12.815/12.816: NOTES ON THERMAL EQUILIBRIUM MODELS

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1. Rotating, Atmosphereless Bodies in Thermal Equilibrium

1.1. **Fast Rotating Bodies.** Let (ϕ, θ) define the $(\text{lat.}, \text{long.})$ of a terrestrial atmosphereless body as measured from the subsolar point. We take $(\phi, \theta) \in [-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$.

For a fast rotating planet, we may determine the equilibrium temperature $T_s(\phi)$ at a given latitude ϕ:

$$\int_{\theta=-\pi/2}^{\pi/2} L_{\odot}(1-A)\left(\frac{R_{\odot}}{a_{AU}}\right)^2 \times \cos\phi\cos\theta \times dA_i(\phi,\theta) = \int_{\theta=-\pi/2}^{\pi/2} \epsilon\sigma T_s^4(\phi) \times dA_e(\phi,\theta) \tag{1}$$

The factors of $\cos\phi\cos\theta$ have been included to account for the contribution of the incident radiation normal to the surface at a given latitude and longitude. For the outgoing case, since radiation is emitted in all directions without preference, we do not need to account for this.

The area elements for incoming radiation dA_i and outgoing radiation dA_e are

$$dA_i = R_p^2 \cos\phi . d\phi . d\theta \tag{2}$$

$$dA_e = 2 \times dA_i \tag{3}$$

Since the planet is considered to be fast rotating, $dA_e = 2 \times dA_i$ since, at any given time, only half the planet sees the sun, while all of its surface emits radiation.

Date: Monday 11 November 2013.

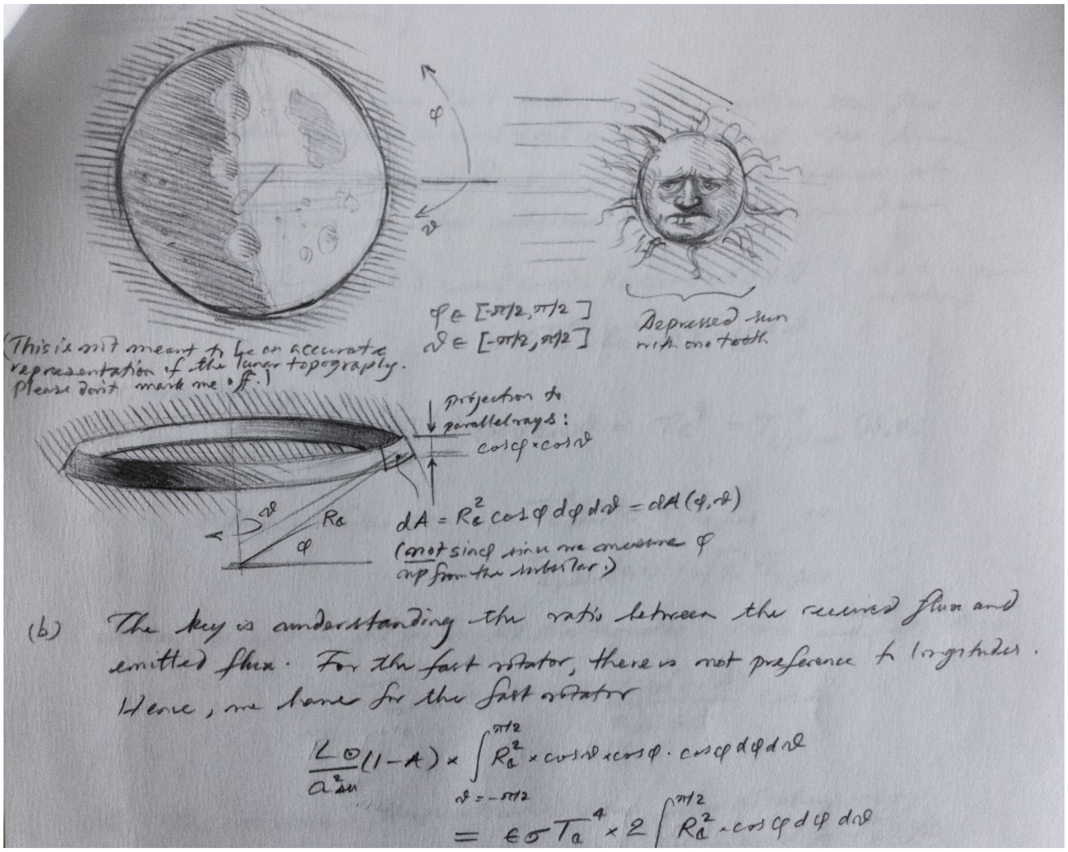

FIGURE 1. Geometry of an atmosphereless planet in equilibrium with the sun.

Substituting for the differential area elements, we get

$$\int_{\theta=-\pi/2}^{\pi/2} L_\odot (1-A) \left(\frac{R_\odot}{a_{AU}}\right)^2 \cos^2\phi \cos\theta . d\phi . d\theta = 2 \int_{\theta=-\pi/2}^{\pi/2} \epsilon\sigma T_s^4 \cos\phi . d\phi . d\theta \tag{4}$$

$$\Rightarrow 2L_\odot (1-A) \left(\frac{R_\odot}{a_{AU}}\right)^2 \cos^2\phi . d\phi = 2\pi T_s^4 (\phi) \cos\phi . d\phi \tag{5}$$

so that, solving for $T_s^4(\phi)$, we get

$$T_s^4(\phi) = \frac{L_\odot (1-A)}{\epsilon\sigma} \left(\frac{R_\odot}{a_{AU}}\right)^2 \frac{\cos\phi}{\pi} \tag{6}$$

We recall that, for the simplified analysis in which we considered the average temperature $\bar{T}_s^4$ of the entire planet that we had

$$\bar{T}_s^4 = \frac{L_\odot (1-A)}{4\epsilon\sigma}\left(\frac{R_\odot}{a_{AU}}\right)^2 \tag{7}$$

Thus, if we average $T_s^4(\phi)$ over all latitudes, we should get $\bar{T}_s^4$. This is indeed the case. Consider the total area of the planet $A_p = 4\pi R_p^2$. Then

$$\begin{aligned}
\frac{1}{A_p}\iint\limits_{\theta,\phi=-\pi/2}^{\pi/2} T_s^4(\phi)\, dA_e &= \frac{2R_p^2}{4\pi R_p^2}\int\limits_{-\pi/2}^{\pi/2} d\theta \int\limits_{-\pi/2}^{\pi/2} T_s^4(\phi)\cos\phi . d\phi \\
&= \frac{1}{2}\int\limits_{-\pi/2}^{\pi/2} \frac{L_\odot (1-A)}{\epsilon\sigma}\left(\frac{R_\odot}{a_{AU}}\right)^2 \frac{\cos\phi}{\pi}\cos\phi . d\phi \\
&= \frac{1}{2}\frac{L_\odot (1-A)}{\epsilon\sigma}\left(\frac{R_\odot}{a_{AU}}\right)^2 \frac{\pi/2}{\pi} \\
&= \frac{L_\odot (1-A)}{4\epsilon\sigma}\left(\frac{R_\odot}{a_{AU}}\right)^2 \\
&= \bar{T}_s^4
\end{aligned} \tag{8}$$

1.2. **Slow Rotating Bodies.** For a slow rotating body, the analysis is considerably simpler. This is because at each (lat., long.), the differential incoming and outgoing area elements are the same (modulo the cosine factors). Thus, we have

$$L_\odot (1-A)\left(\frac{R_\odot}{a_{AU}}\right)^2 \times \cos\phi\cos\theta \times dA(\phi,\theta) = \epsilon\sigma T_s^4(\phi,\theta) \times dA(\phi,\theta) \tag{9}$$

$$\Rightarrow T_s^4(\phi,\theta) = \frac{L_\odot (1-A)}{\epsilon\sigma}\left(\frac{R_\odot}{a_{AU}}\right)^2 \times \cos\phi\cos\theta \tag{10}$$