INTRODUCTION

In most practical situations torsional loads on reinforced concrete beams are accompanied by bending moments and transverse shear forces. A review of the literature indicates that the majority of research in this area has been focused on ultimate strength analysis and design. Nevertheless, it is generally understood that short-time load-deflection characteristics and precracking and postcracking stiffness (or compliance) are important under service conditions. A good example of such a situation is the Torsional Guideway Transit (TGT) system that is presently being investigated by the first and the third writers (4). In the TGT system, a single beam elevated structure is employed for the two-way passage of automated guideway transit vehicles as shown schematically in Fig. 1. The guideway of such a system is loaded in torsion as well as transverse shear and flexure. The ride quality and the guideway dynamic loading, both of which are functions of the total RMS (root mean square) acceleration components at the centroid of vehicle compartment, therefore depend on the torsional and flexural stiffness of guideway. This relationship is shown in Figs. 2(a) and 2(b) for a typical TGT system (4) where it is shown that as the torsional and flexural stiffness increase, vehicle RMS acceleration levels decrease. On the other hand, cost, weight, and topological considerations impose limitations to guideway stiffness. The benefits of closed sections for TGT guideways have been covered in Ref. 4. Furthermore, guideway cost considerations often favor concrete box sections.

The investigation of the postcracking stiffness may be considered more significant than that of the precracking stiffness for two main reasons: (1) The precracking stiffness can be determined with relative ease using elastic theory

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for concrete; and (2) the contribution of the reinforcement of a reinforced concrete beam to the precracking stiffness is negligible. Among the few research papers dealing with the postcracking stiffness of reinforced concrete beams is the work by Hsu (10,11). This experimental investigation considered pure torsional loads on rectangular sections and showed that the postcracking torsional stiffness is approximately a linear function of the total reinforcement ratio. An analytical study useful for dynamic modeling of the TGT guideway is that reported by Lampert (13). He obtained simple expressions for the postcracking stiffness (strictly speaking, the reciprocals of postcracking compliance) of reinforced concrete box sections under combined torsion and bending using a space truss model. On all four sides of the member, 45° helical cracks were assumed to form. The results were verified by tests. A similar analysis was made by Ojah (15), for combined torsion, bending, and shear. Helical crack angle was introduced as a parameter and again the space truss analogy was utilized. Deflections were obtained using Castigliano's energy principle.

While the analytical results generally agree with experimental data, the theoretical analysis of Lampert (13) is based on the assumption that 45° spiral cracks of identical nature form on all four sides of the member at yielding of reinforcement. This assumption facilitates the application of the truss analogy. However several experimental studies, including those by Pandit and Warwaruk (16), Gvozdev, et al. (8), Goode and Helmy (6), and Collins et al. (2) have
indicated that when an external bending moment is present, the helical cracks due to torsion appear only on three sides, whereas a compression zone forms on the fourth side at yielding of the reinforcement of underreinforced members. Hsu (10) observed such compression zones even for pure torsion. The failure theory developed assuming a compression zone on one side, as if bending takes place about an axis making a skew angle less than 90° with the beam axis, is commonly known as the skew bending theory (or ultimate equilibrium method). This theory is credited to Lessig (14,3) and Gvozdev (7,3). The publications related to this theory indicate that this is the most widely accepted failure

FIG. 3.—Definition of Postcracking Compliance (Single-Degree-of-Freedom Case)

FIG. 4.—Reinforcement Details

FIG. 5.—Failure Mode Under Small Transverse Shear

theory for reinforced concrete rectangular sections under combined torsion, bending, and transverse shear.

Lampert's work (13) is significant in that his expressions for postcracking compliance are simple and convenient for implementation in analytic models. However, the assumption with regard to crack formation and the fact that the study did not include the effects of transverse shear in the postcracking model limit the applicability of the results. In the present work, the skew bending theory is used to establish expressions for postcracking compliance of rectangular
reinforced concrete beams subjected to combined torsion, flexure, and transverse shear. Initially the transverse shear is assumed small. Next the problem is approached assuming that the transverse shear is dominant and it influences the failure mode.

**POSTCRACKING COMPLIANCE ANALYSIS**

**Postcracking Stiffness and Compliance.**—In this section rational definitions for postcracking stiffness and postcracking compliance are given. The ambiguity of the terminology is minimal in single degree-of-freedom situations, which are considered first. If a gradually increasing load (torque $T$, bending moment $M$, or transverse shear force $V$) is applied to a reinforced concrete member and the deflection [twist per unit length $d\theta/dx$, curvature $d\psi/dx$, or transverse strain $\epsilon$ (Strain referred to is the average strain of vertical stirrups at a section. This is caused by the stirrup forces which in turn support the shear force at the section under cracked conditions.)] at the point of application is measured until the final failure occurs, the general shape of the corresponding load-deflection curve is sketched in Fig. 3. It is commonly assumed that such members are underreinforced so that yielding of the tensile reinforcement occurs prior to concrete crushing in compression zones. The same assumption is made in the present work. A linearized (in "secant" sense) model, shown by a broken line in Fig. 3, may be used to describe the postcracking behavior of the member between the point of initial cracking (point A in Fig. 3) and the point of initial yielding of any portion of the reinforcement (point B in Fig. 3). In this postcracking region the reinforcement obeys Hooke’s Law. The definitions of postcracking stiffness $K$ and postcracking compliance $L$ are given in Fig. 3.

Under combined loading (multidegree-of-freedom), the linearized postcracking model can be generalized as

\[
\begin{bmatrix}
\frac{d\theta}{dx} \\
\frac{d\psi}{dx} \\
e
\end{bmatrix}
= \begin{bmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
\begin{bmatrix}
T \\
M \\
V
\end{bmatrix}
\]  
\hfill (1)

The inverse relationship is

\[
\begin{bmatrix}
T \\
M \\
V
\end{bmatrix}
= \begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{bmatrix}
\begin{bmatrix}
\frac{d\theta}{dx} \\
\frac{d\psi}{dx} \\
\epsilon
\end{bmatrix}
\]  
\hfill (2)

The compliance matrix and the stiffness matrix are assumed nonsingular. Then

\[
\begin{bmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
= \begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{bmatrix}^{-1}
\]  
\hfill (3)
In these expressions \( L_{ij} \) are the postcracking compliance terms and \( K_{ij} \) are the postcracking stiffness terms. From Eq. 3 it is clear that, in general, \( K_{ij} \neq 1/L_{ij}, \) \((i, j = 1, 2, 3)\). Also, as a direct consequence of Maxwell’s principle, the stiffness and compliance matrices are symmetric, i.e., \( K_{ij} = K_{ji} \) and \( L_{ij} = L_{ji} \).

**Compliance Relations for Small Transverse Shear (Case 1).**—Reinforcement details of the beam considered in the present work are given in Fig. 4. The terms \( A_s' \) and \( A_s \) are the top and bottom longitudinal reinforcement areas, respectively. The value \( A_h \) is the cross-sectional area of each closed stirrup. The longitudinal spacing between stirrups is \( s \). The height and the width of each stirrup loop are \( h_o \) and \( b_o \), respectively.

Consider a \((T, M, V)\) combination that causes cracking of concrete and the initiation of yielding of the reinforcement. This corresponds to point B in Fig. 3 in the single-degree-of-freedom case. In employing the skew bending theory, it is assumed that under combined torsion, flexure and transverse shear, the compression zone forms on the same side as that for pure bending. An exception to this situation is the case in which the transverse shear force is relatively large. This case is considered in the next section. For small transverse shear, the assumed failure mode is shown in Fig. 5 for the indicated load configuration. This failure mode was also considered by other workers (5,17,1) and verified by tests (6,2,18). Further assumptions made with respect to geometry of the failure surface are: (1) The crack angle is 45°; and (2) the height of the compression zone is small at yielding of the reinforcement.

Tests by Kemp (12) showed that assumption 1 is valid for moment/torque ratios as high as 2.8. Assumption 2 was verified and employed by many investigators (5,17). Since the height of the compression zone is small it is reasonable to assume that the compression force \( Z_t \) in the top longitudinal reinforcement, resultant concrete force \( C \) in the compression region, and the net force \( B_z \) in top stirrup legs crossing the compression region, act approximately at the same level \( h_o \) from the bottom longitudinal reinforcement. Force vectors \( Z_t \) and \( C \) form angle \( \beta \). The tensile force in the bottom longitudinal reinforcement is denoted by \( Z_b \). Let \( B_1, B_2, \) and \( B_3 \) be the net forces of all stirrup legs intercepting cracks on sides 1, 2, and 3 as shown in Fig. 5. It is usually assumed that the stirrup stress is constant along the entire hoop (5,13). As a consequence

\[
\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
\]

\( \sigma_i \) and \( \epsilon_i \) are the stress and the strain averaged along side \( i \) in Fig. 5. At cracks there are abrupt transfers of stress from concrete to steel. Due to slip of the reinforcement associated with cracking it is reasonable to assume that for a particular stirrup hoop, the tensile stress in stirrup legs is approximately uniform in the three cracked sides. This tensile stress is, in turn, transferred to the ends of the stirrup leg on the compression side of concrete independent of the presence of this compression zone. However, the tensile in steel is not uniform along this leg unless the bond stresses between steel and concrete are small and the stirrup tension is large. The full Eq. 4 is redundant in the present work. However, where necessary a part of the information available from this equation is used.

Force and moment balance (equilibrium) about centroidal point \( P \) of the small element shown in Fig. 5 can be expressed as
\[ Z_b - Z_t - C \cos \beta = 0 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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\[ Z_t = -A'_s \sigma'_s \]  
\[ Z_b = A_s \sigma_s \]  
\[ (19) \]
\[ (20) \]

In these expressions \( \sigma'_s \) and \( \sigma_s \) are the top and bottom longitudinal reinforcement stresses and \( \epsilon'_s \) and \( \epsilon_s \) are the corresponding strains.

An approximate expression for the shear strain \( \gamma \) on each side of the beam in terms of longitudinal and transverse strains \( \epsilon_l \) and \( \epsilon_h \) may be given as (13)
\[ \gamma = \epsilon_l + \epsilon_h \]  
\[ (21) \]

For a hollow section the angle of twist per unit length can be expressed by
\[ \frac{d\theta}{dx} = \frac{1}{2b_o h_o} \phi \gamma ds \]  
\[ (22) \]
in which \( ds \) is an elemental length along the perimeter of the beam section.

![Diagram](image)

**FIG. 6.—Torsion-Bending Interaction Diagram at Yielding of Reinforcement**

and \( \phi \) represents the integration operator along this closed path (13,9). It is indicated that this expression can also be used as an approximation for solid sections. For the present problem, Eq. 22 becomes
\[ \frac{d\theta}{dx} = \frac{1}{2b_o h_o} \left\{ h_o \left[ \epsilon_1 + \frac{1}{2} (\epsilon_s + \epsilon'_s) \right] + h_o \left[ \epsilon_2 + \frac{1}{2} (\epsilon_s + \epsilon'_s) \right] + b_o (\epsilon_3 + \epsilon_s) + b_o (\epsilon_4 + \epsilon'_s) \right\} \]  
\[ (23) \]

By the use of Hooke’s law for steel and Eqs. 12, 15–20 one obtains
\[
\frac{d\theta}{dx} = \frac{(b_o + h_o)}{2b_o h_o E_s} \left[ \frac{s}{h_o A_h} (B_1 + B_2) + \frac{Z_b}{A_s} - \frac{Z_t}{A_s'} \right] \quad (24)
\]

in which \( E_s \) is the Young’s modulus for steel. It has been assumed that \( \epsilon_s \) and \( \epsilon_s' \) are equal, which is clear from Eq. 4. In view of this and Eqs. 17 and 18

\[
B_4 = \frac{(2h_o + b_o)}{b_o} B_3 \quad (25)
\]

It can be deductively verified that Maxwell’s reciprocity principle is satisfied if and only if \( \cot \beta = 1 \). This, when used in conjunction with Eqs. 5, 6, 12, 25 and 13 results in

\[
Z_b - Z_t = \frac{(b_o + h_o)}{b_o h_o} T \quad (26)
\]

On substituting Eq. 14 into 26

\[
Z_t = \frac{M}{h_o} - \frac{1}{2} \frac{(b_o + h_o)}{b_o h_o} T \quad (27)
\]

At this point, Eqs. 13, 14 and 27 are substituted into Eq. 24 to obtain the result

\[
\frac{d\theta}{dx} = \frac{(b_o + h_o)}{2b_o h_o^2 E_s} \left\{ \left[ \frac{s}{b_o A_h} + \frac{1}{2} \left( 1 + \frac{h_o}{b_o} \right) \left( \frac{1}{A_s'} - \frac{1}{A_s} \right) \right] T \right.
\]

\[
- \left. \left( \frac{1}{A_s'} - \frac{1}{A_s} \right) M \right\} \quad (28)
\]

The beam curvature may be given by the well known expression

\[
\frac{d\psi}{dx} = \frac{1}{h_o} (\epsilon_s - \epsilon_s') \quad (29)
\]

Hooke’s law for steel reinforcement is now used in conjunction with Eqs. 19, 23, 16 and 33 to obtain

\[
\frac{d\psi}{dx} = \frac{1}{h_o^2 E_s} \left[ - \frac{1}{2} \left( 1 + \frac{h_o}{b_o} \right) \left( \frac{1}{A_s'} - \frac{1}{A_s} \right) T + \left( \frac{1}{A_s'} - \frac{1}{A_s} \right) M \right] \quad (30)
\]

For the purpose of comparison with the results given in Ref. 13, the following parameters are defined:

\[
r = \frac{A_s'}{A_s} \quad (31)
\]

\[
u = 2(b_o + h_o) \quad (32)
\]

\[
m = \frac{s}{u A_h} (A_s' + A_s) \quad (33)
\]
\[ \ddot{k} = \frac{2b_o}{(b_o + h_o)} \quad \quad \quad \quad \quad \quad \quad (34) \]
\[ \bar{L}_M = \frac{2}{(h_o^2 A_s E_s)} \quad \quad \quad \quad \quad \quad \quad (35) \]
\[ \bar{L}_T = \frac{us}{4b_o^2 h_o^2 A_h E_s} \quad \quad \quad \quad \quad \quad \quad (36) \]

These parameters are substituted into Eqs. 28 and 30 to obtain the postcracking compliance vector-matrix equation,

\[
\begin{bmatrix}
\frac{d\theta}{dx} \\
\frac{d\psi}{dx}
\end{bmatrix}
= \begin{bmatrix}
\bar{L}_T & -\bar{L}_M \\
\bar{L}_M & \bar{L}_T
\end{bmatrix}
\begin{bmatrix}
T \\
M
\end{bmatrix}
\quad \quad \quad \quad \quad (37)
\]

This expression is identical to that proposed by Lampert (13) for a limited range corresponding to large \( T/M \) ratios. However, due to a more realistic failure mode adopted in the present approach it is shown that the preceding expression may be applied to a wider range of loading as covered in a later section.

**Compliance Relations for Large Transverse Shear (Case 2).**—The assumed failure mode for rectangular reinforced concrete beams under combined torsion, flexure, and large transverse shear consists of a compression zone on one vertical side and helical cracks on the remaining three sides (Fig. 7). This failure mode is a commonly accepted one and has been verified by several experimental studies (2,5,17). This failure mode occurs because the diagonal tensile stresses due to torsion and transverse shear are additive on one vertical side and subtractive on the other. While the former side cracks in tension, a compression zone forms on the latter. The same assumptions cited previously are made here with respect to the crack angle and the size of the compression zone.

The equilibrium equations about the centroid of the small element shown in Fig. 7 are

\[ Z_b - Z_t - C \sin \beta = 0 \quad \quad \quad \quad \quad \quad \quad (38) \]
\[ B_3 - B_4 = 0 \quad \quad \quad \quad \quad \quad \quad (39) \]
\[ B_1 + B_2 - C \cos \beta - V = 0 \quad \quad \quad \quad \quad \quad \quad (40) \]
\[ M - \frac{h_o}{2} Z_b - \frac{h_o}{2} Z_t = 0 \quad \quad \quad \quad \quad \quad \quad (41) \]
\[ T + \frac{b_o}{2} B_1 - \frac{b_o}{2} B_2 - \frac{h_o}{2} B_3 - \frac{h_o}{2} B_4 - \frac{b_o}{2} C \cos \beta = 0 \quad \quad \quad \quad \quad \quad \quad (42) \]
\[
\frac{1}{2} (b_o + h_o) B_3 + \frac{1}{2} (b_o + h_o) B_4 - \frac{b_o}{2} C \sin \beta = 0 \quad \ldots \quad (43)
\]

As in the previous case, an expression for \( T \) in terms of stirrup forces may be derived as follows. Eqs. 16, 17, 19 and 20 still hold but Eqs. 15 and 18 have to be replaced by

\[
B_1 = \frac{(2b_o + h_o)}{s} A_h \sigma_1 \quad \ldots \quad (44)
\]

\[
B_4 = \frac{b_o}{s} A_h \sigma_4 \quad \ldots \quad (45)
\]

The results analogous to Eqs. 12 and 25 in the previous case are

\[
B_2 = \frac{h_o}{2b_o} (B_3 + B_4) \quad \ldots \quad (46)
\]

\[
B_1 = \frac{(2b_o + h_o)}{h_o} B_2 \quad \ldots \quad (47)
\]

Elimination of \( C \) in Eqs. 42 and 43 and substitution of \( B_1 \) and \( B_2 \) from Eqs. 46 and 47 result in

\[
T = h_o (B_3 + B_4) \quad \ldots \quad (48)
\]

As before, the Maxwell's reciprocity principle is satisfied if and only if \( \cot \beta = 1 \). This fact was used in the derivation of Eq. 48. Now, from Eqs. 38, 43, 58, and 41

\[
Z_b = \frac{M}{h_o} + \frac{(b_o + h_o)}{2b_o h_o} T \quad \ldots \quad (49)
\]

\[
Z_t = \frac{M}{h_o} - \frac{(b_o + h_o)}{2b_o h_o} T \quad \ldots \quad (50)
\]

Note that these equations are identical to the relationships derived for the previous case. Eqs. 48–50 in conjunction with 17, 45, and 48 are now substituted in Eqs. 23 to obtain

\[
\frac{d\theta}{dx} = \frac{1}{2b_o h_o E_s} \left\{ 2h_o E_s \varepsilon + \left[ \frac{s}{h_o A_h} + \frac{(b_o + h_o)^2}{2b_o h_o} \left( \frac{1}{A'_s} - \frac{1}{A_s} \right) \right] T \\
- \frac{(b_o + h_o)}{h_o} \left( \frac{1}{A'_s} - \frac{1}{A_s} \right) M \right\} \quad \ldots \quad (51)
\]

Hooke's law for steel has been incorporated in this relationship.

An approximate expression for the average transverse strain \( \varepsilon \) is

\[
\varepsilon = \frac{s (B_1 + B_2)}{2 h_o A_h E_s} \quad \ldots \quad (52)
\]

in which \( B_1 + B_2/2 \) is the average vertical stirrup force (Fig. 7). In view
of Eqs. 40, 43, and 48

\[ B_1 + B_2 = V + \frac{T}{b_o} \] .................................................. (53)

From this expression the interaction equation at yielding of reinforcement in the \( TV \) plane for the case of large transverse shear can be obtained (Fig. 8).

![Figure 7: Failure Mode under Dominant Transverse Shear](image)

![Figure 8: Torsion-Transverse Shear Interaction Diagram at Yielding of Reinforcement (Dominant Shear)](image)

With the use of Eq. 53, Eq. 52 becomes

\[ \epsilon = \frac{s}{2 h_o A_h E_s} \left( \frac{T}{b_o} + V \right) \] .................................................. (54)

Now Eq. 54 is substituted into Eq. 51

\[ \frac{d \theta}{dx} = \frac{(b_o + h_o)}{2 b_o h_o^2 E_s} \left\{ \left[ \frac{s}{b_o A_h} + \frac{1}{2} \left( 1 + \frac{h_o}{b_o} \right) \left( \frac{1}{A'_s} - \frac{1}{A_s} \right) \right] T \right. \\
- \left. \left( \frac{1}{A'_s} - \frac{1}{A_s} \right) M + \frac{sh_o}{(b_o + h_o) A_h} V \right\} \] .................................................. (55)

Also, in view of Eqs. 49 and 50, curvature Eq. 29 becomes

\[ \frac{d \psi}{dx} = \frac{1}{h_o^2 E_s} \left[ \frac{s}{2} \left( 1 + \frac{h_o}{b_o} \right) \left( \frac{1}{A'_s} - \frac{1}{A_s} \right) T + \left( \frac{1}{A'_s} - \frac{1}{A_s} \right) M \right] \] ............ (56)

Finally, by defining additional parameters

\[ \tilde{L}_v = \frac{s}{2 h_o A_h E_s} \] .................................................. (57)

\[ \tilde{L}_{TV} = \frac{s}{2 b_o h_o A_h E_s} \] .................................................. (58)
Compliance equations, Eqs. 52, 54, and 55, can be expressed in the vector-matrix form

\[
\begin{bmatrix}
\frac{d\theta}{dx} \\
\frac{d\psi}{dx} \\
\epsilon
\end{bmatrix}
= 
\begin{bmatrix}
\bar{L}_T & \frac{\bar{L}_M}{2\bar{k}} & -\frac{\bar{L}_M}{2\bar{k}} \\
\frac{1}{1 + \frac{1}{4m}} & \frac{1}{1 + \frac{1}{r}} & -\frac{1}{1 + \frac{1}{r}} \\
\frac{1}{1 + \frac{1}{2k}} & \frac{1}{1 + \frac{1}{r}} & \frac{1}{1 + \frac{1}{r}}
\end{bmatrix}
\begin{bmatrix}
T \\
M \\
V
\end{bmatrix}
\]

\[\ldots \quad (59)\]

Note in particular the interaction between transverse shear and torsion expressed by this result for dominant transverse shear.

**Examination of Results**

Eqs. 37 and 59 represent the two expressions resulting from this study for the postcracking compliance of rectangular reinforced concrete beams under combined torsion, flexure, and transverse shear. The former is for small transverse shear and the latter for large transverse shear. It is seen from Eq. 37 that when the transverse shear force is relatively small (the degree of smallness being determined by the failure mode) it does not influence torsional and flexural postcracking compliance and can be ignored in the stiffness analysis. On this condition Eq. 39 agrees with the fundamental result in Ref. 13. For large transverse shear, a cross compliance term \(\bar{L}_{TV}\) appears in the expression for torsional postcracking compliance and therefore torsional and transverse shear postcracking compliances are no longer uncoupled. However the flexural postcracking compliance remains unaffected.

In considering the flexural postcracking compliance a useful parameter is the torque/moment ratio \(k = \frac{T}{M}\). When \(M\) is large compared to \(T\) (i.e., for small \(k\)) the flexural postcracking compliance in combined loading should approach that for pure bending. However this does not happen in expressions derived in the present work and in Ref. 13 because the crack angle for pure bending is approx 90° and therefore the failure surface is different. Lampert (13) suggests that his expression for flexural postcracking compliance is sufficiently accurate for \(\hat{k} \leq k < \infty\) (This interval is named “Lampert Range II”). The value \(\hat{k}\) is defined by Eq. 34. Note that \(k = \hat{k}\) corresponds to zero stress in the top longitudinal reinforcement as can be seen from Eqs. 14 and 27.
A reason for not suggesting a larger interval of validity for Lampert's expression is that the failure mode assumed in his work, namely 45° cracks on all four sides, has been known to be true for pure torsion \((k \to \infty)\) only. On the other hand, the failure mode used in the present work (compression zone on one side and 45° cracks on the remaining three sides) is satisfactory even for moderately small values of \(k\) as verified, for example, by the tests reported in Ref. 12. It has been reported that this failure mode is valid for \(k\) as small as 0.36.

Noting that for a square section \(k = 1\), it is proposed to extend Range II in Ref. 13 to \(\bar{k}/2 \leq k < \infty\). This interval is termed "Modified Range II."

It is left to establish an expression for flexural postcracking stiffness that is valid in the interval \(0 \leq k \leq \bar{k}/2\) (termed "Modified Range I"). An approximate expression is obtained using an interpolative technique.

The case of pure bending is considered first. The notation used is described in Fig. 9. From strain compatibility, Hooke's law and equilibrium equations

\[
\left( \frac{d\psi}{dx} \right) = \frac{1}{M} \left[ h_o (h - c_o) A_s - \frac{1}{2} b c_o^2 \left( c_o - h + h_o \right) \right] E_s
\]

with \(c_o = \sqrt{(\rho' + \rho)^2 n^2 h^2 + 2 nh (h \rho' + h \rho - h_o \rho')} - nh (\rho' + \rho)\)  \(\ldots (61)\)

in which \(\rho' = \frac{A_s'}{bh}\)  \(\ldots (62)\)

\(\rho = \frac{A_s}{bh}\)  \(\ldots (63)\)

\(n = \frac{E_s}{E_c}\)  \(\ldots (64)\)

where \(E_c\) is the modulus of elasticity of concrete. When the reinforcement is near its state of yielding, \(c_o\) is small compared to \(h_o\). An approximate expression for pure bending postcracking compliance is therefore

\[
\left( \frac{d\psi}{dx} \right) = \frac{1}{M} \left[ h_o (h - c_o) A_s - \frac{1}{2} b c_o^2 \left( c_o - h + h_o \right) \right] E_s
\]
\[ \left( \frac{d\theta}{dx} \right) \approx \frac{1}{\eta^2 E_s A_s} \quad \text{for} \quad k = 0 \quad \ldots \quad (65) \]

\[ \left( \frac{d\psi}{dx} \right) \approx \frac{\bar{L}_M}{2} \quad \text{for} \quad k = 0 \quad \ldots \quad (66) \]

From Eq. 59, at \( k = \bar{k}/2 \)
\[ \left( \frac{d\psi}{dx} \right) \approx \frac{\bar{L}_M}{4} \left( 3 + \frac{1}{r} \right) \quad \ldots \quad (67) \]

An approximate expression for flexural postcracking compliance in Modified Range I is obtained assuming that its variation is linear with respect to \( k \) (13), i.e.
\[ \frac{d\psi}{dx} = (1 + \alpha k) \left( \frac{d\psi}{dx} \right) \approx \frac{\bar{L}_M}{4} \left( 3 + \frac{1}{r} \right) \quad \ldots \quad (68) \]

This equation satisfies the end condition at \( k = 0 \). Unknown parameter \( \alpha \) is determined using Eq. 67 together with Eq. 66 in Eq. 68. Then
\[ \alpha = \frac{1}{k} \left( 1 + \frac{1}{r} \right) \quad \ldots \quad (69) \]
\[ \frac{d\psi}{dx} = \frac{\bar{L}_M}{2} \left[ 1 + \frac{k}{\bar{k}} \left( 1 + \frac{1}{r} \right) \right] \quad \text{for} \quad 0 \leq k \leq \frac{\bar{k}}{2} \quad \ldots \quad (70) \]

The expressions for postcracking compliance established in the present work are shown graphically in Figs. 10(a), 10(b) and 10(c) for the same beam size and the reinforcement ratios as in Ref. 13. Theoretical and experimental results of Lampert (13) are also given in the same figures for comparison. The agreement of the theoretical results with experimental values is satisfactory. In particular, the tests show an increase in torsional postcracking compliance due to transverse shear as predicted by the results established in the present work.

**Summary and Conclusions**

Skew bending theory is used to establish expressions for postcracking compliance of reinforced concrete beams under combined torsion, bending and trans-
verse shear. These expressions are useful in modeling reinforced concrete structural members. Analytical expressions in the presence of small and large transverse shears are provided. The expressions for flexural postcracking compliance given by these equations is in error for large bending moments. For this case, an approximate expression based on a linear interpolation scheme, for flexural postcracking compliance is given. For large shear, the torsional postcracking compliance is found to be significantly affected by shear. However, when the transverse shear is small, its influence on torsional postcracking compliance is negligible, subject to the assumptions made in the present analysis. The results graphically displayed for a particular case are in satisfactory agreement with experiments. In particular, the weakening of torsional strength (as indicated by an increase of torsional postcracking compliance) due to the existence of transverse shear is predicted by the expressions established.

APPENDIX I.—REFERENCES


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