Advances on Matroid Secretary Problems: Free Order Model and Laminar Case

Patrick Jaillet[1]⋆, José A. Soto[2]⋆⋆, and Rico Zenklusen[3]⋆⋆⋆

[1] Dept. of Electrical Engineering and Computer Science, MIT
jaillet@mit.edu
[2] DIM-CMM University of Chile, and Technische Universität Berlin
jsoto@dim.uchile.cl
[3] Dept. of Applied Mathematics and Statistics, Johns Hopkins University
ricoz@jhu.edu

Abstract. The best-known conjecture in the context of matroid secretary problems claims the existence of an $O(1)$-approximation applicable to any matroid. Whereas this conjecture remains open, modified forms of it were shown to be true, when assuming that the assignment of weights to the secretaries is not adversarial but uniformly at random [20,18]. However, so far, no variant of the matroid secretary problem with adversarial weight assignment is known that admits an $O(1)$-approximation. We address this point by presenting a 9-approximation for the *free order model*, a model suggested shortly after the introduction of the matroid secretary problem, and for which no $O(1)$-approximation was known so far. The free order model is a relaxed version of the original matroid secretary problem, with the only difference that one can choose the order in which secretaries are interviewed.

Furthermore, we consider the classical matroid secretary problem for the special case of laminar matroids. Only recently, a $O(1)$-approximation has been found for this case, using a clever but rather involved method and analysis [12] that leads to a 16000/3-approximation. This is arguably the most involved special case of the matroid secretary problem for which an $O(1)$-approximation is known. We present a considerably simpler and stronger $3\sqrt{3}e \approx 14.12$-approximation, based on reducing the problem to a matroid secretary problem on a partition matroid.

1 Introduction

The secretary problem is a classical online selection problem of unclear origin [6,8,9,10,16]. In its original form, the task is to choose the best out of n

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secretaries, also called *elements* or *items*. Secretaries arrive (or are interviewed) one by one in random order. As soon as a secretary arrives, it can be ranked against all previously seen secretaries. Then, before the next one arrives, one has to decide irrevocably whether to choose the current secretary or not. There is a classical algorithm that selects the best secretary with probability $1/e$ [6], and this is known to be asymptotically optimal. In its initial form, the secretary problem was essentially a stopping time problem, and not surprisingly, it mainly attracted the interest of probabilists.

Recently, secretary problems enjoyed a revival, and various generalizations were studied. These developments are strongly motivated by a close connection to online mechanism design, where a good is sold to agents arriving online [13,2]. Here, the agents correspond to the secretaries and they reveal prices that they are willing to pay in exchange for goods. This leads to secretary problems where more than one secretary can be chosen. The most canonical generalization asks to hire k out of n secretaries, each revealing a non-negative weight upon arrival, and the goal is to hire a maximum weight subset of k secretaries. This interesting variant was introduced and studied by Kleinberg [13], who presented a $(1 - O(1/\sqrt{k}))$-approximation for this setting. However, in many applications, additional constraints have to be imposed on the elements that can be chosen. A very general class of constrained secretary problems, where the chosen elements have to form an independent set of a given matroid $M = (N, \mathcal{I})$, was introduced by Babaioff, Immorlica and Kleinberg [2][1]. This setting, now generally termed *matroid secretary problem*, covers at the same time many interesting cases and has a rich structure that can be exploited to design strong approximation algorithms.

To give a concrete example of a matroid secretary problem, and to motivate some of our results, consider the following connection problem. Given is an undirected graph $G = (V, E)$, representing a communication network, with non-negative edge-capacities $c : E \to \mathbb{N}$ and a server $r \in V$. Clients, which are the equivalent of candidates in the secretary problem, reside at vertices of the graph and are interested to connect to the server r via a unit-capacity path. The number of clients and their locations are known. Each client has a price that she is willing to pay to connect to the server. These prices are unknown and no assumptions are made on them except for being non-negative. Clients then reveal themselves one by one in random order, announcing their price. Whenever a client reveals herself, the network operator has to decide irrevocably before the next client appears whether to serve this client and receive the announced price. The goal is to choose a maximum weight subset of clients that can be served simultaneously without exceeding the given capacities c. It is well-known that the constraints imposed by the limited capacity on the clients that can be chosen is a special type of matroid constraint, namely a gammoid constraint [19].

[1] A matroid $M = (N, \mathcal{I})$ consists of a finite set N, called the *ground set*, and a non-empty family $\mathcal{I} \subseteq 2^N$ of subsets of N, called *independent sets*, satisfying: (i) $I \in \mathcal{I}, J \subseteq I \Rightarrow J \in \mathcal{I}$, and (ii) $I, J \in \mathcal{I}, |I| > |J| \Rightarrow \exists f \in I \setminus J$ with $J \cup \{f\} \in \mathcal{I}$. For more information on matroids we refer the reader to [19]

For the classical matroid secretary problem, as discussed above, the currently best approximation algorithm is a $O(\sqrt{\log \rho})$-approximation by Chakraborty and Lachish [4], where ρ is the rank of the matroid. This improved on an earlier $O(\log \rho)$-approximation of Babaioff, Immorlica and Kleinberg [2]. Babaioff et al. conjectured that there is an $O(1)$-approximation for the matroid secretary problem. This conjecture remains open and is arguably the currently most important open question regarding the matroid secretary problem.

Motivated by this conjecture, many interesting advances have been made to obtain $O(1)$-approximations, either for special cases of the matroid secretary problem or variants thereof. In particular, $O(1)$-approximations have been found for graphic matroids [2,15] (currently best approximation factor: $2e$), transversal matroids [2,5,15] (8-approximation), co-graphic matroids [20] ($3e$-approximation), linear matroids with at most k non-zero entries per column [20] (ke-approximation), and most recently laminar matroids [12] (16000/3-approximation). For most of the above special cases, strong approximation algorithms have been found, typically based on very elegant techniques. However for the laminar matroid, only a considerably higher approximation factor is known due to Im and Wang [12], using a very clever but quite involved method and analysis.

Furthermore, variants of the matroid secretary problem have been investigated that assume random instead of adversarial assignment of the weights, and for which $O(1)$-approximations can be obtained without any restriction on the underlying matroid. Recall that the classical matroid secretary problem does not make any assumptions on how weights are assigned to the elements, which means that we have to assume a worst-case, i.e., *adversarial*, weight assignment. However, the order in which the elements reveal themselves is assumed to be random. Soto [20] considered the variant where not only the arrival order of the elements is assumed to be uniformly random but also the assignment of the weights to the elements, and presented a $2e^2/(e-1)$-approximation for this case. More precisely, in this model, the weights can still be chosen by an adversary, but are then assigned uniformly at random to the elements of the matroid. Building on Soto's work, Vondrák and Oveis Gharan [18] showed that a $40e/(e-1)$-approximation can even be obtained when the arrival order of the elements is adversarial and the assignment of weights remains uniformly at random. Hence, this model is somehow the opposite of the classical matroid secretary problem, where assignment is adversarial and arrival order is random.

However, so far, no progress has been made in variants with adversarial assignment. One such variant, suggested shortly after the introduction of the matroid secretary problem [14], assumes that the appearance order of elements can be chosen by the algorithm. More precisely, in this model, which we call the *free order model*, whenever a next element has to reveal itself, the algorithm can choose the element to be revealed. E.g. in the above network connection problem, one could decide at each step which is the next client to reveal its price, by using for this decision the network structure and the elements observed so far. A main further complication when dealing with adversarial assignments—as in the free order model—contrary to random assignment, is that the knowledge of the initial structure of the matroid seems to be of little help. This is due to the fact that an adversary can assign a weight of zero to most elements of the matroid, and only give a non-negative weight to a selected subset $A \subseteq N$ of elements. Hence, the problem essentially reduces to the restriction $M|_A$ of the matroid M over the elements A. However, the structure of $M|_A$ is essentially impossible to guess from M. This is in stark contrast to models with random assignment, e.g., in the model considered by Soto, the mentioned $2e^2/(e-1)$-approximation right at the start exploits the given structure of the matroid M, by partitioning N and solving a standard single secretary problem on each part of the partition. Different approaches are needed for adversarial weight assignments.

We are interested in the following two questions. First, is there an $O(1)$-approximation for the free order model? Second, we are interested in getting a better understanding of the laminar case of the classical secretary problem, with the goal to find considerably stronger and simpler procedures.

As it is common in this context, when we talk about a c-approximation we always compare against the *offline* optimum solution, i.e., the maximum weight independent set. In this type of analysis, known as *competitive analysis*, a c-approximation is also called a *c-competitive algorithm*.

Our results and techniques We present a 9-approximation for the free order model, thus obtaining the first $O(1)$-approximation for a variant of the matroid secretary problem with adversarial weight assignment, without any restriction on the underlying matroid. This algorithm is in particular applicable to the mentioned network connection problem, when the order, in which the network operator negotiates with the clients, can be chosen. Previously, no matroid secretary model with adversarial weight assignment was known to admit an $O(1)$-approximation for this problem setting.

On a high level our algorithm follows a quite intuitive idea, which, interestingly, does not work in the traditional matroid secretary problem. In a first phase, we draw each element with probability 0.5 to obtain a set $A \subseteq N$, without selecting any element of A. Let OPT_A be the best offline solution in A. We call an element $f \in N \setminus A$ *good*, if it can be used to improve OPT_A, in the sense that either $\mathrm{OPT}_A \cup \{f\}$ is independent or there is an element $g \in \mathrm{OPT}_A$ such that $(\mathrm{OPT}_A \setminus \{g\}) \cup \{f\}$ is independent and has a higher value than OPT_A. In the second phase, we go through the remaining elements $N \setminus A$, drawing element by element in a well-chosen way to be specified soon. We accept an element $f \in N \setminus A$ if it is good and does not destroy independence when added to the elements accepted so far. Our approach fails if elements are drawn randomly in the second phase. The main problem when drawing randomly, is that we may accept good elements of relatively low value that may later *block* some high-valued good elements, in the sense that they cannot be added anymore without destroying independence of the selected elements. To overcome this problem, we determine after the first phase a specific order of how elements will be drawn in the second phase. The idea is to first draw elements of $N \setminus A$ that are in the span of elements of A of high weight. More precisely, let $A = \{a_1, \dots, a_m\}$ be the numbering of the elements of A according to decreasing weights. In the second phase we start by drawing elements of $(N \setminus A) \cap \text{span}(\{a_1\})$, then $(N \setminus A) \cap \text{span}(\{a_1, a_2\})$, and so on[2]. Intuitively, if there is a set $S \subseteq N$ with a high density of high-valued elements, then it is likely that many elements of S are part of A. Hence, high-valued elements of A span further high-valued elements in S. Thus, by the above order, we are likely to draw high-valued elements of S early, before they can be blocked by the inclusion of lower-valued elements.

Similar to previous secretary algorithms, we show that our algorithm is a $O(1)$-approximation by proving that each element $f \in \text{OPT}$ of the global offline optimum OPT will be chosen with probability at least $1/9$. However, the way we prove this is based on a novel approach. Broadly speaking, we show that for any element $f \in \text{OPT}$ there is a threshold weight $\overline{\mathrm{w}}_f$ such that with constant positive probability we have simultaneously: (i) $f \notin A$, (ii) f is spanned by the elements in A with weight $\geq \overline{\mathrm{w}}_f$, and (iii) good elements considered in the second phase with weight at least $\overline{\mathrm{w}}_f$ do not block f. From this we can observe that f gets selected with constant probability. Interestingly, several probabilities of interest that appear in our analysis are very hard to compute exactly. E.g., even when all weights are known and a threshold $\overline{\mathrm{w}}_f$ is given, it is in general $\#P$-hard to compute the probability that f is in the span of all elements of A of weight at least $\overline{\mathrm{w}}_f$ [3]. Still, we can show that a good threshold weight $\overline{\mathrm{w}}_f$ exists, which is all we need to guarantee that our algorithm is a $O(1)$-approximation.

Furthermore, we present a new approach to deal with laminar matroids in the classical matroid secretary model. Our technique leads to a $3\sqrt{3}e \approx 14.12$-approximation, thus considerably improving on the $16000/3 \approx 5333$-approximation of Im and Wang [12]. Our main contribution here is to present a simple way to transform the matroid secretary problem on a laminar matroid M to one on a unitary partition matroidM_P by losing only a small constant factor of $3\sqrt{3} \approx 5.2$. The secretary problem on M_P can then simply be solved by applying the classical e-approximation for the standard secretary problem to each partition of M_P. We first observe a constant fraction of all elements, on the basis of which a partition matroid M_P on the remaining elements is then constructed. To assure feasibility, M_P is defined such that each independent set of M_P is as well an independent set of M. To best convey the main ideas of our procedure, we focus on a very simple method to obtain a weaker $27e/2 \approx 36.7$-approximation, which already improves considerably on the 16000/3-approximation of Im and Wang. The $3\sqrt{3}e$-approximation is obtained through a strengthening of this approach by using a stronger partition matroid M_P and a tighter analysis.

We remark that the algorithms we present do not need to observe the exact weights of the items when they reveal themselves, but only need to be able to

[2] We recall that $\text{span}(S)$ for $S \subseteq N$ is the unique maximal set $U \supseteq S$ with the same rank as S.

[3] Consider for example the graphic matroid with underlying graph $G = (V, E)$. Here, the question whether some edge $\{s, t\} \in E$ is in the span of a random set of edge $A \subseteq E$ containing each edge with probability 0.5, reduces to the question of whether A contains an s-t path. This is the well-known $\#P$-hard s-t reliability problem [21].

compare the weights of elements observed so far. This is a common feature of many matroid secretary algorithms and matroid algorithms more generally.

To simplify the exposition, we assume that all weights are distinct, i.e., they induce a linear order on the elements. This implies in particular, that there is a unique maximum weight independent set. The general case with possibly equal weights easily reduces to this case by breaking ties arbitrarily between elements of equal weight to obtain a linear order.

Related work We mention briefly that recently, matroid secretary problems with submodular objective functions have been considered. For this setting, $O(1)$-approximations have been found for knapsack constraints, uniform matroids, and more generally for partition matroids if the submodular objective function is furthermore monotone [3,7,11].

Additionally, variations of the matroid secretary problem have been considered with restricted knowledge on the underlying matroid type. This includes the case where no prior knowledge of the underlying matroid is assumed except for the size of the ground set. Or even more extremely, the case without even knowing the size of the ground set. For more information on such variations we refer to the excellent overview in [18].

Subsequent results We would like to highlight that very recently, Ma, Tang and Wang [17] further improved the currently best approximation ratio for the secretary problem on laminar matroids by presenting a 9.6-approximation.

Organization of the paper Our 9-approximation for the free order model is presented in Section 2. Section 3 discusses our simple $27e/2$-approximation for the classical matroid secretary problem restricted to laminar matroids. Due to space constraints, we defer details of how this algorithm can be strengthened to obtain the claimed $3\sqrt{3}e$-approximation to a long version of this paper.

2 A 9-approximation for the free order model

To simplify the writing we use "+" and "−" for the addition and subtraction of single elements from a set, i.e., $S+f-g = (S \cup \{f\}) \setminus \{g\}$. Algorithm 1 describes our 9-approximation for the free order model. It operates in two phases.

As mentioned previously, a *good* element $f \in N \setminus A$ is an element that allows for improving the maximum weight independent set in A. Using standard results on matroids, an element f is good if either $f \notin \text{span}(A)$, or if there is an index $i \in \{1, \dots m\}$ such that $f \in \text{span}(A_i) \setminus \text{span}(A_{i-1})$ and $\text{w}(f) > \text{w}(a_i)$. Hence, our algorithm indeed only accepts good elements.

To show that Algorithm 1 is a 9-approximation, we show that each element f of the offline optimum OPT will be contained in the set I returned by the algorithm with probability at least $1/9$. We distinguish two cases:

(i) $\Pr[f \in \text{span}(A - f)] \leq 1/3$, and
(ii) $\Pr[f \in \text{span}(A - f)] > 1/3$.

Algorithm 1 A 9-approximation for the free order model.

1. **Draw** each element with probability 0.5 to obtain $A \subseteq N$, without selecting any element of A. We number the elements of $A = \{a_1, \dots, a_m\}$ in decreasing order of weights. Define $A_i = \{a_1, \dots, a_i\}$, with $A_0 = \emptyset$.
   Initialize: $I \leftarrow \emptyset$.
2. **For** $i = 1$ to m:
   draw one by one (in any order) all elements $f \in (\text{span}(A_i) \setminus \text{span}(A_{i-1})) \setminus A$.
   if $I + f \in \mathcal{I}$ and $\text{w}(f) > \text{w}(a_i)$, **then** $I = I + f$.
   For all remaining elements $f \in N \setminus \text{span}(A)$ (drawn in any order):
   if $I + f \in \mathcal{I}$, **then** $I = I + f$.
   Return I

The following lemma handles the simpler first case, which allows us to highlight some ideas that we will also employ to prove the more interesting second case. Notice that in the following statement we do not even have to assume $f \in \text{OPT}$.

Lemma 1. *Let $f \in N$ with $\Pr[f \in \text{span}(A - f)] \leq 1/3$. Then f is selected by Algorithm 1 with probability at least $1/6$.*

Proof. We start by observing that f is selected by Algorithm 1 if the three events $E_1 : f \notin A$, $E_2 : f \notin \text{span}(A - f)$ and $E_3 : f \notin \text{span}((N \setminus A) - f)$ happen simultaneously. Indeed, if $E_1 \cap E_2$ occurs, then f will be considered during the second for-loop of the second phase of Algorithm 1. Furthermore, adding f at that moment will not violate independence since the elements selected so far are a subset of $N \setminus A$, and if E_3 holds we have $f \notin \text{span}((N \setminus A) - f)$. It therefore suffices to show that the probability of E_1, E_2, E_3 happening simultaneously is at least $1/6$.

Notice that E_1 is independent of E_2, E_3. Hence,

$$\Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1] \cdot \Pr[E_2 \cap E_3] = \frac{1}{2} \cdot \Pr[E_2 \cap E_3]. \tag{1}$$

Furthermore, we observe that A and $N \setminus A$ have the same distribution since they contain each element of N with probability 0.5. Hence,

$$\Pr[E_3] = \Pr[E_2] = 1 - \Pr[f \in \text{span}(A - f)] \geq \frac{2}{3}.$$

Denoting by $\overline{E_2}$ and $\overline{E_3}$ the complements of E_2 and E_3, respectively, we thus obtain by the union bound:

$$\Pr[E_2 \cap E_3] = 1 - \Pr[\overline{E_2} \cup \overline{E_3}] \geq 1 - \Pr[\overline{E_2}] - \Pr[\overline{E_3}] = \Pr[E_2] + \Pr[E_3] - 1 \geq \frac{1}{3}.$$

Combining the above with (1) we obtain $\Pr[E_1 \cap E_2 \cap E_3] \geq 1/6$. □

We now consider the case $f \in \text{OPT}$ with $\Pr[f \in \text{span}(A - f)] > 1/3$. Let $N = \{f_1, \dots, f_n\}$ be the numbering of all elements in decreasing order of weights, and let $N_j = \{f_1, \ldots, f_j\}$ with $N_0 = \emptyset$. This time, we want to show that with constant probability, f is chosen in the first for-loop of the second phase of Algorithm 1. More precisely, we want to find a good threshold weight $\overline{\mathrm{w}}_f$ as discussed in the introduction. For this we determine an index $\bar{j} \in \{1, \ldots, n\}$—and $\overline{\mathrm{w}}_f$ will then correspond to $\mathrm{w}(f_{\bar{j}})$—satisfying two properties. First, we want that with constant positive probability, $f \in \mathrm{span}((A \cap N_{\bar{j}}) - f)$. The benefit of having $f \in \mathrm{span}((A \cap N_{\bar{j}}) - f)$ is that if additionally $f \notin A$, then f will be considered in the first for-loop of phase two at some iteration i with $\mathrm{w}(a_i) \geq \mathrm{w}(f_{\bar{j}})$. Hence, up to that point, only elements with weight $\geq \mathrm{w}(f_{\bar{j}})$ have been selected. Thus, when checking whether f can be added without violating independence, only those elements have to be considered. Second, we want that $\Pr[f \in \mathrm{span}((A \cap N_{\bar{j}}) - f)]$ is also bounded away from 1, because this implies that $\Pr[f \notin \mathrm{span}((N_{\bar{j}} \setminus A) - f)] = \Pr[f \notin \mathrm{span}((A \cap N_{\bar{j}}) - f)]$ is some constant > 0. Whenever $f \notin \mathrm{span}((N_{\bar{j}} \setminus A) - f)$ occurs, then f will not violate independence when added to any set of selected elements with weight at least $\mathrm{w}(f_{\bar{j}})$, since they are a subset of $N_{\bar{j}} \setminus A$. Hence, intuitively, for our analysis we want to find an index $\bar{j}$ such that $\Pr[f \in \mathrm{span}((A \cap N_{\bar{j}}) - f)]$ is bounded away from zero and from one. The following lemma shows that such an index indeed exists. In Lemma 3, we then show how the above sketch of our proof can be formalized, and in particular, how to deal with dependencies of the different events discussed above.

For brevity, let $p_j = \Pr[f \in \mathrm{span}((A \cap N_j) - f)]$.

Lemma 2. *Let $f \in N$ with $\Pr[f \in \mathrm{span}(A - f)] \geq 1/3$. There exists an index $\bar{j} \in \{1, \ldots, n\}$, such that $p_{\bar{j}} \in [1/3, 2/3]$.*

Proof. By assumption we have $p_n = \Pr[f \in \mathrm{span}(A - f)] > 1/3$. Furthermore, $p_0 = 0$. Since p_j is increasing in j, to prove the proposition it suffices to show that $p_{j+1} \leq 2/3$, for all $j \in \{0, \ldots, n-1\}$ such that $p_j < 1/3$. This indeed holds due to the following:

$$\begin{aligned} p_{j+1} = & \underbrace{\Pr[f_{j+1} \notin A]}_{=0.5} \cdot \underbrace{\Pr[f \in \mathrm{span}((A \cap N_{j+1}) - f) \mid f_{j+1} \notin A]}_{=p_j} + \\ & \underbrace{\Pr[f_{j+1} \in A]}_{=0.5} \cdot \underbrace{\Pr[f \in \mathrm{span}((A \cap N_{j+1}) - f) \mid f_{j+1} \in A]}_{\leq 1} \leq \frac{1}{2}(p_j + 1). \end{aligned}$$

□

The following completes the proof of the case $\Pr[f \in \mathrm{span}(A - f)] > 1/3$.

Lemma 3. *Let $f \in \mathrm{OPT}$ with $\Pr[f \in \mathrm{span}(A - f)] > 1/3$. Then f is selected by Algorithm 1 with probability at least $1/9$.*

Proof. Let $\bar{j} \in \{1, \ldots, n\}$ be an index with $p_{\bar{j}} \in [1/3, 2/3]$ as claimed by Lemma 2. We start by reasoning that f will be selected by Algorithm 1 if the three events $E_1 : f \notin A$, $E_2 : f \in \mathrm{span}((A \cap N_{\bar{j}}) - f)$, and $E_3 : f \notin \mathrm{span}((N_{\bar{j}} \setminus A) - f)$ happen simultaneously. Notice that $E_1 \cap E_2$ implies that f will be considered during the first for-loop of the second phase of Algorithm 1, at some iteration i with $\mathrm{w}(a_i) \geq \mathrm{w}(f_{\bar{j}})$. Since the elements selected so far—at the time f is considered—must all have a weight of at least $\mathrm{w}(a_i) \geq \mathrm{w}(f_{\bar{j}})$, the occurrence of E_3 guarantees that f can be added without violating independence since the selected elements at that point are a subset of $N_{\bar{j}} \setminus A$. Also notice that since $f \in \mathrm{OPT}$ and $f \in \mathrm{span}(A_i)$, we have $\mathrm{w}(f) > \mathrm{w}(a_i)$, i.e., f is good. Therefore, f indeed gets selected if E_1, E_2, E_3 occur simultaneously. Hence it suffices to show that $E_1 \cap E_2 \cap E_3$ occurs with probability $\geq 1/9$. Again, E_1 is independent of E_2, E_3, and hence

$$\Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1] \cdot \Pr[E_2 \cap E_3] = \frac{1}{2} \cdot \Pr[E_2 \cap E_3]. \tag{2}$$

To deal with the dependence between the events E_2 and E_3 we invoke the FKG inequality (see [1]). Notice that both events E_2 and E_3 are *increasing* in A, i.e., for any two sets $Q, P \subseteq N$ with $Q \subseteq P$, if E_2 (or E_3) occurs for $A = Q$ then it also occurs if $A = P$. The FKG inequality then implies

$$\Pr[E_2 \cap E_3] \geq \Pr[E_2] \cdot \Pr[E_3]. \tag{3}$$

Furthermore, since $A \cap N_{\bar{j}}$ has the same distribution as $N_{\bar{j}} \setminus A$, we have $\Pr[E_3] = 1 - \Pr[E_2]$. Hence, together with (2) and (3) we obtain

$$\Pr[E_1 \cap E_2 \cap E_3] = \frac{1}{2} \cdot \Pr[E_2] \cdot (1 - \Pr[E_2]).$$

Due to our choice of $\bar{j}$, we have $\Pr[E_2] \in [1/3, 2/3]$, and hence $\Pr[E_2] \cdot (1 - \Pr[E_2]) \geq 2/9$, thus leading to $\Pr[E_1 \cap E_2 \cap E_3] \geq 1/9$ as desired. $\square$

Finally, by combining Lemma 1 and Lemma 3 we obtain.

Corollary 1. *Algorithm 1 is a* 9*-approximation for the free order model.*

3 Classical secretary problem for laminar matroids

Let $M = (N, \mathcal{I})$ be a laminar matroid whose constraints are defined by the laminar family $\mathcal{L} \subseteq 2^N$ with upper bounds b_L for $L \in \mathcal{L}$ on the number of elements that can be chosen from $\mathcal{L}$, i.e., $\mathcal{I} = \{I \subseteq N \mid |I \cap L| \leq b_L \ \forall L \in \mathcal{L}\}$. Without loss of generality we assume $b_L \geq 1$ for $L \in \mathcal{L}$, since otherwise we can simply remove all elements of L from M. Furthermore, we assume $N \in \mathcal{L}$, since otherwise a redundant constraint $|I \cap N| \leq b_N$ can be added by choosing a sufficiently large right-hand side b_N.

To reduce the matroid secretary problem on M to a problem on a partition matroid, we first number the elements $N = \{f_1, \dots, f_n\}$ such that for any set $L \in \mathcal{L}$, the elements in L are numbered consecutively, i.e., $L = \{f_p, \dots, f_q\}$ for some $1 \leq p < q \leq n$. Figure 1 shows an example of such a numbering.

For the sake of exposition, we start by presenting a conceptually simple algorithm and analysis, based on the introduced numbering of the ground set, that leads to a $27e/2$-approximation. The claimed $3\sqrt{3}e$-approximation follows the same ideas, but strengthens both the approach and analysis. Algorithm 2

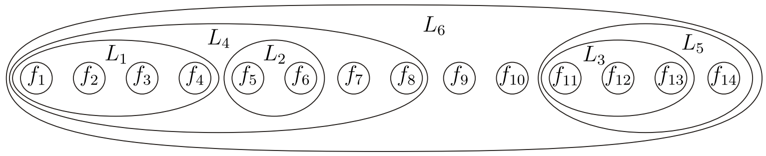

Fig. 1. An example of a numbering of the elements of the ground set such that each set $L \in \mathcal{L} = \{L_1, \ldots, L_6\}$ of the laminar family contains consecutively numbered elements.

Algorithm 2 A $27e/2$-approximation for laminar matroids.

1. **Observe** $\text{Binom}(n, 2/3)$ elements of N, which we denote by $A \subseteq N$.
   Determine maximum weight independent set $\text{OPT}_A = \{f_{i_1}, \ldots, f_{i_p}\}$ in A where $1 \leq i_1 < \cdots < i_p \leq n$. Define $P_j = \{f_k \mid k \in \{i_{j-1}, \ldots, i_j\}\} \setminus A$ for $j \in \{1, \ldots, p+1\}$, where we set $i_0 = 0, i_{p+1} = n$. Let
   $$\mathcal{P}_{\text{odd}}(A) = \{P_j \mid j \in \{1, \ldots, p+1\}, j \text{ odd}\},$$
   $$\mathcal{P}_{\text{even}}(A) = \{P_j \mid j \in \{1, \ldots, p+1\}, j \text{ even}\}.$$
   If $\text{OPT}_A = \emptyset$ **then** set $\mathcal{P} = \{N \setminus A\}$,
   else set $\mathcal{P} = \mathcal{P}_{\text{odd}}(A)$ with probability 0.5, otherwise set $\mathcal{P} = \mathcal{P}_{\text{even}}(A)$.
2. **Apply** to each set $P \in \mathcal{P}$ an e-approximate classical secretary algorithm to obtain an element $g_P \in P$.
   Return $\{g_P \mid P \in \mathcal{P}\}$.

describes our $27e/2$-approximation. Notice that applying a standard secretary algorithm to the sets of $\mathcal{P}$ in step 2 can easily be performed by running $|\mathcal{P}|$ many e-approximate secretary algorithms in parallel, one for each set $P \in \mathcal{P}$. Elements are drawn one by one in the second phase, and they are forwarded to the secretary algorithm corresponding to the set P that contains the drawn element, and are discarded if no set of $\mathcal{P}$ contains the element. Furthermore, observe that A contains each element of N independently with probability 2/3.

We start by observing that Algorithm 2 returns an independent set.

Lemma 4. *Let $A \subseteq N$ with $\text{OPT}_A \neq \emptyset$ and let $\mathcal{P} \in \{\mathcal{P}_{\text{even}}(A), \mathcal{P}_{\text{odd}}(A)\}$. For each $P \in \mathcal{P}$, let g_p be any element in P. Then $\{g_P \mid P \in \mathcal{P}\} \in \mathcal{I}$.*

Proof. Let $I = \{g_P \mid P \in \mathcal{P}\}$ be a set as stated in the lemma. Notice that for any two elements $f_k, f_\ell \in I$ with $k < \ell$ we have $|\text{OPT}_A \cap \{f_k, f_{k+1}, \ldots, f_\ell\}| \geq 2$. Now consider a set $L \in \mathcal{L}$ corresponding to one of the constraints of the underlying laminar matroid. By the above observation and since L is consecutively numbered, at least one of the following holds: (i) $|L \cap I| = 1$, or (ii) $|L \cap \text{OPT}_A| \geq |L \cap I|$. If case (i) holds, then the constraint corresponding to L is not violated since we assumed $b_L \geq 1$. If (ii) holds, then L is also not violated since $|L \cap I| \leq |L \cap \text{OPT}_A| \leq b_T$ because $\text{OPT}_A \in \mathcal{I}$. Hence $I \in \mathcal{I}$. □

Theorem 1. *Algorithm 2 is a $27e/2$-approximation for the laminar matroid secretary problem.*

Proof. Let $\text{OPT} \in \mathcal{I}$ be the maximum weight independent set in N, i.e., the offline optimum. Furthermore, let I be the set returned by Algorithm 2, and let $f \in \text{OPT}$. We say that f is *solitary* if $\exists\, P \in \mathcal{P}$ with $P \cap \text{OPT} = \{f\}$. Similarly we call $P \in \mathcal{P}$ *solitary* if $|P \cap \text{OPT}| = 1$. We prove the theorem by showing that each element $f \in \text{OPT}$ is solitary with probability $\geq 2/27$. This indeed implies the theorem since we can do the following type of accounting. Let X_f be the random variable which is zero if f is not solitary, and otherwise if f is solitary, X_f equals the weight of the element $g \in I$ that was chosen by the algorithm out of P that contains f. By only considering the weights of elements chosen in solitary sets $\mathcal{P}$ we obtain

$$\mathbf{E}[w(I)] \geq \sum_{f \in \text{OPT}} \mathbf{E}[X_f]. \tag{4}$$

However, if each element $f \in \text{OPT}$ is solitary with probability $2/27$, we obtain $\mathbf{E}[X_f] \geq \frac{2w(f)}{27e}$, because the classical secretary algorithm will choose with probability $1/e$ the maximum weight element of the set P that contains the solitary element f. Combining this with (4) yields $\mathbf{E}[w(I)] \geq \frac{2}{27e} w(\text{OPT})$ as desired. It remains to show that each $f \in \text{OPT}$ is solitary with probability $\geq 2/27$.

Let $f_i \in \text{OPT}$. We assume that OPT contains an element with a lower index than i and one with a higher index than i. The cases of f_i being the element with highest or lowest index in OPT follow analogously. Let $f_j \in \text{OPT}$ by the element of OPT with the largest index $j < i$. Similarly, let $f_k \in \text{OPT}$ be the element of OPT with the smallest index $k > i$. One well-known matroidal property that we use is $\text{OPT} \cap A \subseteq \text{OPT}_A$. Hence, if $f_j, f_k \in A$ then also $f_j, f_k \in \text{OPT}_A$, and if furthermore $f_i \notin A$, then f_i will be the only element of OPT in the set $P \in \mathcal{P}_{\text{odd}}(A) \cup \mathcal{P}_{\text{even}}(A)$ that contains f_i. Hence, if the coin flip in Algorithm 2 chooses the family $\mathcal{P} \in \{\mathcal{P}_{\text{odd}}(A), \mathcal{P}_{\text{even}}(A)\}$ that contains P, then f_i is solitary. To summarize, f_i is solitary if $f_j, f_k \in A$, $f_i \notin A$ and the coin flip for $\mathcal{P}$ turns out right. This happens with probability $\left(\frac{2}{3}\right)^2 \cdot \left(1 - \frac{2}{3}\right) \cdot \frac{1}{2} = \frac{2}{27}$. □

One conservative aspect of the proof of Lemma 1 is that we only consider the contribution of solitary elements. Additionally, a drawback of Algorithm 2 itself is that about half of the elements of $N \setminus A$ are ignored as we only select from either $\mathcal{P}_{\text{odd}}(A)$ or $\mathcal{P}_{\text{even}}(A)$. Addressing these weaknesses, the claimed $3\sqrt{3}e$-approximation can be obtained. Details are omitted due to space constraints.

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