On the Optimal Timing of Benefits with Heterogeneous Workers and Human Capital Depreciation*

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Abstract

This paper studies the optimal timing of unemployment insurance benefits when workers are heterogeneous, or when human capital depreciates during unemployment. Our model builds on Shimer and Werning (2005) to distinguish unemployment benefits from consumption during unemployment by allowing workers to save and borrow freely—a crucial feature given our focus on the timing of benefits. We show that, unlike in homogeneous-agent economies without skill depreciation, optimal benefits are typically not constant. We investigate the main determinants for the shape of the optimal benefits schedule.

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1 Introduction

When workers are homogeneous and face a stationary search problem a constant and indefinite unemployment insurance schedule is optimal, or nearly so (Shimer and Werning, 2005). In this paper we examine how the optimal timing of benefits is affected by relaxing these assumptions. In particular, we model an unemployment agency that sets a schedule of unemployment insurance benefits as a function of unemployment duration and faces either: (i) a group of homogenous workers whose human capital depreciation during unemployment; or (ii) a group of workers that are heterogenous.

Both scenarios are empirically relevant and represent a departure from previous work on optimal unemployment insurance, which focuses on homogenous workers with stationary search problems (e.g. Shavell and Weiss, 1979; Hopenhayn and Nicolini, 1997). Our work also differs from most research on optimal unemployment insurance by distinguishing consumption from benefits by giving workers financial freedom. In Shimer and Werning (2005) we argued that this is crucial for understanding the design of unemployment insurance policy: a constant benefit is optimal, but unemployed workers choose a declining path for their consumption. Intuitively, with homogenous workers facing stationary search problems, constant benefits are optimal because the tradeoff between insurance and incentives does not change over time.

This paper shows that the optimal benefit schedule is not flat when we move away from the benchmark with identical workers facing stationary search problems. When workers’ job opportunities deteriorate during the spell, or when the pool of unemployed workers shifts from one type to another, the tradeoff between insurance and incentives changes during the spell—and benefits change with it. We provide a simple and tractable framework for exploring the main determinants of the optimal timing of benefits. In particular, our model is well suited for isolating the impact of human capital depreciation and heterogeneity on the timing of benefits, precisely because benefits are constant in the version of our model without these features.

We show that the optimal time-varying path of benefits depends on the form of human capital depreciation or heterogeneity. Indeed, we illustrated forces for increasing or decreasing benefit schedules. We explore our model numerically to understand these forces, disentangling the various effects. These explorations are preliminary and ongoing, and we have initially focused on human capital depreciation.

Some interesting lessons emerge from our exercises. We consider two forms of human
capital depreciation. In the first, job opportunities arrive at a constant rate but the wage distribution they are drawn from deteriorates in a steady and parallel fashion. This case captures skill depreciation and is close to that in Ljungqvist and Sargent (1998). The second form of depreciation has workers sampling from the same distribution of wages, but the arrival of opportunities gets rarer over time. That is, search friction rise during the unemployment spell. One interpretation for this is that workers become increasingly detached from the labor market, as they initially exhaust their nearest sources for jobs and turn to remoter options. For concreteness, we call the first form of depreciation *skill depreciation*, and the second form *search depreciation*.

In our skill-depreciation exercises we find decreasing optimal unemployment insurance benefits. Constant benefits would give long-term unemployed a higher replacement ratio relative to their potential wages and induce these workers to become overly picky, or even drop out, as stressed by Ljungqvist and Sargent (1998). In our exercises, the declining wage opportunities lead the reservation wage to decline steadily during unemployment. However, benefits decline even faster: the ratio of unemployment insurance benefit to the reservation wage is also decreasing. Interestingly, we also find that skill depreciation creates a force for higher unemployment benefit levels. Intuitively, the shocks to worker’s permanent income from remaining unemployed for an additional week, which we seek to insure, become larger: they are no longer simply the missed current earnings, but also include the lower future earnings.

In sharp contrast, in our exercises we find that search depreciation creates a force for rising benefits. Intuitively, unfortunate workers that remain unemployed for a long time have lower arrival rates of offers and, therefore, demand more insurance to deal with their heightened duration risk. Long-term unemployed workers have lower exit rates from unemployment, but not because they become choosier. Indeed, in our exercises we find declining reservation wages during the spell. The reason for the lower employment rate is that they receive fewer job offers. Since the moral hazard problem becomes less severe, but risks loom greater, this leads to rising insurance benefits.

After briefly discussing some related literature, this rest of the paper is organized into four sections. **Section 2** describes the model. **Section 3** characterizes the worker’s search problem for any given unemployment insurance benefit schedule. **Section 4** studies the optimal unemployment insurance problem when workers face depreciation during unemployment. **Section 5** turns to the case of heterogeneity. When the time is ripe, future drafts will include a concluding section.
Some Related Literature

Ljungqvist and Sargent (1998) emphasize human capital depreciation of unemployed workers to explain higher European unemployment. They model skill loss as stochastic, so their story actually also combines elements of heterogeneity. In particular, during ‘tranquil’ times human capital depreciates steadily during unemployment generating unimportant amounts of heterogeneity among the unemployed. In contrast, during ‘turbulent’ times a fraction of workers lose skills immediately at the moment they are laid off, generating significant amounts of heterogeneity.

Pavoni (2003) and Violante and Pavoni (2005) study optimal unemployment insurance in environments with human capital depreciation, and other elements. However, these papers do not focus on the optimal timing of benefits since they assume that the unemployment insurance agency can control consumption and do not attempt to distinguish consumption from benefits.

2 The Model

We adapt the model from Shimer and Werning (2005, 2006). These papers provide a tractable version of a McCall (1970) search problem enhanced to incorporate risk-averse workers that can save and borrow freely. Time is continuous and infinite $t \in [0, \infty)$. Workers seek to maximize expected discounted utility

$$E_0 \int_0^\infty e^{-\rho t} u(c(t))dt$$

(1)

where $c(t)$ is consumption. We work with Constant Absolute Risk Aversion (CARA) utility functions: $u(c) = -e^{-\gamma c}$ with $c \in \mathbb{R}$. This assumption allows us to solve the model in closed form.

Unemployed workers sample job opportunities at Poisson arrival rates $\lambda(t)$. Jobs are distinguished by their wage $w$ drawn from a distribution $F(w, t)$ with density $f(w, t)$. This

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1 Ljungqvist and Sargent (1998, pg. 548) conclude that “during tranquil times, the depreciation of skills during spells of unemployment […] is simply too slow to have much effect on the amount of long-term unemployed. The primary cause of long-term unemployment in our turbulent times is the instantaneous loss of skills at layoffs. Our probabilistic specification of this instantaneous loss creates heterogeneity among laid-off workers having the same past earnings.”

2 In Shimer and Werning (2005) we verified that CARA preferences provided a good benchmark: the numerical solution with CRRA preferences was very close to the CARA closed form solution.
introduces two potential forms of human capital depreciation. We assume jobs last forever.

**Budget Constraints.** Workers can save and borrow at the market interest $r$. Their budget constraints are

$$\dot{a}(t) = ra(t) + y(t) - c(t),$$

where $a(t)$ are assets and $y(t)$ represents current non-interest income: it equals benefits, $B(t)$, during unemployment and the wage, $w(t)$, during employment. Initial assets $a(0) = a_0$ are given. In addition workers must satisfy the No-Ponzi condition $\lim_{t \to \infty} e^{-rt}a(t) \geq 0$.

**Heterogeneity.** To capture worker heterogeneity we assume that there are $N$ types indexed by $n = 1, 2, \ldots, N$ and index any of the primitives of the worker’s search problem by the type, e.g. $\lambda_n(t), F_n(w, t), \gamma^n$, and so on. To focus on depreciation we initially assume there is only one type and drop the superscripts.

**Unemployment Insurance Policy.** In this paper we are interested in the optimal timing of unemployment insurance benefits, and not in larger, more comprehensive, social welfare reforms. This motivates the policy problem we consider, which is to select a schedule of unemployment benefits $\{B(t)\}_{t \geq 0}$ that stipulates the benefit $B(t)$ received by a worker that remains unemployed at $t$. The duration of unemployment is the only variable for which benefits can be conditioned on.$^3$

The optimal unemployment insurance policy problem we study is to find the best such schedule of benefits. For the case with heterogenous workers one must, in general, specify a welfare criterion. However, to avoid redistributitional concerns we allow initial lump-sum transfers between workers types. These transfers are equivalent to redistributions in terms of initial assets $a_0$. It turns out that, with CARA preferences, the optimal schedule $\{B(t)\}_{t \geq 0}$ is then uniquely pinned down: all Pareto efficient allocations can be achieved with the same schedule by varying the initial lump-sum transfers (initial assets) between workers. Thus, we do not need to specify any particular welfare criterion and our analysis characterizes all Pareto efficient schedules.

### 3 Worker Behavior

In this section we characterize the behavior of a single unemployed worker confronted with any benefit schedule $\{B(t)\}_{t \geq 0}$.

$^3$ We do not consider, for instance, menus of schedules that may self-select and separate worker types; likewise we do not consider constraining workers access to savings.
Let $V(t)$ represent the lifetime utility of an unemployed worker at time $t$,

$$V(t) \equiv \mathbb{E}_t \int_0^\infty e^{-\rho s} u(c(t + s)) ds. \quad (3)$$

We now derive a few properties about $V(t)$.

For any job-acceptance policy, workers solve a standard consumption-savings subproblem, maximizing utility in equation (1) subject to the budget constraint equation (2) and the No-Ponzi condition. This subproblem implies the usual intertemporal Euler equation

$$u'(c(t)) = e^{-(\rho - r)s} \mathbb{E}_t u'(c_{t+s})$$

With CARA preferences $u'(c) = \gamma u(c)$, so this implies

$$u(c(t)) = e^{-(\rho - r)s} \mathbb{E}_t u(c_{t+s})$$

Substituting this into equation (3) gives

$$V(t) = \frac{1}{r} u(c(t)) \quad \text{or equivalently} \quad c(t) = u^{-1}(rV(t)), \quad (4)$$

which conveniently relates lifetime utility to current consumption.

Lifetime utility $V(t)$ can always be decomposed as

$$V(t) \equiv -v(t)u(ra(t)), \quad (5)$$

where $v(t)$ represents the lifetime utility of a worker with zero assets.

Unemployed workers just wait around for job offers. An unemployed worker accepts a job offer if the wage is higher than the wage $\bar{w}(t)$ which makes her indifferent. Since a worker earning $\bar{w}(t)$ with assets $a(t)$ consumes $ra(t) + \bar{w}(t) + (\rho - r)/\gamma r$,

$$V(t) = \frac{u(ra(t) + \bar{w}(t) + (\rho - r)/\gamma r)}{r} \quad (6)$$

$$\Rightarrow \bar{w}(t) = u^{-1}(rV(t)) - ra(t) - \frac{\rho - r}{\gamma r} = u^{-1}(rv(t)) - \frac{\rho - r}{\gamma r} \quad (7)$$

With a reservation rule the lifetime utility during unemployment is a function of time
and evolves according to

\[ \rho V(t) = u(c(t)) + \dot{V}(t) + \lambda(t) \int_{\bar{w}(t)}^{\infty} \left( \frac{u(ra(t) + w + (\rho - r)/\gamma r)}{r} - V(t) \right) dF(w, t) \]

Using equation (4) it follows that we can write this as

\[ \dot{V}(t) = (\rho - r)V(t) - \lambda(t) \int_{\bar{w}(t)}^{\infty} \left( \frac{u(ra(t) + w + (\rho - r)/\gamma r)}{r} - V(t) \right) dF(w, t) \]

Rewriting this in terms of \( v(t) \) using equation (5) gives

\[ -\dot{v}(t) + \gamma v(t)r\dot{a}(t) = (r - \rho)v(t) - \lambda(t) \int_{\bar{w}(t)}^{\infty} \left( \frac{-u(w + (\rho - r)/\gamma r)}{r} + v(t) \right) dF(w, t) \quad (8) \]

As for \( \dot{a} \), the budget constraint equation (2) during unemployment combined with equations (4) and (5) gives

\[ \dot{a}(t) = ra(t) + B(t) - c(t) = ra(t) + B(t) - u^{-1}(rV(t)) \]
\[ = ra(t) + B(t) - u^{-1}(rv(t)) - ra(t) = B(t) - u^{-1}(rv(t)). \]

Substituting this into equation (8) yields

\[ \dot{v}(t) = \gamma v(t)r(B(t) - u^{-1}(rv(t))) - (r - \rho)v(t) \]
\[ + \lambda(t) \int_{\bar{w}(t)}^{\infty} \left( \frac{-u(w + (\rho - r)/\gamma r)}{r} + v(t) \right) dF(w, t). \quad (9) \]

To transform this into a law of motion for the reservation wage, note that equation (7) implies

\[ \dot{\bar{w}}(t) = \frac{1}{w(u^{-1}(rv(t)))}r\dot{v}(t) = \frac{-1}{\gamma u(u^{-1}(rv(t)))}r\dot{v}(t) = -\frac{\dot{v}(t)}{\gamma v(t)}. \]

Then equation (9) becomes

\[ \dot{\bar{w}}(t) = G(\bar{w}, t) - rB(t), \quad (10) \]

where

\[ G(\bar{w}, t) \equiv r\bar{w}(t) - \frac{\lambda(t)}{\gamma} \int_{\bar{w}(t)}^{\infty} (1 + u(w - \bar{w}(t))) dF(w, t). \quad (11) \]

This is a crucial equation for our analysis. In a stationary environment, with constant
benefits $B$, a constant arrival rate $\lambda$, and a constant wage distribution $F(w)$, equation (10) boils down to the reservation wage equations in Shimer and Werning (2005, 2006).

**Relation between $\bar{w}(t)$ and $B(t)$**. We shall use the characterization of the relationship in equation (10) between the reservation wage path $\{\bar{w}(t)\}$ and benefits schedule $\{B(t)\}$ extensively. It is useful to understand what this relation does and does not imply.

Suppose we are in a stationary environment so that $G(\bar{w}, t) = G(\bar{w})$ is independent of time $t$. Then, at a steady state, where $\dot{\bar{w}} = 0$, we have $B = G(\bar{w})/r$ and since $G$ is increasing in $\bar{w}$ it follows that there is a positive relation between benefits and the reservation wage. This is very intuitive since a higher benefit level makes the option of waiting for higher wage draws more attractive without making employment any more desirable. As a result, the worker becomes more picky about what jobs to accept.

However, the steady-state relationship does not imply that along any dynamic path $\bar{w}(t)$ and $B(t)$ will rise and fall in tandem. In equation (16) benefits $B(t)$ are determined not only by $\bar{w}(t)$ but also by $\dot{\bar{w}}(t)$. Informally, if the reservation wage is rising sharply, it indicates that the unemployed worker’s lifetime utility is doing the same; things are better in the near future, so current benefits must be temporarily low.

This implies that there is no simple relation between the paths of $\bar{w}(t)$ and $B(t)$. For instance, suppose $\bar{w}(t)$ is monotonically increasing; then, it may seem reasonable to expect benefits $B(t)$ to also be increasing. This will be the case as long as $\bar{w}(t)$ does not accelerate too much, so that $\dot{\bar{w}}$ does not rise too abruptly; otherwise, equation (16) implies that benefits will decrease over a range where $\dot{\bar{w}}$ rises quickly. Thus, a monotonic $\bar{w}(t)$ does not imply monotonic benefits $B(t)$. The converse, however, is true: if the benefit schedule $B(t)$ is monotonic then $\bar{w}(t)$ is monotonic.

This discussion emphasizes the dynamic nature of worker’s search problem. The reservation wage is not only affected by the current benefit, but also by future benefits. As a result, current benefits may generally provide a poor measure of the current subsidy to unemployment implicit in the entire schedule.\(^4\) Perhaps a better measure is simply to observe the effect on the actual reservation wage, which is forward looking and incorporates the dynamics of future benefits.

\(^4\) To take an extreme example, suppose that benefits are negative during the first week of unemployment but they then jump up to a positive level, much higher than any potential wage. Few would describe the situation faced by the worker in the first period as providing a tax on unemployment that encourages finding a job.
Transversality Condition. The following limiting conditions must also hold

$$\lim_{t\to\infty} e^{-\rho t} \mu(t)V(t) = 0$$
$$\Rightarrow \lim_{t\to\infty} e^{-\rho t} \mu(t)v(t)u(ra(t)) = 0$$

where

$$\mu(t) \equiv \exp \left( -\int_0^t H(\bar{w}(s), s) ds \right)$$

(12)

is the probability of being unemployed at time $t$ and

$$H(\bar{w}, s) \equiv \lambda(s)(1 - F(\bar{w}, s))$$

(13)

is the hazard rate of accepting a job.

4 Optimal Policy: Human Capital Depreciation

In this section we consider the case of a single worker type that faces a non-stationary search problem, with $\lambda$ or $F$ are changing over time.

4.1 Policy Problem

We imagine an unemployment insurance agency that wishes to maximize unemployed workers’ lifetime utility subject to the constraint that it must break even on average. Imagine the agency charging the worker an upfront fee $C$, equal to the expected present value of benefits. Then using equation (6) the worker’s utility is

$$V(0) = \frac{1}{r} u \left( r(a_0 - C) + \bar{w}(0) + \frac{\rho - r}{\gamma r} \right)$$

Thus, the agency chooses a path of benefits to maximize $\bar{w}(0) - rC$, where the expected cost of the unemployment insurance system is

$$C \equiv \int_0^{\infty} e^{-rt} B(t) \mu(t) dt$$

and reservation wages solve equation (10) and the transversality condition.
Use equation (10) to eliminate \( B(t) \) from the objective function
\[
\bar{w}(0) - rC = \bar{w}(0) + \int_{0}^{\infty} e^{-rt} (\bar{w}(t) - G(\bar{w}(t), t)) \mu(t) dt,
\] (14)

Use integration-by-parts to eliminate the term of the integral involving \( \dot{\bar{w}}(t) \):
\[
\int_{0}^{\infty} \dot{\bar{w}}(t) \mu(t) e^{-rt} dt = -\bar{w}(0) - \int_{0}^{\infty} \bar{w}(t) (\dot{\mu}(t) - r \mu(t)) e^{-rt} dt
\]
\[
= -\bar{w}(0) + \int_{0}^{\infty} \bar{w}(t) (r + H(\bar{w}(t), t)) \mu(t) e^{-rt} dt,
\]

since \( \mu(0) = 1 \). Substitute this back equation (14) to simplify the objective function \( \bar{w}(0) - rC \). The planner must choose a sequence of reservation wages to solve
\[
\max_{\{\bar{w}\}} \int_{0}^{\infty} \left( \bar{w}(t) (r + H(\bar{w}(t), t)) - G(\bar{w}(t), t) \right) \mu(t) e^{-rt} dt
\] (15)
subject to \( \dot{\mu}(t) = -H(\bar{w}(t), t) \mu(t) \).

4.2 Constant and Non-Constant Benefits

An interesting property of optimal benefits that follows from our reformulation is that the schedule is entirely forward looking: only future values of \( \lambda \) and \( F \) are relevant. The next result then follows immediately from this observation.\(^5\)

**Proposition 1 (Shimer-Werning, 2005)** With a single worker type facing a stationary problem \( \lambda(t) = \lambda \) and \( F(w, t) = F(w) \) for all \( t \geq 0 \) the optimal benefit schedule is flat: \( B(t) = \bar{B} \) for some \( \bar{B} > 0 \).

To tackle the general problem we write it recursively. Let \( \Phi(\mu, t) \) be the value function for the problem in equation (15). The this value function solves the Bellman equation
\[
 r \Phi(\mu, t) = \max_{\bar{w}} \left( (\bar{w}(r + H(\bar{w}, t)) - G(\bar{w}, t)) \mu - \Phi(\mu, t) H(\bar{w}, t) \mu + \Phi_t(\mu, t) \right)
\]

\(^5\) This result is proven in Shimer and Werning (2005) in a discrete time version of the model using a different argument. That paper shows that the planner does not want to distort savings. In contrast, here we have simply assumed that the policy problem does not consider introducing such distortions.
Note that the value function is homogeneous of degree one in \( \mu \) and so we can define \( \phi(t) \equiv \Phi(\mu, t)/\mu \) solving

\[
\dot{\phi}(t) = \max_{\bar{w}} \left( \bar{w}(r + H(\bar{w}, t)) - G(\bar{w}, t) - \phi(t)H(\bar{w}, t) + \dot{\phi}(t) \right)
\]

Equivalently, we have an ordinary differential equation for \( \phi(t) \):

\[
\dot{\phi}(t) = M(\phi(t), t) \tag{16}
\]

where the law of motion function \( M \) is given by

\[
M(\phi, t) \equiv \min_{\bar{w}} \left( (r + H(\bar{w}, t))(\phi - \bar{w}) + G(\bar{w}, t) \right). \tag{17}
\]

Note that the envelope condition implies that the law of motion function \( M(\phi, t) \) is increasing in \( \phi \). Moreover, since the cross-partial derivative of \( \bar{w} \) and \( \phi \) is negative in the objective function, it follows that the optimal \( \bar{w} \) is increasing in \( \phi \).

To characterize optimal unemployment insurance, we simply need to solve this ordinary differential equation (16), ensuring that the transversality condition holds. When primitives settle down in the long-run one can solve backwards from the long-run steady-state. The optimal reservation wage solves the right hand side of equation (17) at each date. And the benefits that implement this reservation wage are found by inverting equation (10) for \( B(t) \).

It is instructive to verify how the stationary solution in Proposition 1 solves the ODE system in equation (16). Since primitives are constant the law of motion is independent of time: \( M(\psi, t) = M(\psi) \). Since \( M(\psi) \) is increasing there exists a unique steady-state value \( \psi^* \) satisfying \( 0 = M(\psi^*) \). Then note that the stationary solution \( \psi(t) = \psi^* \) solves the ODE system and satisfies the transversality conditions. The reservation wage and benefits implicit in this solution are constant, as in Proposition 1.

A simple non-stationary case, illustrated in Figure 1, is when primitives are constant up to some time \( T \), at which point they switch forever after. That is, we have \( \lambda(t) = \lambda_0 \) and \( F(w, t) = F_0(w) \) for all \( t < T \) and \( \lambda(t) = \lambda_1 \) and \( F(w, t) = F_1(w) \) for all \( t \geq T \). This implies that \( \phi \) evolves according to \( \dot{\phi} = M_0(\psi) \) for \( t < T \), and then \( \dot{\phi} = M_1(\psi) \) for \( t \geq T \).

We know that the optimal solution must reach the steady-state point \( \phi^*_1 \) of \( M_1 \) at \( t = T \). Thus, the initial value \( \phi(0) \) must start somewhere to the right of point \( \phi^*_0 \) and increase—accelerating with the explosive dynamics of \( M_0 \)—until it reaches \( \phi^*_1 \) exactly at time \( t = T \), at which point it remains constant there. The larger is \( T \) the closer \( \phi(0) \) must be to \( \phi^*_0 \);
indeed, as $T \to \infty$ then $\phi(0)$ limits to $\phi_0^*$.

The implications for benefits $B(t)$ are immediate translations of those derived for $\phi(t)$. Let $B_i^*$ denote the optimal constant benefits for the stationary problem with $\lambda_i$ and $F_i(w)$. Then benefits $B(t)$ converge to the optimal constant benefit $B_i^*$ in the long run as $t \to \infty$ and they start somewhere near $B_i^*$. This result is generalized in the next proposition, where we imagine time extending indefinitely on both sides.

**Proposition 2** Suppose that we have $\lambda(t)$ and $F(w,t)$ defined for all $t \in \mathbb{R}$ with well defined limits $\lim_{t \to -\infty} \lambda(t) = \lambda_0$ and $\lim_{t \to -\infty} F(w,t) = F_0(w)$ and $\lim_{t \to -\infty} \lambda(t) = \lambda_1$ and $\lim_{t \to -\infty} F(w,t) = F_1(w)$. Then $B(t)$ is such that $\lim_{t \to -\infty} B(t) = B_0^*$ and $\lim_{t \to \infty} B(t) = B_1^*$, where $B_i^*$ is defined as the optimal constant benefit levels for the economies with constant primitives at $\lambda_i$ and $F_i(w)$.

**4.3 An Aside: Q-Theory Analogy**

Our model can be mapped into the adjustment cost model of investment with constant returns to scale which Hayashi (1982) used to related investment to “Tobin’s Q”.

In the case of certainty the investment model can be formulated as maximizing discounted
profits

\[ \int_0^\infty \pi(i(t), t)K(t)dt \]

subject to \( \dot{K}(t) = i(t)K(t) \), where \( i(t) = \frac{I(t)}{K(t)} \) is the investment rate. There are constant returns to scale in the net profit function, which equals revenues net of investment costs, and constant returns in investment. No assumption of concavity is required. One can show that the value function is homogeneous of degree one, \( V(K) = qK \), so that the marginal and average value of capital, \( q \), often referred to as “Tobin’s \( Q \)”, solves

\[ rq = \max_i \{ \pi(i, t) + iq \} + \dot{q}. \]

The important result for this theory is that the investment rate is a function of \( q \); the entire future is captured by this forward looking variable.

This model maps directly into our framework, with \( \mu \) playing the role of \( K \), with \( \bar{w} \) playing the role of investment \( i \), and with \( \pi \) given by \( \bar{w}(r + H(\bar{w}, t)) - G(\bar{w}, t) \).

### 4.4 Numerical Explorations

This section describes the outcome of two numerical experiments. We first consider skill depreciation, then search depreciation. The purpose of these explorations is not to obtain definite quantitative conclusions. Our goal is to understand the qualitative workings of the model and perhaps get a tentative feel for their quantitative importance.

Our baseline parameterization is close to that in Shimer and Werning (2005). We set \( \gamma = 1, \ r = \rho = .001 \) and \( \lambda = 1 \) and interpret a period to be a week, with an implied annual interest rate of 5.3%. The distribution is assumed to be Fréchet: \( F(w) = \exp(-w^{-\theta}) \). We set \( \theta = 103.5 \).

This baseline calibration has wages concentrated near 1 and delivers an expected duration of around 10 weeks, which is in line with unemployment durations in the United States. The optimal constant benefit level in this economy turns out to be very low, around 0.01. The desire to insurance is small enough, while the moral-hazard problem severe enough, that low benefits result at the optimum. As discussed by Shimer and Werning (2005), liquidity, in contrast, is important: workers are able to smooth their shocks, spreading their impact over time, by dissaving or borrowing.
4.4.1 Skill Depreciation

In this first exercise we keep $\lambda(t) = 1$ constant and instead assume that the distribution of wages shifts downwards in a parallel fashion. Our specification is inspired by the depreciation process used in Ljungqvist and Sargent (1998). Specifically we let

$$
F(w, t) = F(w - \exp(-\delta_F \cdot t)) \quad t < T
$$

$$
F(w, t) = F(w - \exp(-\delta_F \cdot T)) \quad t \geq T
$$

where $F(w)$ is the baseline Fréchet distribution defined above. Thus, at $t = 0$ the wage distribution is simply a rightward shift of the baseline distribution. Over time the distribution shifts continuously to the left, converging back to the baseline distribution. We set the speed of convergence to $\delta_F = 0.01$.

Our approach is to solve the ODE system in equation (16). We first solve the system’s steady state for $t \geq T$. We then solve the ODE backwards up to $t = 0$. This gives us $\phi(t)$. We then compute $\tilde{w}(t)$ and solve equation (10) for $B(t) = \frac{1}{r}(G(\tilde{w}(t), t) - \dot{\tilde{w}}(t))$.

Figure 2 shows the outcome of this exercise for the optimal schedule of benefits. The schedule is decreasing with unemployment duration, starting at benefits just above 0.30 and falling to the steady-state value of 0.01—equal to the value of benefits at the baseline.

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6 The numerical details are as followed. We employed Matlab’s ode45 routine to solve the ODE, along with the fminbnd routine for the required optimization. We computed $\phi(t)$ and then backed out the implied reservation wage $\tilde{w}(t)$. We then fit a piecewise cubic shape-preserving spline through $\tilde{w}(t)$ (using Matlab’s interp1 routine with the pchip option) to obtain the derivative $\dot{w}(t)$ from it.
Figure 4 shows that these benefits induce the reservation wage to fall during the unemployment spell. The rate at which the reservation wage drops, however, does not keep up with the rate of decline in the distribution of wages. This is shown in Figure 5 where we plot the acceptance probability of job opportunities, $1 - F(\bar{w}(t))$. The optimal schedule induces workers to become pickier: the probability of accepting a job offer falls from around 80% to 10%. Note that in this case, since $\lambda(t) = 1$ the acceptance probability equals the hazard rate of out unemployment, $\lambda(t)(1 - F(\bar{w}(t)))$. Benefits, however, all even faster: Figure 3 plots the ratio $B(t)/\bar{w}(t)$, it is decreasing.

As Figure 5 illustrates the hazard rate out of unemployment is high, especially at the beginning of the spell where depreciation is greatest. Workers become more willing to accept bad matches to prevent their skills from depreciating. We computed the solution for $\theta = 15$ which increases the dispersion of wages making search more attractive. Figure 6 plots optimal benefits for this case. Once again we see that benefits are decreasing. Figure 7 shows that the hazard rate out of unemployment is much lower now, and close to the hazard at the benchmark without depreciation.

4.4.2 Search Depreciation

In this second exercise we assume that the wage distribution is fixed at the baseline’s Fréchet specification, but that the arrival rate falls continuously over time. Specifically, we let $\lambda(t) = \bar{\lambda}_0 \exp(-\delta_\lambda \cdot t) + \bar{\lambda}_1$ for $t \leq T$ and constant thereafter; we set $\delta_\lambda = 0.01$, $\bar{\lambda}_0 = 0.9$ and $\bar{\lambda}_1 = 0.1$, so that the arrival rate starts at our benchmark of one offer a week and falls continuously towards one offer every 10 weeks.
Figure 6: Optimal Benefits with $\theta = 15$.

Figure 7: Hazard rate.

Figure 8 shows the results of this exercise for optimal benefits. We find an increasing schedule, which is in line with Proposition 2, since a lower arrival rate increases the duration risk of unemployment prompting higher benefits. The level of benefits is quite low in this case throughout, so the increase in benefits is not very spectacular. Figure 9 shows that even though benefits rise the acceptance probability (blue line) rises with duration as job opportunities become rarer. Workers become less picky and lower their reservation wages. However, the resulting hazard rate out of unemployment $\lambda(t)(1 - F(\bar{w}(t)))$, also shown (green line), comes out to be slightly declining.

We have found that the limiting long-run benefit level is quite sensitive to the long-run value $\lambda_1$ for low enough values, and become quite large for $\lambda_1$ near zero. To illustrate this Figure 10 shows optimal benefits when $\lambda_1 = .01$. Note that benefits rise only moderately for about 6 years—similar to what we found in Figure 8 for higher $\lambda_1$. However, when $\lambda(t)$ gets very close to zero, benefits rise sharply and asymptote to a high level, around 0.73. Of course, with these parameters only an insignificant fraction of workers make it this far into long-term unemployment. Nevertheless, this illustrates that the magnitude of the increase in benefits depends on parameters of the problem.

As explained in Section 3, when the reservation wage accelerates there may be regions where benefits are decreasing. We found that this occurs over an intermediate region of time for the extreme parameter value of $\lambda_1 = 0$. Figure 12 shows that benefits are decreasing over an intermediate region. Indeed, they become slightly negative there because they were low and close to zero initially. The nonmonotonicity occurs after around 6 years, a duration that an insignificant fraction of workers will reach. However, this illustrates the point we made
earlier: that benefits may be a poor measure of the subsidy to unemployment since workers are forward looking and incorporates the dynamics of future benefits.

5 Optimal Policy: Heterogeneity

In this section we formulate the planning problem for the case where the agency faces heterogeneous workers indexed by $n = 1, 2, \ldots, N$. To bring out the role of heterogeneity we suppose that for each worker type the arrival rate of offers $\lambda^n$ and the distribution of wages $F^n$ does not vary with time $t$.

For any given common benefit schedule the pool of unemployed worker types typically varies over time. That is, worker types that tend to have lower hazard rates become more prevalent as time passes. This is an important effect we wish to focus on. For now, we only formulate the problem. We shall study it in future versions of this draft.

5.1 Problem Statement

The agency selects a single benefit schedule to maximize the sum of the reservation wages net of the discounted cost the program. The problem can be formulated as the optimal control problem:

$$
\max_B \sum_{n=1}^{N} \left( \bar{w}^n - \int_0^{\infty} e^{-rt} B(t) \mu^n(t) dt \right)
$$
subject to,

\[ \dot{\mu}^n(t) = -H(\bar{w}^n(t)) \]
\[ \dot{\bar{w}}^n(t) = G^n(\bar{w}^n(t)) - rB(t) \]

with \( \mu^n(0) \) given. One way to attack this problem is to study the cost minimization part of the problem as an optimal control problem. We will pursue a different strategy.

\subsection{5.2 Two Types: Recursive Formulation}

For the case with two types we now show how to reduce the problem to a two-dimensional state variable system. Define the difference in reservation wages

\[ \Delta(t) \equiv \bar{w}^2(t) - \bar{w}^1(t). \]

Then the law of motion for this difference is

\[ \dot{\Delta}(t) = M_\Delta(\bar{w}^1(t), \Delta), \]
(18)

where \( M_\Delta(\bar{w}^1, \Delta) \equiv G^2(\bar{w}^1 + \Delta) - G^1(\bar{w}^1) \). Also define the ratio

\[ \alpha(t) \equiv \frac{\mu^1(t)}{\mu^1(t) + \mu^2(t)}, \]
with the law of motion

$$\dot{\alpha}(t) = \alpha(t)(1 - \alpha(t)) \left( H^1(\bar{w}^1(t)) - H^2(\bar{w}^1(t) + \Delta(t)) \right). \quad (19)$$

Following the arguments from Section 4 we can rewrite the objective function as

$$\int_0^\infty e^{-rt} \left( \alpha(t) J(\bar{w}^1(t)) + (1 - \alpha(t)) J(\bar{w}^1(t) + \Delta(t)) \right) \mu(t) dt, \quad (20)$$

where we introduced the functions $J^n(\bar{w}) \equiv \bar{w}(r + H^n(\bar{w}^n)) - G^n(\bar{w}^n)$ for notational convenience.

Thus, the planning problem is to maximizing equation (20) over $\bar{w}^1(t), \Delta(t), \alpha(t)$ and $\mu(t)$ subject to equation (18), equation (19) and

$$\dot{\mu}(t) = -\mu(t)(\alpha(t)H^1(\bar{w}^1(t)) + (1 - \alpha(t))H^2(\bar{w}^1(t) + \Delta(t))).$$

It is useful to think of $\bar{w}^1(t)$ as the control and $\Delta(t), \alpha(t)$ and $\mu(t)$ as state variables.

Let $\Phi(\Delta(t), \alpha(t), \mu(t))$ be the value function. This value function is homogenous of degree 1 in $\mu(t)$, so we write

$$\Phi(\Delta(t), \alpha(t), \mu(t)) = \phi(\Delta(t), \alpha(t))\mu(t)$$

Then $\phi(\Delta(t), \alpha(t))$ satisfies a Hamiltonian-Jacobi-Bellman partial-differential equation, where the maximization is over the control $\bar{w}^1$. 

Figure 12: Optimal Benefits with Search Depreciation when $\lambda_1 = .01$. 

Figure 13: Acceptance Probability and Hazard Rates when $\lambda_1 = .01$. 

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Then $\phi(\Delta(t), \alpha(t))$ satisfies a Hamiltonian-Jacobi-Bellman partial-differential equation, where the maximization is over the control $\bar{w}^1$. 

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5.3 Optimal Control Formulation

We take an optimal control approach to the planning problem. To simplify we solve a finite horizon version of the problem where policy is restricted to some simple form after some time $T$. Let $\Psi(\alpha, \Delta)$ be the continuation value from a the set of such policies that implements $\Delta$. A simple set of policies might be to find the constant benefit schedules that implements $\Delta$, if possible, but we could consider other simple policies.

Let
\[
\bar{J}(\alpha(t), \bar{w}^1(t), \Delta(t), t) \equiv \alpha(t)J(\bar{w}^1(t)) + (1 - \alpha(t))J(\bar{w}^1(t) + \Delta(t))
\]

Then the policy problem can then be written as
\[
\int_0^T e^{-rt} \bar{J}(\alpha(t), \bar{w}^1(t), \Delta(t), t)\mu(t)dt + e^{-rT}\Psi(\alpha(T), \Delta(T), T)\mu(T).
\]

The Hamiltonian for this problem can be written as (omitting the arguments in functions to save on notation)
\[
\mathcal{H} \equiv (\bar{J} + k_\mu M^\mu + k_\alpha M^\alpha + k_\Delta M^\Delta)\mu
\]

where
\[
M^\mu(\alpha, \bar{w}^1, \Delta, t) \equiv -\alpha H^1(\bar{w}^1) + (1 - \alpha)H^2(\bar{w}^1 + \Delta)
\]
\[
M^\alpha(\alpha, \bar{w}^1, \Delta, t) \equiv \alpha(1 - \alpha)(H^1(\bar{w}^1) - H^2(\bar{w}^1 + \Delta))
\]
\[
M^\Delta(\bar{w}^1, \Delta, t) \equiv G^2(\bar{w}^1 + \Delta, t) - G^1(\bar{w}^1, t)
\]

**Maximum Principle.** The optimality conditions are then
\[
\max_{\bar{w}^1} \mathcal{H}
\]
\[
(k_\mu \dot{\mu}) = \dot{k}_\mu \mu + k_\mu \dot{\mu} = rk_\mu - \mathcal{H}
\]
\[
(k_\Delta \dot{\mu}) = \dot{k}_\Delta \mu + k_\Delta \dot{\mu} = rk_\Delta - \mathcal{H}_\Delta
\]
\[
(k_\alpha \dot{\mu}) = \dot{k}_\alpha \mu + k_\alpha \dot{\mu} = rk_\alpha - \mathcal{H}_\alpha
\]
We can rewrite this as

\[
\dot{k}_\mu = (r - M_\mu)k_\mu - \mathcal{H} \\
\dot{k}_\Delta = (r - M_\Delta)k_\Delta - \mathcal{H}_\Delta/\mu \\
\dot{k}_\alpha = (r - M_\alpha)k_\alpha - \mathcal{H}_\alpha/\mu
\]

Since \( \Delta(0) \) is a free variable its costate should be initially zero

\[ k_\Delta(0) = 0 \quad (22) \]

At \( t = T \) we must meet the terminal conditions that

\[
\begin{align*}
k_\mu(T) &= \Psi(\alpha(T), \Delta(T), T) \\
k_\Delta(T) &= \Psi_\Delta(\alpha(T), \Delta(T), T) \\
k_\alpha(T) &= \Psi_\alpha(\alpha(T), \Delta(T), T)
\end{align*}
\]

Algorithm. The idea is to start at \( T \) and work backwards. The control variable is \( \bar{w}^1 \) and the states are \( \mu, \Delta \) and \( \alpha \); along with their costates \( k_\mu, k_\Delta \) and \( k_\alpha \). As before, \( \mu \) plays no role in the dynamics and it can be ignored.

One way to solve this is to guess the values of \( \alpha(T) \) and \( \Delta(T) \). One can then obtain the values of the costates at \( T \) using equations (23)–(25). This is then sufficient to solve the system backwards for \( \bar{w}^1, \alpha, \Delta, k_\mu, k_\Delta, k_\alpha \). One can then compute the implied value of \( k_\Delta(0) \) and \( \alpha(0) \). One can then search for the initial guess \( \alpha(T) \) and \( \Delta(T) \) that has \( k_\Delta(0) = 0 \) and \( \alpha(0) = \alpha_0 \), that is to satisfy the optimality condition equation (22) and the given initial condition for \( \alpha \).
References


_ and Iván Werning, “Reservation Wages and Unemployment Insurance,” 2006. Mimeo, MIT.