Two-Terminal Routing Games with Unknown Active Players *

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Abstract. We analyze 2-terminal routing games with linear cost functions and with unknown number of active players. We deal with both splittable and unsplittable models. We prove the existence and uniqueness of a symmetric safety-level equilibrium in such games and show that in many cases every player benefits from the common ignorance about the number of players. Furthermore, we prove new theorems on existence and uniqueness of equilibrium in 2-terminal convex routing games with complete information.

1 Introduction

The study of congestion games [24, 20, 17] is central to game theory, transportation engineering, multi-agent systems, CS/AI, electronic commerce, and communication networks. Indeed, the study of congestion games has become a central ingredient in work connecting the above disciplines (see e.g. the following papers and their lists of references: [13, 15, 9, 11, 23, 14, 21, 12, 3, 22, 4, 6]). Most of the related studies assume complete information about the set (and in particular the number) of participants in the system.[3] However, in many settings, although the set of registered/potential participants may be known, the actual set of active participants is unknown. Hence, incorporating uncertainty about the set of actual participants into congestion settings is a desirable task.

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[3] Some recent works deal with incomplete information about other parameters in the Bayesian setting [7, 8].

A routing game is defined by a congestion network and a set of players. A congestion network consists of a directed graph, a vector of edge cost functions, and a set of source-target pairs. Each player is associated with a source-target pair. Every player has to move one unit of good from her source to her target. In a splittable routing game, the player can split her unit amongst the paths that connect the source to the target, and in an unsplittable routing game the players cannot split their units. The edge cost functions determine the cost of every user of the edge as a function of the number of users. Routing games are special type of congestion games, which are defined by congestion forms.[4] In this paper we focus on routing games, that are determined by two-terminal congestion networks. That is, there is a single source-target pair, which is associated with all users.[5] In this paper, when dealing with splittable routing games we assume that the edge cost functions are increasing, continuously differentiable, and convex. When dealing with unsplittable routing games we assume only that they are non-decreasing. In a routing game with complete information every player knows the network structure, the cost functions, and the number of users. In a routing game with incomplete information discussed in this paper, every active player knows all of the above except for the number of active players; She does know the number of potential players.[6] We assume there does not exists a commonly known prior probability over the possible number of active players. Hence we suggest to model behavior of the players in these game by the concept of safety-level equilibrium, which was recently defined for games with incomplete information in [1]. A safety-level equilibrium is a strategy profile in which each agent minimizes her worst case cost over all possible states of the environment, assuming the other agents stick to their

[4] A congestion form is a particular structure consisting of a set of resources together with a class of subsets of this set. Every congestion form, and a finite set of users uniquely define a congestion game. However, distinct congestion forms can generate identical congestion games. A particular type of form is the one generated by a congestion network.

[5] It is not known wether every congestion game is a routing game, but it can be derived from [16] that every symmetric congestion game is a 2-terminal routing game.

[6] In our model there is a finite number of agents. The initial research of congestion games [24], as well as much of the recent research [23, 21] discuss congestion games with continuum of agents, which are called non-atomic congestion games.

prescribed strategies. In the context of routing games discussed in this paper, the possible states of the environment correspond to the possible sets of active players. Let $c(k)$ be the cost of each player in equilibrium in the complete information case when there are k players, and $c(k,n)$ be the cost of each player in a safety-level equilibrium in the related game with incomplete information when there are k active players and n potential players. We define the value of ignorance to be $\nu(k,n) = c(k) - c(k,n)$. If this value is non-negative, ignorance is beneficial (in the weak sense) for the players.[7] In order for the above index of the value of ignorance to make sense the cost in equilibrium at each of the above settings should be uniquely defined. Therefore, parts of this paper are devoted to proving existence and uniqueness of equilibrium results. While analysis of the value of ignorance is performed only for congestion networks with linear cost functions, our existence and uniqueness theorem are proved for more general cost functions.

Our results concerning the value of ignorance in models with linear cost functions are as follows: Fix the number, $k \geq 1$, of active players. We show the following structure of the function $\nu(k,n)$ for $k \leq n < \infty$:

In symmetric splittable routing games: It is non-negative and non-increasing at the interval $[k, 2k-1]$, and it is non-increasing at the interval $[2k-1, \infty)$. More refined structure is proved for the subclass of

splittable parallel routing games: if k is sufficiently large, it is proved that $\nu(k,n) \geq 0$ for $k < n \leq k(k+1)+1$, and it vanishes at $n = k(k+1)-1$. Consequently, for $n > k(k+1)-1$, the value of ignorance is not positive. That is, knowledge is a desirable good. Finally, we prove for

unsplittable parallel routing games: For sufficiently large k, $\nu(k,n) \geq 0$ for every $n > k$, and it is maximized over $n \in [k, \infty)$ at $n = 2k-1$.

Our results have interesting implications in the context of protocol design in congestion settings with incomplete information. Consider an organizer who knows the number of participants at each

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[7] We chose the index of "value of ignorance" rather then "value of knowledge" because it turns out that ignorance is beneficial to the players in most cases.

given point, and wishes to maximize social surplus. That is, the organizer's goal is to minimize the agents' costs. In ranges in which the value of ignorance is positive (e.g., when the number of potential participants is not too large with respect to the number of active participants) the organizer should not reveal the number of actual participants. Analogously, in ranges in which the value of ignorance is negative the organizer should reveal the number of actual participants. Note that if the costs are paid to a revenue-maximizing organizer, the above policies should be reversed.

The paper is organized as follows: Sections 2-4 are devoted to the analysis of 2-terminal splittable routing games with complete information. In Sections 5-6 we analyze the value of ignorance in 2-terminal splittable routing games. In Section 7 we analyze parallel unsplittable routing games (which we call resource selection games) with complete information, and in Section 8 we discuss the value of ignorance for such games.

2 Congestion Networks

A congestion network consists of a directed graph, a set of source-target pairs, and a vector of edge cost functions. In this paper we deal only with 2-terminal congestion networks, i.e. congestion networks which posses a single source-target pair.

2.1 2-Terminal Congestion Networks

Let $G = (V, E, v_s, v_t)$ be a 2-terminal directed graph without self edges, where V is a finite set of nodes, E is a finite set of edges, and $v_s, v_t \in V$ are two distinct nodes called the source node and target node, respectively. For every $v \in V$ we denote by $Out(v)$ and $In(v)$ the set of out-going and in-coming edges of v, respectively. A *route* is a directed path with distinct nodes that connects v_s to v_t. For every edge $e \in E$ and a route R we write $e \in R$ whenever e is part of the route R. Let RO be the set of routes. We assume that $RO \neq \emptyset$.

Every edge $e \in E$ is associated with a cost function $d^e : \Re \rightarrow \Re$ which, unless we say otherwise, satisfies the following properties:

- d^e is continuously differentiable, convex, increasing, and $d^e(x) > 0$ for every $x > 0$.

$d^e(x)$ is interpreted as the cost per unit that is moved through e when the load on e is x.[8] A congestion network is called *linear* if for every edge e there exist constants a^e, b^e such that $d^e(x) = a^e x + b^e$ for every $e \in E$ and for every $x \in \Re$. Obviously, in a linear cost function satisfying the above conditions $a^e > 0$ and $b^e \geq 0$ for every $e \in E$.

Let $\mathbf{d} = (d^e)_{e \in E}$ be the vector of edge cost functions. The tuple $\mathcal{N} = (G, \mathbf{d})$ is called a *2-terminal congestion network*. A 2-terminal congestion network is called *parallel* if $V = \{v_s, v_t\}$.

2.2 Route Flows and Edge Flows

Consider an agent who has to move a continuously divisible unit of good from the source to the target. A splitting policy for such an agent is therefore a function $g : RO \to [0,1]$ with $\sum_{R \in RO} g(R) = 1$. That is, for every route R, $g(R)$ is interpreted as the proportion of the unit sent through the route R. Such a splitting policy is also called a *route flow*. For every route flow g and for every $e \in E$ we let $f_g^e = \sum_{R \in RO | e \in R} g(R)$. That is, f_g^e is the number of units routed through e. It is well-known that for every route flow g the following two conditions hold for the vector $f = (f^e)_{e \in E} = (f_g^e)_{e \in E}$:

$$\sum_{e \in Out(v)} f^e = \sum_{e \in In(v)} f^e + r^v \quad \text{for every} \quad v \in V. \tag{1}$$

$$f^e \geq 0, \quad \text{for every} \quad e \in E, \tag{2}$$

where

$$r^v = \begin{cases} 1 & v = v_s \\ -1 & v = v_t \\ 0 & otherwise. \end{cases}$$

Every vector $f = (f^e)_{e \in E}$ that satisfies the above two conditions is called an *edge flow*, and f_g is called the edge flow induced by the route flow g. The set of route flows is denoted by $\Delta(RO)$, and the set of edge flows is denoted by F. Hence every route flow $g \in \Delta(RO)$ induces an edge flow $f_g \in F$, but it is obvious, and well-known that not every edge flow is induced by some route flow. A sufficient

[8] The values of $d^e(x)$ for $x < 0$ are not relevant to any of our discussions, but it is technically useful to let d^e be defined over the whole real line.

condition for an edge flow to be induced by a route flow is given below. A cycle in G is a simple closed directed path. Let f be an edge flow, and let C be a cycle. We say that C is positive with respect to f if $f^e > 0$ for every $e \in C$.

Lemma 1. *Let $\mathcal{N}$ be a 2-terminal congestion network. Every flow f with no positive cycles is induced by some route flow.*

Proof. The proof follows from a more general theorem named - the flow decomposition theorem (see e.g. [2]).□

Note that an edge flow may be induced by several distinct route flows, as can be seen in the following example.

Example 1. Consider the following graph (Figure 1). Let g_a and g_b be two route flows defined as follows: $g_a(v_s - a - b - v_t) = 0.1$, $g_a(v_s - a - c - d - b - v_t) = 0.2$, $g_a(v_s - c - d - v_t) = 0.7$. $g_b(v_s - a - b - v_t) = 0.1$, $g_b(v_s - a - c - d - v_t) = 0.2$, $g_b(v_s - c - d - b - v_t) = 0.2$, $g_b(v_s - c - d - v_t) = 0.5$. Observe that both g_a and g_b induce the edge flow shown in the figure.

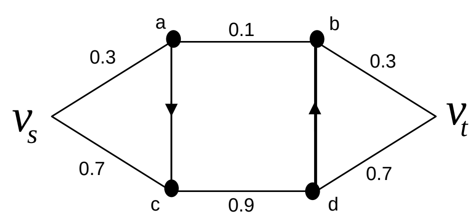

Figure 1: Single-source single-destination congestion network.

3 Routing Games

Every congestion network, a set of players, and an association of a single source-target pair to each player defines a congestion game, which is called a *routing game*. In this game every player has to move a unit of good from her source to her target. When every player can split her unit amongst the available routes, the associated game is called a *splittable routing game*. When the units are not divisible we get an *unsplittable routing game*. When the congestion network is 2-terminal we get a *2-terminal routing game*. Obviously, a 2-terminal routing game is a symmetric game.[9] Similarly, a linear congestion network generates a *linear routing game*, and a parallel congestion network generates a *parallel routing game*.

4 Equilibrium in 2-Terminal Splittable Routing Games with Complete Information

Let $\mathcal{N} = (G, \mathbf{d})$ be a 2-terminal congestion network, and let I be a nonempty finite set of players. Whenever it is convenient and harmless we assume that $I = \{1, \cdots, n\}$, $n \geq 1$. We are about to define actions and cost functions in the corresponding 2-terminal splittable routing game denoted by $\Gamma_{\mathcal{N}}(I)$. In this game, every player i chooses a route flow g_i, and thus a *route flow profile* $\mathbf{g} = (g_1, g_2, \cdots, g_n) \in \Delta(RO)^I$ is generated. Each such profile of route flows generates a profile of edge flows, $\mathbf{f_g} = (f_{g_1}, f_{g_2}, \cdots, f_{g_n}) \in F^I$. The cost function of every player, $c_i(\mathbf{g})$ would depend on the profile of edge flows $\mathbf{f_g}$ via the formula

$$c_i(\mathbf{g}) = C_i(\mathbf{f_g}),$$

where $C_i : F^I \to \Re$ is a function defined over profiles of edge flows as follows:

$$C_i(\mathbf{f}) = \sum_{e \in E} f_i^e d^e(\mathbf{f}^e), \tag{3}$$

where

$$\mathbf{f}^e = \sum_{i=1}^{n} f_i^e \quad \text{for every } e \in E.$$

A route flow profile $\mathbf{g}$ is in *equilibrium* in $\Gamma_{\mathcal{N}}(I)$ if

$$c_i(\mathbf{g}) \leq c_i(h_i, \mathbf{g}_{-i})$$

for every player i and for every route flow h_i, where $\mathbf{g}_{-i}$ denotes the profile of route flows of all players but i.

[9] A congestion form is a generalization of a congestion network. Every congestion form defines two type of games, a splittable congestion game and an unsplittable congestion game. It is not known whether every congestion game is a routing game, but it can be derived from [16] that every symmetric congestion game is a 2-terminal routing game.

In the following theorem we show that every 2-terminal splittable routing game possesses an equilibrium. We further show that although the game may have multiple equilibria, the concept of *equilibrium cost* is well-defined. That is, there exists a level of cost $c(n)$ such that in every equilibrium profile $\mathbf{g}$ in $\Gamma_{\mathcal{N}}(I)$, every player pays $c(n)$. That is, $c_i(\mathbf{g}) = c(n)$ for every player i.

Theorem 1. *Let $\mathcal{N} = (G, \mathbf{d})$ be a 2-terminal congestion network, let n be a positive integer, and let I be a set of n players.*

1. *$\Gamma_{\mathcal{N}}(I)$ possesses an equilibrium.*
2. *There exists a symmetric profile of edge flows, $\mathbf{f}[n] = (f[n], \cdots, f[n])$ such that every equilibrium route flow profile in $\Gamma_{\mathcal{N}}(I)$ induces $\mathbf{f}$. That is, for every equilibrium $\mathbf{g}$, $\mathbf{f}_{\mathbf{g}} = \mathbf{f}[n]$.*
3. *Consequently, there exists a level of cost, $c(n)$ such that in every equilibrium of $\Gamma_{\mathcal{N}}(I)$ every player pays $c(n)$; $c(n)$ is called the equilibrium cost in $\Gamma_{\mathcal{N}}(I)$.*

Proof. It is useful to extend the splittable routing game to a game in which the players can choose edge flows directly. In this game, which we call the *edge flow splittable routing game* and denote it by $\tilde{\Gamma}_{\mathcal{N}}(I)$, every player is able to choose an edge flow rather then just a route flow. Hence, the action set of every player is F, and the cost function of player i is given in (3).

It was proved in Theorem 5 in [19] that there exists a unique equilibrium in $\tilde{\Gamma}_{\mathcal{N}}(I)$. Obviously every permutation of this equilibrium profile is also an equilibrium. Therefore the unique equilibrium must be symmetric. In order to complete the proof of the theorem we have to relate route equilibrium profiles in the splittable routing game $\Gamma_{\mathcal{N}}(I)$ to the unique edge equilibrium profile in $\tilde{\Gamma}_{\mathcal{N}}(I)$.

Let $\mathbf{f}_{-i}$ be a profile of edge flows of all players but i. An edge flow f_i is called a best response for i versus $\mathbf{f}_{-i}$ if $\min_{h_i \in F} C_i(h_i, \mathbf{f}_{-i})$ is attained at $h_i = f_i$. Because the edge cost functions are positive in $(0, \infty)$, such a best response f_i cannot have a positive cycle. Therefore, by Lemma 1, f_i is induced by some route flow g_i, that is $f_{g_i} = f_i$. Hence the following claim holds:
Claim 1

1. For every equilibrium profile, $\mathbf{f} \in F^I$ in the edge flow routing game $\tilde{\Gamma}_{\mathcal{N}}(I)$ there exists a route flow profile $\mathbf{g} \in \Delta(RO)$ that

induces $\mathbf{f}$, that is $\mathbf{f_g} = \mathbf{f}$. Moreover, every such route flow profile $\mathbf{g}$ is in equilibrium in the splittable routing game $\Gamma_{\mathcal{N}}(I)$.
2. Let $\mathbf{g} \in \Delta(RO)^I$ be an equilibrium route flow profile in $\Gamma_{\mathcal{N}}(I)$. Then $\mathbf{f_g}$ is an equilibrium profile in $\tilde{\Gamma}_{\mathcal{N}}(I)$.

Combining the existence and uniqueness of equilibrium in $\tilde{\Gamma}_{\mathcal{N}}(I)$ with Claim 1 completes the proof. □

5 Equilibrium in 2-Terminal Splittable Routing Games with an Unknown Set of Active Players

Splittable routing games with unknown active players, are pre-Bayesian games as discussed in Section **??**. Let $\mathcal{N}$ be a 2-terminal congestion network, and let $I = \{1, 2, \cdots, n\}$ be a finite set of *potential players.*
Terminology: During our discussion we will deal with splittable routing games of the form $\Gamma_{\mathcal{N}}(K)$, where $K \subseteq I$ is a nonempty subset of players. The cost function of player $i \in K$ at this game should be denoted by c_i^K. However, we will denote it by c_i^k, where $k = |K|$. This is a useful and harmless abuse of notations. Furthermore, whenever the set of players and their number is clear we may also omit the superscript k.

A *state* is a nonempty subset of players, K. That is, the set of states is $\Omega = 2^I \setminus \{\emptyset\}$. The set of active players at the state K is K itself. An active player knows that he is active, but he does not know the true state. Hence, an active player knows nothing about the other players (except for an upper bound determined by the number of potential players), and in particular he does not know the number of active players. In a *2-terminal splittable routing game with unknown active players* denoted by $H_{\mathcal{N}}(I)$, at every state K the players in K are playing the game $\Gamma_{\mathcal{N}}(K)$, but they do not know it. The lack of knowledge about the set of active players does not have an effect on the set of actions available to each potential player. A strategy for every potential player i in $H_{\mathcal{N}}(I)$ is a route flow g_i, which he will use once he is active. Note however, that an active player cannot compute his cost even if he knows the complete route flow profile $\mathbf{g} = (g_i)_{i=1}^n$. All he knows is that he will get $c_i^k(\mathbf{g_K})$ if the set of active players is K, where $\mathbf{g_K} = (g_i)_{i \in K}$. When players are considering worst-case scenarios regarding the missing information about the set of active players, and they are in equilibrium, they form a *safety-level equilibrium* as defined in [1].

Formally, in our context, a profile of route flows $\mathbf{g}$ is a *safety level equilibrium* in $H_{\mathcal{N}}(I)$ if for every player i the minimal value of $\max_{\{K\subseteq I|i\in K\}} c_i^k(h_i, \mathbf{g_{K\setminus\{i\}}})$ over all $h_i \in \Delta(RO)$ is obtained at $h_i = g_i$.

Since all cost functions are increasing the worst case scenario, that is
$\max_{\{K\subseteq I|i\in K\}} c_i^k(h_i, \mathbf{g_{K\setminus\{i\}}})$ is obtained in state $K = I$. Therefore, we obtain the following result:

Lemma 2. *Let $\mathcal{N} = (G, \mathbf{d})$ be a 2-terminal congestion network, and let I be a finite set of players. Let $\mathbf{g} \in \Delta(RO)^I$ be a route flow profile. $\mathbf{g}$ is a safety-level equilibrium in the associated routing game with incomplete information $H_{\mathcal{N}}(I)$ if and only if $\mathbf{g}$ is an equilibrium in the associated game with complete information $\Gamma_{\mathcal{N}}(I)$.*

Proof. Assume $\mathbf{g} \in \Delta(RO)^I$ is an equilibrium in $\Gamma_{\mathcal{N}}(I)$. Let i be an active player. By the comment we made before the statement of this lemma,

$$\min_{h_i\in\Delta(RO)} \max_{\{K\subseteq I|i\in K\}} c_i^k(h_i, \mathbf{g_{K\setminus\{i\}}}) = \min_{h_i\in\Delta(RO)} c_i^n(h_i, \mathbf{g}). \tag{4}$$

Because $\mathbf{g}$ is an equilibrium in $\Gamma_{\mathcal{N}}(I)$, the min in the right-hand-side of Equation 4 is attained at g_i. Therefore, $\mathbf{g}$ is a safety-level equilibrium in $H_{\mathcal{N}}(I)$. An analogous argument proves the if part of the lemma.□

Theorem 1 and Lemma 2 imply that when there are n potential players and k active players each of the active players is using at every safety-level equilibrium a route flow that induces the edge flow $f[n]$, which is the edge flow induced in equilibrium in the complete information game with n players. Let $c(k, n)$ be the actual cost of each of the active k players when each of them is using $f[n]$. That is, for an arbitrary player i,

$$c(k, n) = C_i^k(\mathbf{f[n]_K}).$$

6 The Value of Ignorance – Splittable Games

We proceed to analyze the value of ignorance in 2-terminal splittable routing games as a function of the relationship between the number of active participants, k, and the number of potential participants, n.

Consider a 2-terminal congestion network $\mathcal{N} = (G, \mathbf{d})$, and the associated splittable routing game with unknown active players, $H_{\mathcal{N}}(I)$, where $|I| = n$. Suppose that the real state of the world is K where $|K| = k$ and $k < n$. If this state is commonly known then each player $i \in K$ pays $c(k)$. If the real state is unknown then every active player pays $c(k, n)$ as defined at the end of the previous section. Therefore it is natural to call the difference, $c(k) - c(k, n)$ the *value of ignorance.* We denote the value of ignorance by $\nu(k, n)$. That is,

$$\nu(k, n) = c(k) - c(k, n).$$

The value of ignorance indicates how much players "enjoy" the ignorance about the actual set of players. Observe that ignorance is beneficial (in a weak sense) for the players if and only if $\nu(k, n) \geq 0$. In the following example we demonstrate the value of ignorance in a parallel routing game.

Example 2. Consider the congestion network $\mathcal{N}$ in Figure 2. Let $I = \{1, 2, 3\}$, i.e. there are 3 potential players. Let the real state be $K = \{1, 2\}$. Hence, there are two active players. First we find the equilibrium in the routing game with complete information with two players. Assume the first player sends $y \geq 0$ on the upper edge and $1 - y \geq 0$ on the lower edge. Then the second player's objective is to minimize $x(x+y) + (1-x)(1-x+1-y+1)$, where x is the amount she will send on the upper edge. The solution to this is $x = \frac{2-y}{2}$. Since by Theorem 1 the induced edge flow profile in equilibrium is symmetric, it must be that $x = y$. Therefore $x = \frac{2}{3}$. The cost for each of the players in this case is $c(2) = \frac{2}{3} * \frac{4}{3} + \frac{1}{3} * (\frac{2}{3} + 1) = \frac{13}{9}$.

We next find the equilibrium in the splittable routing game with complete information with three players. Assuming the total amount two players send in the upper edge is $y \geq 0$, then the third player's objective is to minimize $x(x+y) + (1-x)(1-x+2-y+1)$, where x is the amount she will send on the upper edge. The solution to this is $x = \frac{5-2y}{4}$. By the symmetry of the induced edge flow profile in equilibrium we obtain $x = \frac{5-4x}{4}$ and therefore $x = \frac{5}{8}$. If the state K is not known to the players, then by playing the safety-level equilibrium each of the players in K will send $\frac{5}{8}$ in the upper edge an therefore their costs will be $c(2,3) = \frac{5}{8} * \frac{10}{8} + \frac{3}{8}(\frac{6}{8} + 1) = \frac{23}{16}$.

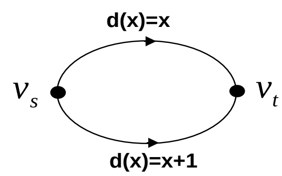

Figure 2.

Hence the value of ignorance is $\nu(2,3) = \frac{13}{9} - \frac{23}{16} > 0$.

In Example 2 we showed that the value of ignorance may be positive. In our next results we show a rich class of games in which this phenomena occurs. In the next section we proceed to estimate the value of ignorance in linear splittable routing games in general 2-terminal congestion networks. More refined results will be obtained in the section following it for parallel networks.

6.1 Linear 2-Terminal Splittable Routing Games

Theorem 2. *Let $\mathcal{N} = (G, \mathbf{d})$ be a linear 2-terminal congestion network. Let $k, n \geq 1$ be integers. If $2k-1 \geq n > k$ then $\nu(k,n) \geq 0$, and $\nu(k, n-1) \leq \nu(k,n)$. Moreover, if there exists an edge e for which $f^e[n]$ admits at least two different values at the interval $n \in [k, 2k-1]$, the above inequalities are strict.*

We need some preparations for the proof of Theorem 2. Recall that an edge flow is a vector $f \in R^E$ that satisfies conditions (1) and (2). However, the right-hand-side of (3) is well-defined for every vector indexed by the edges. This enables us to extend the cost functions C_i to $(\Re^E)^I$.

Let $\mathbf{f} \in (\Re^E)^I$. The marginal cost of each user $i \in I$ on the edge e with respect to f_i^e is

$$\frac{\partial C_i(\mathbf{f})}{\partial f_i^e} = d^e(\mathbf{f}^e) + f_i^e \frac{\partial d^e(\mathbf{f}^e)}{\partial \mathbf{f}^e}.$$

We further need the following notation. For every fictitious edge flow profile $\mathbf{f}$ and every couple of reals $\alpha, \beta \in \Re$ let $K_i(\mathbf{f}, \alpha, \beta) = \frac{\partial C_i(\mathbf{f})}{\partial f_i^e} + \alpha - \beta$.

Let $\mathcal{N}$ be a congestion network, and let I be a finite set of players. Let $\mathbf{f_{-i}}$ be a profile of edge flows of all players but i. We say that f_i is a best response to $\mathbf{f_{-i}}$ if f_i is an optimal solution for the following minimization problem:

$$PR_i : \quad \min_{z_i \in F} C_i(z_i, \mathbf{f_{-i}}).$$

Note that PR_i can be specifically written as the following minimization problem with the decision variables z_i^e, $i \in I, e \in E$

$$PR_i \begin{cases} \min \sum_{e \in E} z_i^e d^e(z_i^e + \sum_{j \neq i} f_j^e) \\ \text{s.t.} \quad z_i \in \Re^E \quad \text{and} \\ \sum_{e \in Out(v)} z_i^e = \sum_{e \in In(v)} z_i^e + r_v^i \qquad v \in V \\ z_i^e \geq 0, \quad \text{for every} \quad e \in E. \end{cases}$$

As the objective function in PR_i is convex, and all constraints are defined by linear inequalities and equalities, PR_i is a convex minimization problem with linear constraints.

Therefore by KKT theorem in Section 9, necessary and sufficient conditions for optimality are provided by the Karush-Kuhn-Tucker (KKT) conditions. Thus, $f_i \in F$ is an optimal solution for PR_i if and only if there exist Lagrange multipliers λ_i^v, $v \in V$ such that for every edge $e \in E$:

$$K_i(\mathbf{f}, \lambda_i^{\hat{t}(e)}, \lambda_i^{\hat{h}(e)}) \geq 0, \quad \text{and} \quad K_i(\mathbf{f}, \lambda_i^{\hat{t}(e)}, \lambda_i^{\hat{h}(e)}) f_i^e = 0, \tag{5}$$

where $\hat{t}(e)$ and $\hat{h}(e)$ are the tail and head nodes of the edge e respectively.

Therefore by Claim 1 (appearing at the proof of Theorem 1) an edge flow profile $\mathbf{f} \in F^I$ is induced by a route flow equilibrium profile if and only if for every player $j \in I$ there exist lagrange multipliers λ_j^v, $v \in V$ such that for every edge $e \in E$:

$$K_j(\mathbf{f}, \lambda_j^{\hat{t}(e)}, \lambda_j^{\hat{h}(e)}) \geq 0, \quad \text{and} \quad K_j(\mathbf{f}, \lambda_j^{\hat{t}(e)}, \lambda_j^{\hat{h}(e)}) f_j^e = 0. \tag{6}$$

As $d^e(x) = a^e x + b^e$, and $I = \{1, ..., n\}$,

$$K_j(\mathbf{f}, \lambda_j^{\hat{t}(e)}, \lambda_j^{\hat{h}(e)}) = a^e f_j^e + a^e \sum_{i=1}^{n} f_i^e + b^e + \lambda_j^{\hat{t}(e)} - \lambda_j^{\hat{h}(e)} \qquad (7)$$

for every $j \in I$.

By Theorem 1, $(f[n], ..., f[n])$ is the unique edge flow profile induced by every equilibrium. Therefore, there exist Lagrange multipliers λ^v, $v \in V$ such that for all $e \in E$:

$$\begin{aligned} &(n+1)a^e f^e[n] + b^e + \lambda^{\hat{t}(e)} - \lambda^{\hat{h}(e)} \geq 0, \quad \text{and} \\ &[(n+1)a^e f^e[n] + b^e + \lambda^{\hat{t}(e)} - \lambda^{\hat{h}(e)}]f^e[n] = 0. \end{aligned} \qquad (8)$$

For every real $t \geq 0$ consider the following convex optimization problem $(SYM)_t$:

$$(SYM)_t: \quad min_{f \in F} \sum_{e \in E} a^e t (f^e)^2 + b^e f^e.$$

Remark: the problem $(SYM)_n$ can be interpreted as finding the minimal cost for every player when all players are restricted to use the same edge flow.

By the KKT Theorem in Section (9), $f \in F$ is an optimal solution for $(SYM)_t$ if and only if there exist Lagrange multipliers λ^u, $u \in V$ such that for every edge $e \in E$:

$$\begin{aligned} &2ta^e f^e + b^e + \lambda^{\hat{t}(e)} - \lambda^{\hat{h}(e)} \geq 0 \quad \text{and} \\ &[2ta^e f^e + b^e + \lambda^{\hat{t}(e)} - \lambda^{\hat{h}(e)}]f^e = 0. \end{aligned} \qquad (9)$$

We are now able to prove the following lemma.

Lemma 3. *Let $\mathcal{N} = (G, \mathbf{d})$ be a linear congestion network and let $n \geq 1$. $f \in F$ is a solution to $(SYM)_n$ if and only if $f = f[2n-1]$. In addition*

$$\sum_{e \in E} f^e[n](a^e \frac{n+1}{2} f^e[n] + b^e) \leq \sum_{e \in E} f^e[k](a^e \frac{n+1}{2} f^e[k] + b^e)$$

for every integer $k \geq 1$.

Proof. The proof follows by observing that by setting $t = \frac{n+1}{2}$ in (9) we get (8).$\square$

Proof of Theorem 2: We show that $c(k, n+1) \leq c(k, n)$ for every n such that $2k - 2 \geq n \geq k$. For every integer $\hat{n} > 0$ we extend the function $c(\cdot, \hat{n})$ to non-integer positive numbers α as follows: $c(\alpha, n) = \sum_{e \in E} f^e[\hat{n}](a^e \alpha f^e[\hat{n}] + b^e)$.
Note that

$$c(k, n) = A_k(f[n]) + B(f[n]),$$

where

$$A_k(f[n]) = k \sum_{e \in E} a^e (f^e[n])^2$$

and

$$B(f[n]) = \sum_{e \in E} b^e f^e[n].$$

Let $2k - 2 \geq n \geq k$. By lemma 3, $c(\frac{n+1}{2}, n) \leq c(\frac{n+1}{2}, n+1)$, and also $c(\frac{n+2}{2}, n) \geq c(\frac{n+2}{2}, n+1)$. Therefore

$$c(\frac{n+2}{2}, n) - c(\frac{n+2}{2}, n+1) =$$

$$A_{\frac{n+2}{2}}(f[n]) + B(f[n]) - A_{\frac{n+2}{2}}(f[n+1]) - B(f[n+1]) =$$

$$s[A_{\frac{n+1}{2}}(f[n]) - A_{\frac{n+1}{2}}(f[n+1])] + B(f[n]) - B(f[n+1]) \geq 0,$$

where $s = \frac{n+2}{2} / \frac{n+1}{2}$.
Let $D(t) =$

$$t[A_{\frac{n+1}{2}}(f[n]) - A_{\frac{n+1}{2}}(f[n+1])] + B(f[n]) - B(f[n+1]).$$

We showed that $D(1) \leq 0$ and $D(s) \geq 0$. Therefore $D(t) \geq 0$ for every $t \geq s$ by the monotonicity of $D(t)$ in t. However $k \geq \frac{n+2}{2}$. Therefore by setting $t = k / \frac{n+1}{2}$ we obtain the desired result since $t \geq s$. If $f^e[n] \neq f^e[n+1]$ for some $e \in E$ then $c(k, n+1) < c(k, n)$ by the convexity of the program $(SYM)_n$. $\square$

Next we show that $\nu(k, n)$ is non-increasing in n for $n \geq 2k - 1$.

Theorem 3. *Let $\mathcal{N} = (G, \mathbf{d})$ be a linear congestion network. Let $k \geq 1$ be an integer. $\nu(k, n+1) \leq \nu(k, n)$ for every n such that $n \geq 2k - 1$.*

Proof. We need the following claim.

Claim 2 Let $\rho_k \geq 0$ $k = 1, 2, ...$ be an increasing sequence of real numbers. Let $F : \mathbb{R}_+^m \to \mathbb{R}_+$ and let $G : \mathbb{R}_+^m \to \mathbb{R}_+$. Define $H : \mathbb{R}_+^m \times N \to \mathbb{R}_+$ by $H(x, k) = F(x) + \rho_k G(x)$. Let x_k be a solution of the minimization problem of $H(x, k)$ s.t. $x \in D \subset \mathbb{R}_+^m$ where D is a bounded convex set. $F(x_k) \leq F(x_{k+1})$ for every $k \geq 1$.

Proof (claim 2): Notice that for every $k \geq 1$ $H(x_k, k) \leq H(x_{k+1}, k+1)$ since $\rho_k \leq \rho_{k+1}$. We next show that $G(x_k) \geq G(x_{k+1})$. Observe that

$$F(x_k) + \rho_k G(x_k) \leq F(x_{k+1}) + \rho_k G(x_{k+1})$$

and that

$$F(x_{k+1}) + \rho_{k+1} G(x_{k+1}) \leq F(x_k) + \rho_{k+1} G(x_k).$$

Therefore

$$(\rho_{k+1} - \rho_k) G(x_k) \geq (\rho_{k+1} - \rho_k) G(x_{k+1})$$

which yields $G(x_k) \geq G(x_{k+1})$. Since

$$F(x_{k+1}) + \rho_k G(x_{k+1}) \geq F(x_k) + \rho_k G(x_k)$$

it must be that $F(x_k) \leq F(x_{k+1})$. $\square$

We proceed with the main proof. Let i be some arbitrary player. For every edge flow $f \in F$ we define $F(f) = \sum_{e \in E} f^e(a^e k f^e + b^e)$ and $G(f) = \sum_{e \in E} a^e (f^e)^2$. Let $H(f, m) = F(f) + \frac{m}{2} G(f)$. by lemma 3, for every $m = 0, 1, 2, ...$ the optimization problem $\min_{f \in F} H(f, m)$ is minimized at $f = f[2k+m-1]$. Therefore by the claim $F(f[n+1]) \geq F(f[n])$ for every $n \geq 2k-1$. Observe that $c(k, n)) = F(f[n])$. Hence $c(k, n+1)) \geq c(k, n))$ for every $n \geq 2k - 1$. $\square$

To summarize: Fix the number, $k \geq 1$, of active players. Together, Theorems 2 and 3 imply the following structure of the function $\nu(k, n)$ for $k \leq n < \infty$: It is non-negative and non-increasing at the interval $[k, 2k - 1]$, and it is non-increasing at the interval $[2k - 1, \infty)$.

More information about this function is obtained for parallel routing games as is shown at the next section. For such games, if k is sufficiently large, it is proved that $\nu(k, n)$ vanishes at $n = k(k+1)-1$. Consequently, for $n \geq k(k+1)$, the value if ignorance is not positive. That is, knowledge is a desirable good.

6.2 Linear Splittable Parallel Routing Games

In this section we deal with a linear parallel congestion network, $\mathcal{N} = (G, \mathbf{d})$ in which $d^e(x) = a^e x + b^e$ for every edge e.

Theorem 4. *Let $\mathcal{N} = (G, \mathbf{d})$ be a parallel and linear congestion network . There exist an integer $T(\mathcal{N})$ such that for every $k \geq T(\mathcal{N})$.*

1. *$\nu(k, n) \geq 0$ for every $k(k+1) - 1 > n > k$. Moreover, the inequality is strict if and only if there exists $e_1, e_2 \in E$ such that $b_{e_1} \neq b_{e_2}$.*
2. *$\nu(k, n) \leq 0$ for every $n > k(k+1) - 1$. Moreover, the inequality is strict if and only if there exists $e_1, e_2 \in E$ such that $b_{e_1} \neq b_{e_2}$.*
3. *For $n = k(k+1) - 1$, $\nu(k, n) = 0$.*

In order to prove Theorem 4 we need the following lemma:

Lemma 4 ([10]). *Let $\mathcal{N} = (G, \mathbf{d})$ be a parallel linear congestion network . For every n let $\Gamma_{\mathcal{N}}(n)$ be the associated splittable routing game with n players. Let $A = \sum_{e \in E} \frac{1}{a^e}$ and let $B = \sum_{e \in E} \frac{b^e}{a^e}$.*

1. *If at equilibrium each player sends a positive amount on each edge, that is $f^e[n] > 0$ for every $e \in E$, then*
$$f^e[n] = \frac{1}{a^e A}[1 + \frac{B - b^e A}{(n+1)}] \quad \text{for every} \quad e \in E. \tag{10}$$
2. *$f^e[n] > 0$ for every $e \in E$ if and only if*
$$\frac{1}{A}[1 + \frac{B}{n+1}] > \max_{e \in E} \frac{b^e}{n+1}. \tag{11}$$

Proof of Theorem 4: By part 2 of Lemma 4 there exists an integer T depending on $\mathcal{N}$ such that for every $n \geq T$ inequality (11) holds. Let $T(\mathcal{N}) = T$, and let $n > k \geq T(\mathcal{N})$. Denote $C = \sum_{e \in E} \frac{b^{e2}}{a^e}$. We are about to prove that
$$\nu(k, n) = \frac{(AC - B^2)((n-k)(k^2 + k - n - 1))}{A(n+1)^2(k+1)^2}, \tag{12}$$
where A and B are defined in the statement of Lemma 4. Since $b^{e2} + b^{l2} \geq 2b^e b^l$ for every $e, l \in E$, $AC - B^2 \geq 0$. Moreover, $AC - B^2 > 0$

if and only if there exist a couple of edges, $\hat{e}, \hat{l}$ such that $b_{\hat{e}} \neq b_{\hat{l}}$. In addition $k^2 + k - n - 1$ is positive for $n < k(k+1) - 1$, negative for $n > k(k+1) - 1$ and zero otherwise. Therefore the proof of the theorem follows from (12). We have to prove (12). Indeed, by Lemma 4 and because (11) holds, for every $e \in E$, $f^e[n] = \frac{1}{a^e A}[1 + \frac{B - b^e A}{(n+1)}]$, and $f^e[k] = \frac{1}{a^e A}[1 + \frac{B - b^e A}{(k+1)}]$. Therefore,

$$\nu(k,n) =$$

$$\sum_{e \in E} k a^e [(f^e[k])^2 - (f^e[n])^2] + \sum_{e \in E} b^e (f^e[k] - f^e[n]). \tag{13}$$

As

$$\sum_{e \in E} b^e (f^e[k] - f^e[n]) =$$

$$\sum_{e \in E} \frac{(B - b^e A)(n-k) b^e}{(n+1)(k+1) a^e A} =$$

$$\frac{(B^2 - AC)(n-k)}{(n+1)(k+1)A}, \tag{14}$$

and

$$\sum_{e \in E} k a^e [(f^e[k])^2 - (f^e[n])^2] =$$

$$k \sum_{e \in E} a^e [(\frac{1}{a^e A} + \frac{B - b^e A}{(k+1) a^e A})^2 -$$

$$k \sum_{e \in E} a^e [(\frac{1}{a^e A} + \frac{B - b^e A}{(n+1) a^e A})^2] =$$

$$k \sum_{e \in E} [\frac{2(B - b^e A)(n-k)}{(n+1)(k+1) a^e A^2}] +$$

$$k \sum_{e \in E} [\frac{(B - b^e A)^2}{(k+1)^2 a^e A^2} - \frac{(B - b^e A)^2}{(n+1)^2 a^e A^2}] =$$

$$k \sum_{e \in E} [\frac{(B - b^e A)^2}{(k+1)^2 a^e A^2} - \frac{(B - b^e A)^2}{(n+1)^2 a^e A^2}] =$$

$$k[\frac{CA-B^2}{(k+1)^2A} - \frac{CA-B^2}{(n+1)^2A}].$$

We obtain that $\nu(k,n) =$

$$\frac{(AC-B^2)(k(n+1)^2 - k(k+1)^2}{A(n+1)^2(k+1)^2} -$$

$$\frac{(AC-B^2)(n-k)(n+1)(k+1))}{A(n+1)^2(k+1)^2}.$$

Since

$$(k(n+1)^2 - k(k+1)^2 - (n-k)(n+1)(k+1)) =$$

$$k[(n-k)(n+k) + 2(n-k)] - (n-k)(n+1)(k+1)] =$$

$$(n-k)(k^2+k-n-1),$$

(12) follows.□

7 Unsplittable Routing Games

In an unsplittable routing game a player cannot split her unit, and therefore she has to choose a single route that connects her source to her target. Unsplittable routing games are a special type of congestion games as defined by [20]. Therefore, by [20] each such game has a pure strategy equilibrium. However, a symmetric equilibrium in a symmetric unsplittable routing game cannot, in general, be pure. Hence, in general, an unsplittable symmetric routing game will have more than one equilibrium profile, and moreover, it can be easily verified that it would have more than one equilibrium cost. Hence, the existence of a unique equilibrium cost, which was crucial for our analysis of the value of ignorance in the splittable model is not guaranteed in the unsplittable model. One can hope that when restricting attention only to symmetric (necessarily mixed-action) equilibrium we will have a unique equilibrium cost. This is indeed our conjecture. However, we have been able to prove this conjecture only for parallel unsplittable routing games. Therefore, in what follows we will deal only with parallel routing games. Parallel routing games are also called, for obvious reasons, resource selection games, and we will refer to them with the later name.

7.1 Equilibrium in Resource Selection Games with Complete Information

Let $\mathcal{N} = (G, \mathbf{d})$ be a parallel congestion network. The graph G is practically defined by the set of parallel edges $E = \{1, 2, \cdots, m\}$. Therefore we will use the notation $\mathcal{N} = (E, \mathbf{d})$. Every edge $j \in E$ is called a *resource*. When we dealt with splittable models we assumed that every cost function d^j is defined over $[0, \infty)$ and it is positive, increasing, convex and continuously differentiable. When dealing with unsplittable models we assume d^j is defined only for positive integers, and that it is increasing and non-negative. Let I be a set of n players. Let $\Gamma_{\mathcal{N}}(I)$ be the unsplittable parallel routing game defined by $\mathcal{N}$ and I, which we call a *resource selection game*.

The action set of every player i in $\Gamma_{\mathcal{N}}(I)$ is the set of resources E. For every profile of resources $\mathbf{x} \in E^n$ let $\sigma_j(\mathbf{x})$ be the number of all players $i \in I$ for which $x_i = j$. Let $c_i^n(\mathbf{x}) = d^{x_i}(\sigma_{x_i}(\mathbf{x}))$ be the cost of player i when the players use the resource profile $\mathbf{x}$.

Let $p \in \Delta(E)$ be a mixed action of an arbitrary player. That is, $p = (p_1, \cdots, p_m)$, where p_j is the probability that a player who uses the mixed action p will select resource j. We denote the support of p by $supp(p)$. That is $supp(p) = \{j \in E | p_j > 0\}$. Denote by $c^n(p, j)$ the expected cost of a player that chooses resource j when each of the other $n-1$ players in $\Gamma_{\mathcal{N}}(I)$ is using p. Let $c^n(p)$ be the expected cost of every player when each of the n players in $\Gamma_{\mathcal{N}}(I)$ is choosing p.

For every $n \geq 1$, and for every $0 \leq \alpha \leq 1$. Let $Y_\alpha^n \sim Bin(n, \alpha)$ be a binomial random variable. That is, $f_\alpha^n(k) = P(Y_\alpha^n = k) = \binom{n}{k}\alpha^k(1-\alpha)^{n-k}$ for every $0 \leq k \leq n$. Let $F_\alpha^n(k) = P(Y_\alpha^n \leq k)$ be the distribution function of Y_α^n. Obviously

$$c^n(p, j) = \mathbf{E}(d^j(1 + Y_{p_j}^{n-1})), \tag{15}$$

where $\mathbf{E}$ stands for the expectation operator. That is,

$$c^n(p, j) = \sum_{s=0}^{n-1} d^j(s+1) f_{p_j}^{n-1}(s). \tag{16}$$

Let $(q, \cdots, q) \in \Delta(E)^n$ be a symmetric mixed-action equilibrium profile in $\Gamma_{\mathcal{N}}(I)$. We will refer to q as a *symmetric-equilibrium mixed action*.

Theorem 5. *Every resource selection game with at least two players and with increasing* [10] *resource cost functions possesses a unique symmetric mixed-action equilibrium.*

In order to prove Theorem 5 we need some preparations.

Lemma 5. *Let $n \geq 1$. $F^n_\alpha(k)$ is a strictly decreasing function of α for every $0 \leq k \leq n-1$.*

Proof. We show that the derivative of $F^n_\alpha(k)$ by α is negative for every $0 \leq k \leq n-1$.

$$\frac{\partial F^n_\alpha(k)}{\partial \alpha} = \sum_{i=0}^{k} \binom{n}{i} [i\alpha^{i-1}(1-\alpha)^{n-i} + (n-i)\alpha^i(1-\alpha)^{n-i-1}] =$$

$$\sum_{i=0}^{k} \binom{n}{i} [i\alpha^{i-1}(1-\alpha)^{n-i} - (n-i)\alpha^i(1-\alpha)^{n-i-1}].$$

Let $f(k) \triangleq \frac{\partial F^n_\alpha(k)}{\partial \alpha}\alpha(1-\alpha)$. Therefore

$$f(k) = \sum_{i=0}^{k} \binom{n}{i} [(1-\alpha)i\alpha^i(1-\alpha)^{n-i} - \alpha(n-i)\alpha^i(1-\alpha)^{n-i}] =$$

$$\sum_{i=0}^{k} \binom{n}{i} [\alpha^i(1-\alpha)^{n-i}(i-n\alpha)].$$

Observe that $\sum_{i=0}^{k} \binom{n}{i}\alpha^i(1-\alpha)^{n-i}i = \sum_{i=1}^{k} p(Y^n_\alpha \geq i) - k(1-p(Y^n_\alpha \leq k))$. Therefore

$$f(k) = \sum_{i=0}^{k} \binom{n}{i}\alpha^i(1-\alpha)^{n-i}(i-n\alpha) = \sum_{i=1}^{k} p(Y^n_\alpha \geq i) - k + (k-n\alpha)p(Y^n_\alpha \leq k).$$

Obviously $\sum_{i=1}^{k} p(Y^n_\alpha \geq i) - k < 0$. We distinguish between two cases:

1. $k - n\alpha \leq 0$: This case will yield immediately that $f(k)$ is negative.

[10] That is, $d^j(k) < d^j(k+1)$ for all j and k.

2. $k - n\alpha > 0$: Observe that $f(n) = 0$. We look at the difference between $f(k+1)$ and $f(k)$. $f(k+1) - f(k) = (k+1-n\alpha)p(Y_\alpha^n = k+1) > 0$. Since the differences are positive and $f(n) = 0$ then $f(k) < 0$.

□

Lemma 6. *Let $\Gamma_{\mathcal{N}}(I)$ be a resource selection game with at least two players. Let $q, p \in \Delta(E)$ be mixed actions, and let $j \in E$ be a resource such that d^j is increasing in $\{1, 2, \cdots, n\}$. If $p_j > q_j$ then $c^n(p, j) > c^n(q, j)$.*

Proof. We have to show that $c^n(p, j)$ is increasing in p_j.

By manipulating (16)

$$c^n(p,j) = \sum_{k=1}^{n-1}\left((d^j(k) - w^j(k+1))\sum_{l=0}^{k-1} f_{p_j}^{n-1}(l)\right) + d^j(n)\sum_{l=0}^{n-1} f_{p_j}^{n-1}(l) =$$

$$d^j(n) - [\sum_{k=1}^{n-1}(d^j(k+1) - d^j(k))F_{p_j}^{n-1}(k-1)],$$

where the last equality follows from the fact that $\sum_{k=0}^{n-1} f_{p_j}^{n-1}(k) = 1$.

By Lemma 5, $F_{p_j}^{n-1}(k)$ is strictly decreasing in p_j for every $k = 0, ..., n-2$. In addition, d^j is strictly increasing, and therefore $c^n(p, j)$ is strictly increasing in p_j. □

Proof of Theorem 5 We only need to prove the uniqueness. [11] Suppose in negation that there is more than one mixed-action symmetric equilibrium in $\Gamma_{\mathcal{N}}(I)$. Let q and p be two symmetric equilibrium actions with $p \neq q$. Since $p \neq q$ there exists $j \in E$ with $q_j \neq p_j$. W.l.o.g $q_j > p_j$. Therefore there exist a resource $r \in E$ such that $r \neq j$ and $q_r < p_r$. We get a contradiction from the following sequence of inequalities: $c^n(q, j) > c^n(p, j) \geq c^n(p, r) > c^n(q, r) \geq c^n(q, j)$, where the strict inequalities follow from Lemma 6 and the other inequalities hold because q and p are equilibrium actions.

For every $n \geq 1$ we will denote the unique symmetric equilibrium mixed action in $\Gamma_{\mathcal{N}}(I)$ by p^n, and we denote by $c(n) = c^n(p^n)$ the equilibrium cost of a player in $\Gamma_{\mathcal{N}}(I)$.

[11] Existence is proved in [18].

We say that a resource cost function d^j is *convex* if it can be extended to a convex function on $[1, \infty)$.

The following lemma will be useful later.

Lemma 7. *Let $\mathcal{N} = (E, \mathbf{d})$ be a parallel congestion network with increasing and convex cost functions. There exists an integer $T \geq 2$, $T = T(\mathcal{N})$ such that for every $n \geq T$, the unique symmetric-equilibrium mixed action in the game $\Gamma_{\mathcal{N}}(I)$, $p^n \in \Delta(E)$, has a full support. That is, $p^n_r > 0$ for every $1 \leq r \leq m$.*

Proof. Recall that p^n is the unique symmetric-equilibrium mixed action in $\Gamma_{\mathcal{N}}(I)$, and that $c(n) = c^n(p^n)$ is the symmetric-equilibrium cost of every player. As p^n is an equilibrium mixed action, $c^n(p^n, j) = c(n)$ for every $j \in supp(p^n)$. For every resource j we denote by d^j the convex extension of d^j to $[0, \infty)$. As d^j is convex,

$$c^n(p^n, j) = \mathbf{E}(d^j(1 + Y^{n-1}_{p^n_j})) \geq d^j(1 + \mathbf{E}(Y^{n-1}_{p^n_j})) =$$

$$d^j(1 + p^n_j(n-1)),$$

where the first equality follows from (15), the inequality follows from the convexity of d^j, and the last equality follows from the well-known fact that

$$\mathbf{E}(Y^n_\alpha) = \alpha n. \tag{17}$$

Obviously, there exists $j \in supp(p)$ for which $p^n_j \geq \frac{1}{m}$. For this resource j

$$c(n) = c^n(p^n, j) \geq d^j(1 + \frac{1}{m}(n-1)) \geq$$

$$\min_{r=1}^{m} d^r(1 + \frac{1}{m}(n-1)).$$

Since d^j is increasing and convex, $\lim_{n \to \infty} d^j(n) = \infty$ for every resource j. Therefore $\lim_{n \to \infty} c(n) = \infty$. Hence, there exists T such that for every $n \geq T$ $c(n) > \max_{j=1}^m d^j(1)$. We claim that for every $n \geq T$, $p^n_r > 0$ for every $1 \leq r \leq m$. Indeed, if $p^n_r = 0$ for some r then because $c(n) > d^r(1)$, a player will decreases her cost by deviating from p^n to r (assuming every other player is using p^n). This contradicts p^n being a symmetric-equilibrium mixed action.$\square$

7.2 Resource Selection Games with an Unknown Set of Active Players

Consider a fixed parallel congestion network $\mathcal{N} = (E, \mathbf{d})$ with the set of resources $E = \{1, \cdots, m\}$, $m \geq 1$, and resource cost functions $\mathbf{d} = (d^j)_{j=1}^m$. Let $I = \{1, 2, \cdots, n\}$, $n \geq 1$ be the set of potential players.

Analogously to what we did in Section 5, we proceed to describe the associated resource selection game with unknown set of active players.

A *state* is a nonempty subset, K of players. That is, the set of states is $\Omega = 2^I \setminus \{\emptyset\}$. The set of active players at the state K is K itself. An active player knows that he is active, but he does not know the true state. Hence, an active player knows nothing about the other players (except for an upper bound determined by the number of potential players), and in particular he does not know the number of active players. In a *resource selection game with unknown active players* denoted by $H_{\mathcal{N}}(I)$, at every state K the players in K are playing the game $\Gamma_{\mathcal{N}}(K)$, but they do not know it. The lack of knowledge about the set of active players does not have an effect on the set of strategies available to each potential player. A strategy for every potential player i in $H_{\mathcal{N}}(I)$ is a resource $x_i \in E$, which he will use once he is active. A mixed strategy for i is, therefore, a probability distribution $q[i] \in \Delta(E)$. Note however, that an active player cannot compute his cost even if he knows the complete resource profile $\mathbf{x} = (x_i)_{i=1}^n \in E^n$. All he knows is that he will get $c_i^k(\mathbf{x_K})$ if the set of active players is K, where $\mathbf{x_K} = (x_i)_{i \in K}$. When players are considering worst-case scenarios regarding the missing information about the set of active players, they are using mixed strategies, and they are in equilibrium, they form a *mixed strategy safety-level equilibrium* as described in Section **??**. According to this definition, a profile $\mu = (q[1], \cdots, q[n])$ of mixed strategies in $H_{\mathcal{N}}(I)$ is a mixed strategy safety-level equilibrium if for every player i the minimal value of $\max_{\{K \subseteq I | i \in K\}} c_i^k(p[i], \mu_{K \setminus \{i\}})$ over all $p[i] \in \Delta(E)$ is obtained at $p[i] = q[i]$.

Since all cost functions are increasing the worst case scenario, that is $\max_{\{K \subseteq I | i \in K\}} c_i^k(p[i], \mu_{\mathbf{K} \setminus \{\mathbf{i}\}})$ is obtained in state $K = I$.

Therefore, we obtain the following result whose proof is omitted because of its similarity to the proof of Lemma 2

Lemma 8. *Let $\mathcal{N} = (E, \mathbf{d})$ be a parallel congestion network, and let I be a finite set of players. Let $\mu \in \Delta(E)^I$ be a mixed action profile. μ is a mixed strategy safety-level equilibrium in the resource selection game with incomplete information $H_{\mathcal{N}}(I)$ if and only if μ is a mixed action equilibrium in $\Gamma_{\mathcal{N}}(I)$.*

We proceed to prove a uniqueness result:

Theorem 6. *Let $\mathcal{N}$ be a parallel congestion network in which the resource cost functions are increasing, and let I be a set of n players. $H_{\mathcal{N}}(I)$ has a unique mixed strategy symmetric safety-level equilibrium. In this mixed strategy symmetric safety-level equilibrium every player is using the mixed strategy p^n, where p^n is the unique symmetric-equilibrium mixed action in $\Gamma_{\mathcal{N}}(I)$.*

Proof. The proof follows directly from Theorem 5 and Lemma 8.□

By Theorem 6, each of the players in $H_{\mathcal{N}}(I)$ is using the mixed strategy p^n, where p^n is the unique symmetric-equilibrium mixed action in $\Gamma_{\mathcal{N}}(I)$. However, the cost of each active player in $H_{\mathcal{N}}(I)$ is not $c(n) = c^n(p^n)$, it depends on the true state. If the true state is K, that is K is the set of active players, and $|K| = k$, the cost of each active player i is $c^k(p^n)$. Denote this cost by $c(k, n)$.

8 The Value of Ignorance – Unsplittable Routing Games

As we did in the splittable routing model we denote the value of ignorance by $\nu(k, n)$. That is,

$$\nu(k, n) = c(k) - c(k, n).$$

Theorem 7. *Let $\mathcal{N}$ be a linear and parallel congestion network with increasing resource cost functions. There exist an integer $T = T(\mathcal{N})$, $T \geq 2$ such that for all $n > k \geq T$:*

1. $\nu(k, n) \geq 0$.

2. All inequalities above are strict if and only if there exists $j_1, j_2 \in E$ such that $d^{j_2}(1) \neq d^{j_1}(1)$

Proof. For every resource j let $d^j(x) = a^j x + b^j$ with $a^j > 0$ and $b^j \geq 0$.

1. Let $j \in E$. By Lemma 7 there exist an integer T such that for every $n \geq T$ the unique symmetric equilibrium mixed action in $\mathit{\Gamma}_{\mathcal{N}}(I)$ has full support for every set of players, I, with n players. We show that $c(k) \geq c(k,n)$ for every $n > k \geq T$. By (7.1) and (17),
$$c(k) = c^k(p^k) = d^j(1) + (k-1)a^j p^k_j. \tag{18}$$
Similarly,
$$c(k,n) = c^k(p^n) = \sum_{j=1}^m p^n_j c^k(p^n, j) =$$
$$\sum_{j=1}^m p^n_j[c(n) - (n-k)a^j p^n_j] = c(n) - (n-k)\sum_{j=1}^m a^j (p^n_j)^2.$$
It remains to show that
$$c(n) - c(k) \leq (n-k)\sum_{j=1}^m a^j (p^n_j)^2. \tag{19}$$
By Equation (18) applied to $c(k)$ and $c(n)$
$$p^n_j = \frac{c(n) - d^j(1)}{a^j(n-1)}, \quad p^k_j = \frac{c(k) - d^j(1)}{a^j(k-1)}.$$
Let $A = \sum_{j=1}^m \frac{1}{a^j}$ and $B = \sum_{j=1}^m \frac{d^j(1)}{a^j}$. Since $\sum_{j=1}^m p^k_j = \sum_{j=1}^m p^n_j = 1$ we have that $c(n) = \frac{(n-1)+B}{A}$ and $c(k) = \frac{(k-1)+B}{A}$. Hence
$$c(n) - c(k) = \frac{n-k}{A}.$$
Because of (19) it remains to show that
$$A\sum_{j=1}^m a^j (p^n_j)^2 \geq 1.$$

Let $L = \sum_{j=1}^{m} \frac{(d^j(1))^2}{a^j}$. Hence we have

$$A\sum_{j=1}^{m} a^j (p_j^n)^2 = A\sum_{j=1}^{m} \frac{(c(n)-d^j(1))^2}{a^j(n-1)^2} = \frac{A}{(n-1)^2}[(c(n))^2 A - 2c(n)B + L] =$$

$$\frac{1}{(n-1)^2}[((n-1)+B)^2 - 2((n-1)+B)B + LA] =$$

$$\frac{1}{(n-1)^2}[(n-1)^2 - B^2 + LA] = 1 - \frac{B^2 - LA}{(n-1)^2}. \qquad (20)$$

It remains to show that $LA - B^2 \geq 0$. This is immediate since for every couple of resources $j, r \in E$ such that $j \neq r$, $(d^j(1))^2 + (d^r(1))^2 \geq 2d^j(1)d^r(1)$.

2. Observe that $LA - B^2 = 0$ if and only if $d^j(1) = d^r(1)$ for every $j, r \in E$.

□

We further analyze the properties of the function $\nu(k,n)$ in linear models:

Theorem 8. *Let $\mathcal{N}$ be a parallel congestion network with linear and increasing resource cost functions. There exist T such that the following assertions hold:*

1. *For every $n \geq T$ $p_j^n = \frac{n-1+B-d^j(1)A}{Aa^j(n-1)}$, where $A = \sum_{j=1}^{m} \frac{1}{a^j}$, and $B = \sum_{j=1}^{m} \frac{d^j(1)}{a^j}$.*
2. *For every $k \geq T$, for a set of k players, K, the minimal social cost in $\Gamma_{\mathcal{N}}(K)$ attained with symmetric mixed-action profiles is attained at p^{2k-1}. Consequently, $\nu(k,n) = c(k) - c(k,n)$ is maximized for a fixed k at $n = 2k-1$.*

Proof. Let $q_j^n = \frac{1}{Aa^j} + \frac{B-d^j(1)A}{Aa^j(n-1)}$ for every $j \in E$. Let T be the smallest integer t such that $q_j^t > 0$ for every $j \in E$.

1. Observe that for every $n \geq T$ $q_j^n > 0$ for every $j \in E$. Notice that for every n, $\sum_{j=1}^{m} q_j^n = 1$. Let $n \geq T$. In order to prove that q^n is in equilibrium it suffices to prove that for every player i, if all

players but i play the mixed-action q^n then player i is indifferent between all resources. That is we want to show that

$$c^n(q^n, j) = c^n(q^n, r)$$

for every $j, r \in E$. Indeed, $c^n(q^n, j) = d^j(1) + (n-1)a^j q_j^n = \frac{n-1}{A} + \frac{B}{A}$, which doesn't depend on j.

2. Let $k \geq T$. We need to show that $c^k(p) = \sum_{j=1}^m p_j c^k(p, j)$ is minimized over $p \in \Delta(E)$ at $p = p^{2k-1}$. Note that,

$$\sum_{j=1}^m p_j c^k(p, j) = \sum_{j=1}^m p_j^2 d^j(k-1) + p_j d^j(1).$$

Hence, minimizing $c^k(p)$ over $p \in \Delta(E)$ is a convex program with a differentiable objective function. Therefore, by the Karush-Kuhn-Tucker theorem in Section 9, every $p \in \Delta(E)$ for which there exists a Lagrange multiplier λ that satisfies

$$2p_j d^j(k-1) + d^j(1) = \lambda. \quad \forall j \in E \tag{21}$$

is an optimal solution. By (21) and since $\sum_{j=1}^m p_j = 1$, $\lambda = \frac{2(k-1)+B}{A}$. Therefore $p_j = \frac{2(k-1)+B-d^j(1)A}{2(k-1)Aa^j}$. Observe that p_j has the same form of p^n where $n = 2(k-1)+1 = 2k-1$, which completes the proof.

□

9 Appendix - The Karush-Kuhn-Tucker (KKT) conditions

In this section we describe the relevant theory of the Karush-Kuhn-Tucker (KKT) conditions that is required in our proofs. The material is taken from [5].

Consider the following problem:

$$(IC) \quad \min\{f(x) : g_j(x) \leq 0, j = 1, ..., m,$$

$$h_k(x) = 0, k = 1, ..., p, \quad x_i \geq 0, i = 1, .., n, \quad x \in \mathcal{R}^n\}.$$

We say that (IC) is a convex program if $f, g_1, ..., g_m$ are real valued convex and differentiable functions on $\Re^n$, and $h_1, ..., h_p$ are linear.

For every $x \in \Re^n$ let

$$L(x) = f(x) + \sum_{j=1}^{m} \mu_j g_j(x) + \sum_{k=1}^{p} \lambda_k h_k(x).$$

The following are the well known *Karush-Kuhn-Tucker* (KKT) conditions at a feasible point x^*:

$$KKT \begin{cases} \text{There exists lagrange multipliers } \mu_j \;\; j = 1, ..., m \\ \text{and } \lambda_k \;\; k = 1, ..., p \quad \text{ such that} \\ \mu_j g_j(x^*) = 0, \quad \mu_j \geq 0 \quad j = 1, ..., m, \\ \quad \frac{\partial L(x^*)}{\partial x_i} \geq 0, \qquad x_i^* \frac{\partial L(x^*)}{\partial x_i} = 0 \quad i = 1, ..., n \end{cases}$$

The Karush-Kuhn-Tucker (KKT) Theorem:

Let (IC) be a convex program and let x^* be a feasible solution to (IC). If there exists $\bar{x} \in \mathcal{R}^n_+$ such that at $\bar{x}$ the nonlinear g_j are strictly negative, and linear g_j are non-positive then the KKT conditions are both necessary and sufficient for x^* to be optimal for (IC).

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