NONLINEAR DYNAMIC RESPONSE OF REINFORCED CONCRETE UNDER IMPULSIVE LOADING: 
RESEARCH STATUS AND NEEDS

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Current research status is presented for the multidimensional nonlinear analysis of reinforced concrete subjected to impulsive load conditions. Strategy for the solution of nonlinear dynamic equations is discussed. The description of the development of models for material behavior is given. Further research needs and interests for the development of improved analysis capabilities are indicated.

1. Introduction

Methods of nonlinear dynamic analysis along with their advantages and disadvantages are summarized in a recent paper [1].

The analysis results for dynamic response of reinforced concrete to impulsive loads can be substantially affected by the extent to which the nonlinear material behavior up to some point of dynamic failure is accounted for. Major sources of nonlinearity in reinforced concrete are the progressive cracking of concrete in tension, the nonlinear response of concrete under multiaxial compression, and other nonlinearities related to reinforcement and its interaction with concrete. At present, very little analysis capability exists for a multidimensional nonlinear dynamic analysis of reinforced concrete, taking into account laws of material behavior.

Starting from the corresponding differential equations, Stangenberg [2] presented dynamic ultimate load calculations for reinforced concrete beams and plates. He employed an explicit time integration technique and finite differences for the spatial discretization. The rotational inertia of single beam or plate elements as well as shear deformations were included. The finite propagation velocity of bending and shear waves was taken into account. As an application to this approach, analysis of a reinforced concrete plate subjected to a jet force was included. It was found that the effect of the transverse shear stresses was very large. Plastic bending strength, however, was far from being exhausted. Three-dimensional material behavior is significant in massive reinforced concrete structures. The nonlinear material properties in three-dimensional analysis must be taken into account. In view of the size and computational efforts involved for a three dimensional problem, isoparametric elements represent the only feasible choice [3,4,5]. To retain the efficiency of the isoparametric element, the propagation of changes across the element must be allowed rather than the entire element change properties at one time. This is accomplished by using the Gaussian integration points as the check points for material behavior.

A dynamic analysis capability for reinforced concrete is reported by Rebora et al. [6]. The structure was discretized into twenty-node isoparametric brick elements for concrete and one- or two-dimensional membrane elements for steel. For compression, a three-dimensional nonlinear elastic constitutive law was used for concrete. Two- and three-dimensional cracking surfaces were incorporated. An implicit time integration scheme was adopted with a consistent mass matrix approach. A built-in circular reinforced concrete plate was analyzed for evenly distributed load. A constant load with a rise time equal to half the fundamental period was applied dynamically.

A review of available finite element and finite dif-
ference computer programs [1] indicates that the existing dynamic analysis capabilities are not sufficiently sophisticated to take into account realistic laws of material behavior of reinforced concrete. The analytical models are still being extended and refined. The present paper summarizes the analysis procedures developed to date at the Civil Engineering Department of the Massachusetts Institute of Technology as part of a research project currently being undertaken to predict the dynamic response of reinforced concrete to impactive forces. Discussion of these procedures as related to other reported studies is provided. Suggestions for further research are emphasized.

Sect. 2 of this paper summarizes the solution strategy. Specific impact considerations are included in sect. 3. Sect. 4 discusses the modeling of concrete material behavior including cracking and compressive nonlinearity. Modeling of steel reinforcement and liners is described in sect. 5. A discussion of further research needs and interests are contained in sect. 6. Finally, the last section cites the conclusions.

2. Solution strategy

2.1. Numerical integration scheme

The equation of dynamic equilibrium for a multi-degree-of-freedom system can be written as

\[ M \ddot{U} = P_E - P_I, \quad (1) \]

or in the state-space form

\[ M \ddot{U} = P_E - P_I, \quad (2) \]

where \( M \) = mass matrix, \( U \) = displacement vector, \( \dot{U} \) = velocity vector, \( \ddot{U} \) = acceleration vector, \( P_E \) = consistent external force vector, \( P_I \) = vector of consistent nodal forces due to internal stresses \( \int \int B^T \Delta \sigma \, dV \), where \( B \) = strain-displacement transformation matrix and \( \Delta \sigma \) = vector of stress increment.

Direct step-by-step integration is the only appropriate method for solving the coupled equations of motion (1) for nonlinear problems such as the transient response of reinforced and prestressed concrete structures. Starting with (2), an explicit "staggered" time integration technique is developed based on the trapezium (trapezoidal rule) scheme which, in its implicit form, is a one-step second-order method with the smallest local truncation error of all second-order methods. The result is a central difference type of scheme. For convenience this technique will be referred to hereafter as the "split-time" or ST method.

In the ST scheme, the time integration scale for the computation of the velocity vector \( \dot{V} \) is shifted by \( \frac{1}{2} \Delta t \) relative to the time integration scale for the computation of the displacement vector \( U \) as illustrated in fig. 1, where \( \Delta t \) = time step.

Applying standard central differences, (2a) and (2b) become

\[ U_{n+1} - U_n = \Delta t \, V_{n+1/2}, \quad (3a) \]

\[ M(V_{n+3/2} - V_{n+1/2}) = \Delta t (P_{E,n+1} - P_{I,n+1}), \quad (3b) \]

where

\[ P_{I,n+1} = P_{I,n} + \int \int B^T \Delta \sigma \, dV, \quad (4) \]

and

\[ \Delta \sigma = \sigma_{n+1} - \sigma_n = \text{stress increment due to displacement increment} \ (U_{n+1} - U_n). \quad (5) \]

Stress increment \( \Delta \sigma \) is obtained by integrating the general rate equation

\[ \Delta \sigma = \int_{\epsilon_n}^{\epsilon_n+\Delta \epsilon} D \, d\epsilon, \quad (6) \]

where \( D \) = the material rigidity matrix, and \( \epsilon \) = strain-vector. Some form of numerical approximation is required in order to evaluate this integral. The strain increment \( \Delta \epsilon \) is obtained from

\[ \Delta \epsilon = B(U_{n+1} - U_n) \quad (7) \]

and is evaluated at each interior element integration point. Since geometric nonlinearities are not accounted for, \( B \) contains only prescribed spatial functions. Substituting (4) into (3b) results in

\[ V_{n+3/2} = V_{n+1/2} + M^{-1} \Delta t \left( P_{E,n+1} - P_{I,n} - \int \int B^T \Delta \sigma \, dV \right). \quad (8) \]

Fig. 1. Staggered time integration.
penetration with no damage to the missile. The contact force between the missile and the target is controlled by the strength of the target. Intermediate missiles produce a combination of the hard and soft missile behavior. For typical concrete properties involved in most nuclear structures, a wooden pole would be classified as a soft missile and a steel rod as hard. A steel pipe would be considered as an intermediate missile.

3.1. Current practice

It has been traditional to treat the local and global effects separately. The local behavior in the form of missile buckling, fragmentation, or penetration, however, controls the interface pressure between the missile and target. The basic strategy adopted in both the United States and Europe, to deal with local effects, has been somewhat empirical in nature. The results of wartime ballistics have been mainly used for penetration/spalling phenomena. The empirical expressions used for predicting local effects have been reported in detail [9]. A wide variety of force—time curves exist for the various formulae. It is apparent that these expressions were developed mainly from data analysis and little or no thought was given to basic mechanics. The computation of global structural effects is generally based on two approaches: (1) the definition of a force-time curve and (2) classical “billiard ball” impact theory (generally perfectly elastic). The generation of a force—time curve permits one to proceed with the structural analysis in a consistent fashion. The methods used for generated or assumed force—time curves ranged from simple one-degree-of-freedom systems as in [10] to various finite element methods [11,12]. An alternative method used in [10] is to set the initial velocity conditions on the target using classical plastic impact theory. This method has very limited application owing to its inherent simplicity.

It is possible to define a force—time curve consistent with target penetration or missile fragmentation, at least in principle. This approach allows a more unified treatment of local effects and structural response. Alternatively, given a force—time curve, the local effects can be backfigured for hard and soft missiles.

The intermediate missile is treated as a hard missile while penetrating and as a soft missile while deforming. In conjunction with the research for nonlinear dynamic
Interfacing of the local impact analysis program with the global dynamic response analysis program.

- **Structural Input**
- **Missile Impact**
- **Dynamic Solver**

**Program Mode**
- **Missile Impact Solver**
- **Dynamic Solver**

**Data Check**
- **Stop**

**Structural Input:** Read, check, store all R/C information
**Missile Impact Input:** Read, check, store all missile data
**Dynamic Solver:** Analyze R/C member
**Missile Impact Solver:** Provide contact force of missile/target to dynamic solver. Obtain correct missile/target interface velocity

Fig. 5. Interfacing of the local impact analysis program with the global dynamic response analysis program.

Analysis of reinforced concrete, work is in progress at Massachusetts Institute of Technology to analyze the impact of a deformable missile on a target wall. The missile is modeled as a one-dimensional assemblage of a number of lumped masses connected by bilinear springs, and the target is modeled as a single mass, as shown in fig. 4. A one-dimensional elasto-plastic wave mechanics model is used for the local impact effects. This model provides the contact duration and the forcing function. The objective of the model is to describe the local behavior and provide relevant data or input into a global analysis. For global analysis, this model is interfaced with the structural dynamic analysis program as illustrated in fig. 5. Preliminary results indicate that the following parameters are of interest in the missile impact process: the impact velocity, the yield/crushing strength ratio of missile and target, the missile length, the target thickness, and the missile density and stress–strain law. The stress–strain law is especially important for global response of the target.

4. Modeling of concrete behavior

4.1 Spatial discretization

In view of the size and computational efforts involved in a three-dimensional nonlinear finite element analysis, isoparametric elements, fig. 6, represent a feasible choice for modeling of concrete. The use of these elements allows an efficient treatment of nonlinear concrete behavior across the element at specified integration points. The performance of this quadratic element, which has twenty nodes, with respect to accuracy, efficiency, and economy has been reported [13,14]. These elements are known to result in slightly stiff behavior. However, this stiffening effect can be decreased by reducing the order of numerical integration [4,15]. The use of a suitable number of integration points produces an efficient scheme for tracing large stress gradients and cracking across the element.

4.2 Stress–strain law in compression

The law of material behavior plays a key role in the nonlinear analysis of reinforced concrete. Realistic

Fig. 6. Twenty-node solid finite element.
constitutive laws for dynamic analysis must be formulated. Even though increasing strain rates, the concrete strength increases slightly, the ultimate compressive deformation of concrete may be substantially reduced. In turn, the corresponding dynamic relations between internal stresses and deformations may be disadvantageous to the overall structural strength. Unfortunately, there is at present no verified knowledge on multiaxial dynamic strength and deformation behavior of concrete. Therefore, the results of static tests have to be used in estimating the dynamic material behavior. In the static regime, several alternative ways of defining the stress–strain law for concrete have been explored. Most effort has been directed to the biaxial case because of the availability of carefully obtained test data. Nonlinear elastic relations have been proposed in [16,17] for the biaxial case and in [18] for the triaxial one. Using a nonlinear uniaxial stress–strain relationship and assuming that concrete behaves as an orthotropic material, Sarne [5] developed similar procedures for the triaxial case. In this approach, the triaxial stress–strain relationship is established by organizing around an equivalent uniaxial stress–strain curve for which the ultimate compressive strength is a function of the stresses in the other two principal directions. For the ascending portion of the compressive uniaxial curve, the expression proposed by Saenz [19] is used, and for the descending segment a straight line is used, (fig. 7). In most recent studies [20,21] it has been shown that the biaxial deformation behavior of concrete can be predicted through an elasto–plastic strain-hardening model. Shape of the initial yield surface, appropriate hardening rules and subsequent loading surfaces must then be assumed. Extension of this approach to the triaxial case requires further experimental investigations in order to confirm the assumptions regarding loading surfaces, unloading, and normality conditions.

4.3. Failure surface

In the literature, numerous triaxial compressive failure criteria have been suggested [22]. In this case the range of discrepancy is much greater than for the biaxial case. More experimental investigations are needed for a better understanding of the behavior. The criterion suggested in [23] involves utilization of stress invariants. The proposed failure surface equation is

$$I_1^{1/2} + \frac{n-1}{2n-1} I_1 + \frac{n}{2n-1} f_c' = 0$$

(15)

where \( n \) = ratio of the biaxial compressive strength to the uniaxial strength (≈1.25), \( I_1 = \sigma_1 + \sigma_2 + \sigma_3, \) \( I_2 = \frac{3}{2} \left[ (\sigma_1 - \bar{\sigma})^2 + (\sigma_2 - \bar{\sigma})^2 \right], \) \( \sigma = (\sigma_1 + \sigma_2 + \sigma_3)/3, \) and

![Fig. 7. Uniaxial stress–strain curve for concrete.](image)

![Fig. 8. Triaxial compressive failure surface.](image)
Eq. (8) along with (3a), (6) and (7) constitute the complete set of equations needed to implement the ST method.

It is noted that (3a) and (8) are established at a system level and eqs. (6) and (7) are at an element level. The ST method is an explicit scheme. Thus there are no iterations at the system level as a result of the nature of the time operator. All computations required to solve for the vector $\int \int \int B^T \Delta \sigma \, dV$ are done at the element level using eq. (6), thereby eliminating the need for constructing and updating a total (global) stiffness matrix at each time step. This direct calculation of the internal forces requires much fewer computations and reduces core requirements in the computer.

2.2. Time step

For the central difference type of operator employed here, the customary approach to insure numerical stability of the solution is to limit the time step $\Delta t$ by the linear von Neumann criteria

$$\Delta t \leq \min \alpha \frac{l}{C_L},$$

where $\alpha$ = reduction coefficient varying between 0.2 and 0.9, $l$ = an element dimension ($l_x, l_y, l_z$), $C_L =$ pressure wave velocity in the material with upper limit $= (E/\rho)^{1/2}$, where $E =$ instantaneous modulus of elasticity and $\rho =$ material density. The upper limit of $C_L$ equals the propagation velocity of a one-dimensional longitudinal tension wave. The use of $\alpha$ (0.2 to 0.9) is necessary due to the destabilizing effects of round-off errors and rapidly varying material properties. The ratio ($l/C_L$) represents the time required for a pressure wave to traverse an element and obviously the element with the smallest traversal time should be used in determining the time step.

The ST method, like the conventional central difference method, is not self-starting. An Euler method scheme may be employed to start the solution. For the first step

$$U_{n+1} - U_n = h_E V_n,$$  \hspace{1cm} (10a)

$$V_{n+1} = V_n + M^{-1} h_E [P_{E,n} - P_{I,n-1} - \int \int \int B^T \Delta \sigma \, dV].$$  \hspace{1cm} (10b)

where $h_E =$ the Euler time step and $M =$ mass matrix. There is no $P_I$ term at $t = 0$ because of static equilibrium in the structure. For the second and all subsequent Euler time steps, the difference equations are

$$U_{n+1} - U_n = h_E V_n,$$  \hspace{1cm} (11a)

$$V_{n+1} = V_n + M^{-1} h_E [P_{E,n} - P_{I,n-1} - \int \int \int B^T \Delta \sigma \, dV].$$  \hspace{1cm} (11b)

Figs. 2 and 3 show how the Euler and ST schemes are meshed and the computational flow.

2.3. Mass matrix

As shown by Krieg and Key [7], from the viewpoint of both accuracy and computational efficiency, the lumped mass approach is preferable to the consistent mass when an explicit time integration scheme is used.

Among numerous possibilities for deriving lumped masses, a recommended procedure is that offered by Hinton et al. [8]. This procedure consists of computing the diagonal terms of the consistent mass matrix:

$$M^e = \int \int \int_{\psi} \rho^e N^T N \, dV^e,$$  \hspace{1cm} (12)

where $M^e =$ element consistent mass matrix, $\rho^e =$ element material density and $N =$ element interpolation function. Then, these terms are scaled so as to preserve the total mass of the element.

3. Missile impact considerations

The effects of impacting missiles on reinforced concrete targets are classified as local and global. Local response includes penetration, perforation and spalling. Damage is confined to a limited area, essentially confined to a local region around the impacted zone on both the front and back faces. The structural response is concerned with the overall flexure, shear, ductility, and reactions involved. A basic classification of missile types has been given in [9]. The missiles can be considered in three broad groups defined by the combined missile-target interface behavior. Soft missiles involve damage to the missile without front face penetration of the target. The contact force between the missile and target is controlled by the strength of the missile. Hard missiles involve target
\( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are the principal stresses. Any compressive state of stress which is on or outside the surface, fig. 8, represents a compressive failure. Theoretically, if the deviatoric stresses are zero (hydrostatic stress state), there will be no compressive failure.

The brittle behavior of concrete in tension can be modeled in the analysis by reducing the coefficients in the material property matrix as they relate to the stiffness in the direction of the principal tensile stress when that stress exceeds the selected strength criterion. A concrete element is checked for tensile failure at each of its integration points where stress and strain history is kept. Thus, partial cracking of the element is possible.

The assumed tension failure surface for the triaxial stress state is shown in fig. 9. As seen, the tensile strength decreases linearly with increasing compressive stress in a perpendicular direction. No strength change is assumed in the triaxial tension zone. The Young’s modulus for concrete in tension is taken as a constant equal to the initial elastic value.

Failure due to a principal tensile stress leads to a crack which is perpendicular to that stress direction. In this case, no normal stress in tension acting in the plane of the crack is admitted and a biaxial situation is created. Additional cracking can take place, causing up to two planes of cracks, or failure in compression can appear.

In a recent study [24], the stiffness of the element has been decayed over some strain level at the cracked point, rather than reducing the stiffness immediately to zero in the direction normal to the crack. This softening rather than assuming a sharp break has been rationalized on the grounds of an anticipated statistical variation in material strengths so the cracking will not completely sever the element at one time. After the first crack appears, the concrete is orthotropic, the direction being determined by the crack plane. Then the shear modulus becomes an independent value. Tests indicate that the shear resistance of cracked concrete is not negligible [25,26]. However, further experimental research is needed to establish a basis for accurately modeling the role of dowel action and aggregate interlock in determining this shear resistance. The popular approach, at present, is to employ a reduced shear modulus based on experiments [26] to represent the shear resistance of the cracked concrete.

4.4. Triaxial unloading analysis

In modeling the concrete behavior, unloading and reloading is checked and followed for each principal direction separately. For this, concrete in compression is treated differently from concrete in tension. For compressive behavior, determination of unloading or reloading is based on a comparison in signs of the current incremental strain and the previous total accumulated strain for each principal strain direction. Stress is not a unique quantity on the uniaxial stress—strain curve for concrete. Once unloading is initiated in a principal direction, the initial Young’s modulus is used (elastic unloading/reloading), and the direction cosines are fixed until reloading returns the concrete to the stress—strain state which existed in that principal direction when unloading began. If only one principal direction is unloading, the unit vectors for the two loading directions are then constrained to be in a plane perpendicular to the fixed unloading direction. If two directions are unloading, then all three directions are fixed during the unloading/reloading phase. Unloading and reloading of cracked concrete in tension is treated in the following way: if the strain across a crack is still positive, the crack is considered to be open and no rigidity is accounted across the crack at that integration point. The crack is closed when the strain becomes negative and concrete rigidity in com-
pression is calculated for the negative strain. Although the crack is closed and the concrete is in compression, this principal direction remains fixed. Repeated material unloading and reloading can cause strength losses in concrete structures and this may have a significant effect on design for earthquakes. However, for the analysis of response to loads with one distinct maximum value in the solution time history such as blast and impact forces, these strength losses may be insignificant.

5. Modeling of steel reinforcement

The idealization of the behavior of steel reinforcement as an elastic—perfectly-plastic material is quite acceptable. If strain hardening is introduced, an experimental stress—strain diagram is needed. When modeling reinforcement layers, the thin shell elements, fig. 10, may be used to represent either a single one-directional reinforcement layer or two layers of reinforcement with axial stiffness in orthogonal directions. When plasticity is initiated in one direction, this need not influence the orthogonal direction. If desired, the scissors action that the concrete between the steel grid feels may be included.

6. Research needs

The results of a nonlinear analysis of reinforced concrete under extreme dynamic loads are influenced by many factors including assumed loading conditions, time and spatial discretization, and assumed material behavior. Review of the literature indicates that the application of numerical analysis methods to this type of problem is still in the early stages of development.

Extensive research is needed to provide better capability for response predictions. In the following, several major research areas are discussed.

6.1. Discretization

The surest and most direct means of computation are required in practical calculations where finite time and computer resources are available.

Independent examinations of the error in spatial discretization and the accuracy of time integrators may not give the information needed for a choice. Various combinations must be examined since the numerical results are a product of the two. For example, explicit, conditionally stable, central difference time integration produces efficient computation of transient response when used with a diagonal mass matrix for the finite element formulation [7]. However, alternative combinations should be explored to improve the accuracy, for example, a non-diagonal mass representation that lies between the consistent mass and lumped mass results.

In the use of explicit temporal operators, which are desirable for rapid loading, and large nonlinearities, great care must be taken to avoid numerical instabilities. Numerical instabilities may be detected after a solution is completed by large, obviously spurious growths in velocities. However, in dissipative materials this is not always the case. The energy generated by an instability may rapidly be dissipated as the material becomes nonlinear, and since the stability limits on the time step for plastic elements are far greater, the solution reaches stability. This phenomenon usually results in an erroneous solution. An energy balance procedure has been proposed [27] in order to guard against this problem. Numerical means should be developed to incorporate such procedures in computer programs employing explicit time integration techniques.

It is desirable to study the problem-dependent aspects of nonlinear dynamic analysis. For this, it is necessary to develop numerical procedures capable of performing analyses with combinations of explicit or implicit integration schemes with various formulations for masses.

6.2. Impact effects

For structural response, a spectrum of contact force—
time curves is required between the missile and the target. The significant parameters are the shape, peak load and duration of the load. The sensitivity of a given structure to these parameters can be examined. The peak load is controlled by either the missile strength or the target strength and the impact duration. Some studies on the influence of pulse peak loads, durations and shapes for various structures have been described [28,29]. This approach has distinct promise in the provision of design standards for structural response.

A question of major importance is a suitable description of concrete behavior under sudden loading, as occurs in the impact problem. Results of studies have been reported [30]. Some results of analytical studies on back face spall behavior have recently been reported [31].

Studies are currently in progress at Massachusetts Institute of Technology concerning the structural behavior of the target. Initially it is proposed that the currently available experimental data be analyzed for verification purposes. Of major interest is the flow of momentum through the target and the resulting bending moments, shears, and reactions. These results should then be applicable in the provision of simplified methods of direct use in design.

6.3. Material behavior of concrete

The results of a dynamic analysis and therefore the design of structural members can be substantially influenced by the way and the extent to which the nonlinear material behavior is incorporated. Economic advantages can be gained from stress redistribution as well as from the higher capacity of energy absorption due to plasticity. At present, experimentally verified complete knowledge on the dynamic behavior of reinforced concrete is not available.

Definitions of behavior of reinforced concrete is subjected to bending and shear up to the point of failure are needed considering high rates of strain variations. Realistic constitutive laws for dynamic problems have to be formulated.

Current procedures make use of static test results for the definition of constitutive laws for dynamically loaded reinforced concrete structures. The degree of validity of such assumptions has yet to be verified experimentally. Basic experimental research necessary to provide needed information on dynamic behavior of reinforced concrete should include (1) constitutive relations and failure criteria for concrete under multiaxial stress states, (2) bond slip relations, (3) aggregate interlocking behavior and dowel action, and (4) temperature and long-term effects. Formulation of such information in a suitable form for use in the analytical techniques is essential. Parametric sensitivity studies must be initiated to determine the significance of various factors involved in establishing constitutive laws, cracking and shear transfer mechanisms, stress reversals, influence of reinforcement, etc.

7. Conclusion

The current status of research is presented for the multidimensional nonlinear dynamic analysis of reinforced concrete structures as well as the solution strategy for the nonlinear dynamic equations. Impact considerations are given, and modeling of nonlinear behavior of concrete is discussed. Recommendations are made for further research.

The review of the reported studies indicate that, at present, the existing nonlinear dynamic analysis capabilities for reinforced concrete are not sufficiently developed to account for realistic laws of material behavior. Experimental research is required on the dynamic strength and deformation behavior of concrete to provide information for the development of improved analytical solutions. Formulation of such information in a suitable form is essential. Analytical sensitivity studies must be performed to understand and interpret the observed behavior from experiments.

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Nomenclature

\[
\begin{align*}
B & = \text{strain-displacement transformation matrix} \\
C \text{L} & = \text{pressure wave velocity}
\end{align*}
\]
\[ D \] = material rigidity matrix
\[ h_E \] = Euler time step
\[ I_1, I_2 \] = first and second stress invariants
\[ l \] = element dimension
\[ M \] = mass matrix
\[ N \] = element interpolation function
\[ P_E \] = consistent external nodal force vector
\[ P_I \] = consistent internal nodal force vector
\[ U \] = displacement vector
\[ \dot{U} \] = velocity vector
\[ \ddot{U} \] = acceleration vector
\[ V \] = volume
\[ \dot{V} \] = velocity vector
\[ \sigma \] = stress vector
\[ \sigma \] = mean stress
\[ \sigma_1, \sigma_2, \sigma_3 \] = principal stresses
\[ \varepsilon \] = strain vector
\[ \alpha \] = reduction coefficient
\[ \rho \] = material density
\[ \Delta t \] = time step
\[ \Delta \sigma \] = vector of stress increment
\[ \Delta \varepsilon \] = vector of strain increment

References