, even colleges have a limited scope for manipulation, but this conjecture does not directly follow from our analysis, which only establishes that the core is small.

D Average rank estimates

Table 1 in Section 4 includes values of the following function for each unbalanced market.

$$\mathrm{EST} = \mathrm{EST}(|\mathcal{M}|, |\mathcal{W}|) = \begin{cases} \frac{|\mathcal{W}|}{|\mathcal{M}|} \log\left(\frac{|\mathcal{W}|}{|\mathcal{W}|-|\mathcal{M}|}\right) & \text{for } |\mathcal{W}| > |\mathcal{M}| \\ |\mathcal{W}| \Big/ \left(1 + \frac{|\mathcal{M}|}{|\mathcal{W}|} \log\left(\frac{|\mathcal{M}|}{|\mathcal{M}|-|\mathcal{W}|}\right)\right) & \text{for } |\mathcal{W}| < |\mathcal{M}| \end{cases} \tag{11}$$

The definition of EST is based on Theorem 2, as justified by the following facts.

Remark 2. *Fix any* $\epsilon > 0$.

- *For the case $|\mathcal{W}| > |\mathcal{M}|$, Theorem 2 and its proof imply that, with high probability (asymptotically in $|\mathcal{M}|$), $R_{\text{MEN}}/\text{EST} \in (1-\epsilon, 1+\epsilon)$ under all stable matches (including the MOSM and WOSM). Note that the upper bound on R_{MEN} is part of the statement of Theorem 2, whereas the lower bound follows from the proof (though it is not part of the statement). Hence, we think of* EST *as a heuristic estimate for R_{MEN} in finite markets with $|\mathcal{W}| > |\mathcal{M}|$.*

- *For the case $|\mathcal{M}| > |\mathcal{W}|$, Theorem 2 implies that, with high probability (asymptotically in $|\mathcal{W}|$), $R_{\text{MEN}}/\text{EST} \geq 1-\epsilon$ under all stable matches (including the MOSM and WOSM). Hence, we think of* EST *as a heuristic lower bound for R_{MEN} in finite markets with $|\mathcal{W}| < |\mathcal{M}|$.*

E Chernoff bounds

Fact E.1 Chernoff bounds (see Durrett (2010))**.** *Let $X_i \in \{0,1\}$ be independent with $\mathbb{P}[X_i = 1] = \theta_i$ for $1 \leq i \leq n$. Let $X = \sum_{i=1}^n X_i$ and $\lambda = \sum_{i=1}^n \theta_i$.*

(i) Fix any $\delta \in (0,1)$. Then

$$\mathbb{P}(|X-\lambda| \geq \lambda\delta) \leq 2\exp\{-\delta^2\lambda/3\}. \tag{12}$$

(ii) For any $R \geq 6\lambda$, we have

$$\mathbb{P}(X \geq R) \leq 2^{-R}. \tag{13}$$