Playing Konane Mathematically with Combinatorial Game Theory

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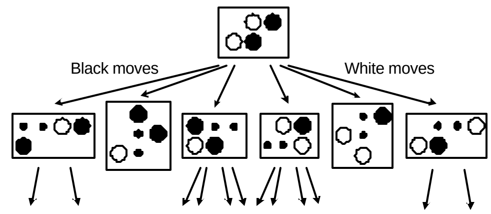
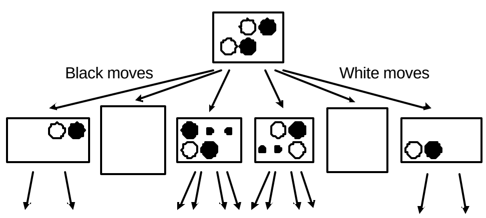
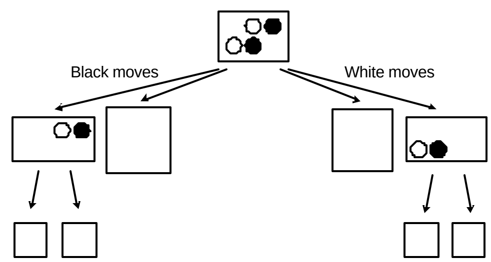
What is a game?



Simplified game tree

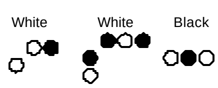


Fully simplified game tree



Questions about a game

- Who wins?
- By how much?
- What is the best move?
- How to combine games?



Combinatorial game theory (CGT) answers these questions precisely.

A game's value tells how many moves of advantage and can be compared, added, etc.

Applicability of CGT

Complete information
No chance
Players moves alternately
First player unable to move loses
Game must end

CGT values (informally)

- Positive: Black wins ○●○ has value **1**
- Negative: White wins ●○●○● has value **-2**
- Zero: second player wins □ has value **0**

  ○●○● has value **0**
- Fuzzy: first player wins ○● has value *****

  ***** is less than any positive value
  greater than any negative value
  incomparable to zero

CGT values (formally)

A game's meaning is its simplified game tree, written { *black-moves* | *white-moves* }

□ = { | } = **0**
○●○ = { ● · · ○ , ○ · · ● | }
= { **0**, **0** | } = { **0** | } = **1**
●○●○● = { | ○ · · ●○●, ●○● · · ○ }
= { | **-1**, **-1** } = { | **-1** } = **-2**
○● = { ● · · | · · ○ }
= { **0** | **0** } = *****

Arithmetic

Value of noninterfering combination of games = sum of values

Example: **2** + **-1** + **0** = **1**

○●○●○ + ●○● + ○●○● = ○●○

A = *B* means *A* + -*B* = **0**

Fractions

○●○ · ○● = { **0**, **-1** | **1** } = { **0** | **1** } = $\frac{1}{2}$

In { **left** , **right** } , choose the simplest number between **left** and **right** .

- integers are simpler than fractions
- among integers, smaller abs value is simpler
- among fractions, smaller denominator (always a power of 2)

{**5**|**22**}=**6** {**-22**|**-7**}=**-8** {**-22**|**3**}=**0** {1/[illegible]|3/[illegible]}=5/[illegible]

Why these rules?

Infinitesimals

● ●○ = { **0** | ***** } = -

Smaller than any positive number

Greater than zero

How does it compare to *****?

(There are even smaller infinitesimals.)

Simplifying a game

Delete dominated options: { ~~**5**~~, **6** | … }

Bypass reversible moves:

P = { … | R, … }
R = { P', … | … }
P' = { … | X, Y, Z }
If P' > P , then P = { … | X, Y, Z, … }

If Right moves to R, then Left will certainly move to P' (or something better), so Right's new options will be X, Y, Z.

Why CGT?

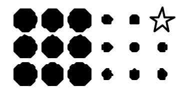
- Reduce the search space by summing subgames
- Simplify game values into equivalent but simpler games
- Provide vocabulary for talking about game values
- Tell which move is best, not just which one wins

Separating stone positions: how far can a stone move?

(To determine noninterfering subgames.)

Idea: potential function

Example: no stone can get to the star



Potential function:

Initial potential is 20

Goal (star) potential is 21

No jump increases potential

2	3	5	8	13	21
1	2	3	5	8	13
1	1	2	3	5	8

CGT and competitive game-playing

CGT is useless in the opening and middle game

Analysis is tractable only for the endgame

CGT gives an exact answer

Do you need the best move, or just a good one?

CGT is a lot of work to program

CGT wins if you can separate a game into pieces

16 stones, branching factor of 4: $4^{16} = 4$ billion

2 groups, 8 stones each, branching 2: $2 * 2^8 = 512$

Learning more

Paper and computer program:

http://sdg.lcs.mit.edu/~mernst/pubs

Combinatorial game theory (most formal last):

- *Surreal Numbers*, Knuth
- *Winning Ways*, Berlekamp, Conway, & Guy
- *Combinatorial Games*, Guy
- *On Numbers and Games*, Conway