An average-case asymptotic analysis of the Container Relocation Problem

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The Container Relocation Problem (CRP) involves finding a sequence of moves of containers that minimizes the number of relocations needed to retrieve all containers in a given order. In this paper, we focus on average case analysis of the CRP when the number of columns grows asymptotically. We show that the expected minimum number of relocations converges to a simple and intuitive lower-bound for which we give an analytical formula.

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1. Introduction

Due to limited space in the storage area in container terminals, containers are stacked in columns on top of each other. As shown in Fig. 1, several columns of containers create bays of containers. If a container that needs to be retrieved (target container) is not on the topmost tier of a column and is covered by other containers, the blocking containers must be relocated to other slots. As a result, during the retrieval process, one or more relocation moves are performed by the yard cranes. Finding the sequence of moves that minimizes the total number of relocations while retrieving containers from a bay in a pre-defined order is referred to as the Container Relocation Problem (CRP) or the Block Relocation Problem (BRP). For reviews and classification surveys of the existing literature on the CRP, we refer the reader to [3,6].

A common assumption of the CRP is that only the containers that are blocking the target container can be relocated. We refer to the CRP with this setting as the restricted CRP. In this paper, unless stated otherwise, CRP refers to the restricted CRP. On the other hand, if we relax this assumption, we will refer to the problem as unrestricted CRP.

In this paper, we study the CRP for large randomly distributed bays and we show that the ratio between the expected minimum number of relocations and a simple lower bound (developed by [2] and denoted below by $S_0$) approaches 1. While the problem is known to be NP-hard, this gives strong evidence that the CRP is “easier” to solve for large instances, and that heuristics can find near-optimal solutions.

Let us define the problem more formally: we are given a bay with $C$ columns, $P$ tiers; initially $N$ containers are stored in the bay with exactly $h$ containers in each column, where $h \leq P - 1$, so $N = h \times C$. We denote such a bay $B_{h,C}$. We label the containers based on their required departure order, i.e., container 1 is the first one to be retrieved. The CRP corresponds to finding a sequence of moves to retrieve containers 1, 2, . . ., $N$ (respecting the order) with a minimum number of relocations. For bay $B_{h,C}$, we denote the minimum number of relocations by $z_{\text{opt}}(B_{h,C})$. We focus on an average case analysis when the number of columns grows asymptotically. In our model, since $N = h \times C$, when $C$ grows to infinity, $N$ also grows to infinity. Average case analysis of CRP is fairly new. The only other paper found in the literature is by [5]. They also provide a probabilistic analysis of the asymptotic CRP when both the number of columns and tiers grow to infinity. They show that there exists a polynomial time algorithm that solves this problem close to optimality with high probability. Our model departs from theirs in two aspects: (i) We keep the maximum height (number of tiers) a constant whereas in [5] the height also grows. Our assumption is motivated by the fact that the maximum height is limited by the crane
2. Background

2.1. Uniformly random bay

We view a bay as an array of \( P \times C \) slots (see Fig. 2). The slots are numbered from bottom to top, and left to right from 1 to \( P \times C \). The goal is to generate a bay \( B_{h,C} \) with uniform probability, meaning each container is equally likely to be anywhere in the configuration, with the restriction that there are \( h \) containers per column. We first generate a uniformly random permutation of \( \{1, \ldots, h \times C\} \) called \( \pi \). Then we assign a slot for each container with the following relation: \( B_{h,C}(i,j) = \pi(h \times (j-1) + i) \) for \( i \leq h \) and \( B_{h,C}(i,j) = 0 \) for \( i > h + 1 \). One can see that each bay is generated with probability \( 1/N! \). There is a one to one mapping between configurations with \( C \) columns and permutations of \( \{1, \ldots, h \times C\} \), denoted by \( B_{h,C} \). Finally, we compute the expectation of random variable \( X \) over this uniform distribution by \( \mathbb{E}_{h,C}[X] \).

2.2. The counting lower bound \( S_0 \)

This bound was introduced in [2] and it is based on the following simple observation. In the initial configuration, if a container is blocking, then it must be relocated at least once. Thus we count the number of blocking containers in \( B_{h,C} \), we denote it as \( S_0(B_{h,C}) \), and we have \( z_{\text{opt}}(B_{h,C}) \geq S_0(B_{h,C}) \). Note that if a container blocks more than one container, it is counted only once. In Lemma 1 we give an explicit formula for the expectation of \( S_0 \) under the uniform distribution.

**Lemma 1.** Let \( C, h \in \mathbb{N} \) and \( S_0 \) be the counting lower bound defined above, we have

\[
\mathbb{E}_{h,C}[S_0(B_{h,C})] = \alpha_h \times C,
\]

where \( \alpha_h = h - \sum_{i=1}^{h-1} 1/i \) is the expected number of blocking containers in one column.

**Fact 2.** Note that \( \alpha_h \) only depends on \( h \).

**Proof of Lemma 1.** Let \( S_0'(B_{h,C}) \) be the number of blocking containers in column \( i \). By the linearity of expectation, we have \( \mathbb{E}_{h,C}[S_0'(B_{h,C})] = \mathbb{E}_{h,C} \left[ \sum_{i=1}^{C} S_0'(B_{h,C}) \right] = \sum_{i=1}^{C} \mathbb{E}_{h,C}[S_0'(B_{h,C})] = \alpha_h \times C \), where \( \alpha_h = \mathbb{E}_{h,C}[S_0'(B_{h,C})] = \mathbb{E}_{h,1}[S_0(B_{h,1})] \). This relies on the fact that each column is identically distributed.

Now let us compute \( \alpha_h \). It is clear that \( \alpha_1 = 0 \). For \( h \geq 2 \), by conditioning on the event that the topmost container is the smallest number in the column or not, we obtain the recursive equation \( \alpha_h = \alpha_{h-1} + (h - 1)/h \). Finally by induction we have \( \alpha_h = h - \sum_{i=1}^{h} 1/i \) which completes the proof.

2.3. The heuristic \( H([1]) \)

Suppose \( n \) is the target container located in column \( c \), and \( r \) is the topmost blocking container in \( C \). For convenience, we denote by \( \min(C) \) the minimum of column \( c \) (note that \( \min(C) = N + 1 \) if \( C \) is empty). \( H \) uses the following rule to determine \( c^* \), the column where \( r \) should be relocated to. If there is a column \( c \) with \( |c| < P \), where \( \min(c) \) is greater than \( r \), then \( H \) chooses such a column where \( \min(c) \) is minimized, since columns with larger minimums can be useful for larger blocking containers (as \( r \) will never be relocated again, we say this relocation of \( r \) is a "good" move). If there is no column satisfying \( \min(c) > r \) (any relocation of \( r \) can only result in a "bad" move), then \( H \) chooses the column where \( \min(c) \) is maximized in order to delay the next unavoidable relocation of \( r \) as much as possible. We will refer to this heuristic as heuristic \( H \) and denote its number of relocations by \( z_H(B_{h,C}) \). Notice that \( z_{\text{opt}}(B_{h,C}) \leq z_H(B_{h,C}) \). Finally we state the following simple fact, which does not require a formal proof:

**Fact 3.** For any configuration \( B \) with \( C \) columns and at most \( C \) containers, we have

\[
S_0(B) = z_{\text{eq}}(B) = z_H(B).
\]

2.4. Definition of "special" columns

For \( h, C \in \mathbb{N} \), a column in \( B_{h,C} \) is called "special" if all of its containers belong to the \( C \) highest. Given this definition, a column in \( B_{h,C+1} \) is "special" if all containers belong to the \( C + 1 \) highest, or equivalently, if each of its containers has index at least \( \alpha_h \), where \( \alpha_h = (h - 1)(C + 1) + 1 \). We will also consider the following event:

\[
\Omega_{h,C} = \{B_{h,C+1} \text{ has at least } 1 \text{ \"special\" column}\}.
\]

**Lemma 4** states that the event \( \Omega_{h,C} \) has a probability that increases exponentially fast to 1 as a function of \( C \). The proof of Lemma 4 can be found in the Appendix.

**Lemma 4.** Let \( h, C \in \mathbb{N} \) such that \( C \geq h + 1 \) and \( \Omega_{h,C} \) be the event defined by Eq. (3), then we have

\[
\mathbb{P}(\Omega_{h,C}) \leq e^{-\theta_h (C+1)},
\]

where

\[
\theta_h = \frac{1}{8h} \left( \frac{2}{h(h + 1)} \right)^{2h} > 0.
\]
3. An average-case asymptotic analysis of CRP

The major result of this paper states that when the number of columns increases to infinity, the expected optimal number of relocations is asymptotically proportional to the expected number of blocking containers.

**Theorem 5.** Let $S_0$ be the counting lower bound and $z_{opt}$ be the optimal number of relocations defined in Section 1. Then for $C, h \in \mathbb{N}$ such that $C \geq h + 1$, we have

$$1 \leq \frac{E_{h,C} \left[ z_{opt}(B_{h,C}) \right]}{E_{h,C} \left[ S_0(B_{h,C}) \right]} \leq f_h(C),$$

where

$$f_h(C) = 1 + \frac{K_h}{C} \to 1,$$

where $K_h$ is a constant defined by Eq. (13).

**Proof of Theorem 5.** The basic intuition is that, as the number of columns grows, for any blocking container, we can find a “good” column with high probability. This implies that with high probability, each container is only relocated once. More formally, as $C$ grows, with high probability $E_{h,C} \left[ z_{opt}(B_{h,C+1}) \right] - E_{h,C} \left[ z_{opt}(B_{h,C}) \right]$ is exactly $\alpha_h$. Therefore, for large enough $C$, $E_{h,C} \left[ z_{opt}(B_{h,C}) \right]$ essentially behaves like $\alpha_h \times C$, which is equal to $E_{h,C} \left[ S_0(B_{h,C}) \right]$ (according to Lemma 1).

Since for all configurations $B_{h,C}$, $z_{opt}(B_{h,C}) \geq S_0(B_{h,C})$ then

$$\frac{z_{opt}(B_{h,C})}{S_0(B_{h,C})} \geq 1.$$

Moreover, we have:

$$E_{h,C} \left[ z_{opt}(B_{h,C}) \right] = 1 + \frac{E_{h,C} \left[ z_{opt}(B_{h,C}) \right] - E_{h,C} \left[ S_0(B_{h,C}) \right]}{E_{h,C} \left[ S_0(B_{h,C}) \right]}$$

$$= 1 + \frac{1}{\alpha_h} - \alpha_h$$

Now we claim that there exists a constant $\theta_h > 0$ (defined in Eq. (5)) such that

$$E_{h,C} \left[ z_{opt}(B_{h,C+1}) \right] \leq E_{h,C} \left[ z_{opt}(B_{h,C}) \right] + \alpha_h$$

$$+ h(P - 1)(C + 1)e^{-\theta_h(C+1)}, \quad \forall C \geq h + 1. \quad (10)$$

Eq. (10) studies how $E_{h,C} \left[ z_{opt}(B_{h,C}) \right]$ evolves and states that it increases almost linearly in $\alpha_h$, which shows that the function $g_h(.)$ is essentially bounded. Before proving Eq. (10) we conclude the proof of the theorem. Using Eq. (10), we have for all $C \geq h + 1$:

$$E_{h,C+1} \left[ z_{opt}(B_{h,C+1}) \right] \leq E_{h,C} \left[ z_{opt}(B_{h,C}) \right] + \alpha_h$$

$$+ h(P - 1)(C + 1)e^{-\theta_h(C+1)}$$

$$\implies E_{h,C+1} \left[ z_{opt}(B_{h,C+1}) \right] \leq S_0 \left[ z_{opt}(B_{h,C}) \right]$$

$$\implies g_h(C + 1) \leq g_h(C) + h(P - 1)(C + 1)e^{-\theta_h(C+1)}$$

$$\implies g_h(C) \leq g_h(C + 1) + h(P - 1) + \sum_{i=h+2}^{C} (i e^{-\theta_h i})$$

$$\implies g_h(C) \leq g_h(C + 1) + h(P - 1) + \sum_{i=h+2}^{\infty} (i e^{-\theta_h i})$$

Now let us prove Eq. (10). Recall that a column is defined to be “special” if none of its containers are smaller than $\alpha_h$ and that $\Omega_{h,C} = \{B_{h,C} \mid C \geq h \}$ has at least one “special” column).

The intuition is the following: the probability of having a “special” column grows quickly to 1 as a function of $C$, implying that the event $\Omega_{h,C}$ happens with high probability. Now, conditioned on $\Omega_{h,C}$, we more easily express the difference between bays of size $C + 1$ and $C$ in the following way. We claim that

$$E_{h,C+1} \left[ z_{opt}(B_{h,C+1}) \right] \leq E_{h,C} \left[ z_{opt}(B_{h,C}) \right] + \alpha_h.$$
After $t'$, we run heuristic $H$ on
\[
B_{h,C}^{t+1} = \left( [n_1 \ldots n_b]^T \cup B_{h,C}^t \right).
\]

We claim that $z_\sigma$ (number of relocations performed by the feasible sequence $\sigma$ constructed above) is exactly the right side of Eq. (15). There are at most $C + 1$ containers in $B_{h,C}^{t+1}$, therefore using Fact 3, we know that if we apply the heuristic $H$ to this configuration, then the number of relocations done by $H$ is $S_0(B_{h,C}^{t+1}) = S_0(B_{h,C}^t) + S_0 \left( [n_1 \ldots n_b]^T \right)$. Therefore $z_\text{opt}(B_{h,C}^{t+1}) \leq z_\sigma(B_{h,C}^{t+1})$ and
\[
z_\text{opt}(B_{h,C}^{t+1}) = \left\{ \begin{array}{l}
\# \text{ relocations up to } t' \\
\text{done by } \sigma_\text{opt}(B_{h,C})
\end{array} \right\} + S_0(B_{h,C}^t) + S_0 \left( [n_1 \ldots n_b] \right)
\]
which gives us
\[
z_\text{opt}(B_{h,C}^{t+1}) \leq z_\text{opt}(B_{h,C}) + S_0 \left( [n_1 \ldots n_b] \right),
\]
and proves Eq. (15).

Now we can take the expectation from both sides of Eq. (15) over a uniform distribution of the rest of the $h \times C$ containers that are not in the first column. We claim that the first term on the right hand-side of Eq. (15) is exactly $\mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}) \right]$. For any configuration that appears in $B_{h,C}$ we can map it to a unique configuration $B_{h,C}$ where all containers are between 1 and $hC$, and vice versa. Thus,
\[
\mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}^{t+1}) \right] \left[ \begin{array}{c} n_1 \\
\vdots \\
n_b \end{array} \right] \leq \mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}) \right] + S_0 \left( [n_1 \ldots n_b] \right).
\]  
(18)

Next, we take the expectation of both sides of Eq. (18) over possible first columns, which is a “special” column. Now notice that if $B_{h,C}^{t+1}$ is generated uniformly in the sets of bags of size $C + 1$, then conditioned on $\Omega_{h,C}$, the probability of having a certain column $[n_1 \ldots n_b]^T$ is identical for any $n_1 \neq \ldots \neq n_b \geq \omega_{h,C}$ and it is given by
\[
P \left( \begin{array}{c}
\vdots \\
n_b \\
\vdots \\
n_1 \\
\vdots
\end{array} \right| \Omega_{h,C} \right) = \frac{(C + 1 - h)!}{(C + 1)!} = \frac{1}{(C + 1)h!}.
\]
Therefore we can write:
\[
\mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}^{t+1}) \right] \left[ \begin{array}{c} n_1 \\
\vdots \\
n_b \end{array} \right] \leq \mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}) \right] + S_0 \left( [n_1 \ldots n_b] \right).
\]
\[
\sum_{(n_1 \ldots n_b) \in \mathbb{N}_h \times \mathbb{N}_b \cap \Omega_{h,C}} \mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}^{t+1}) \right] \left[ \begin{array}{c} n_1 \\
\vdots \\
n_b \end{array} \right] + S_0 \left( [n_1 \ldots n_b] \right) \sum_{(n_1 \ldots n_b) \in \mathbb{N}_h \times \mathbb{N}_b \cap \Omega_{h,C}} P \left( \begin{array}{c} n_1 \\
\vdots \\
n_b \end{array} \right| \Omega_{h,C} \right)
\]
\[
\sum_{(n_1 \ldots n_b) \in \mathbb{N}_h \times \mathbb{N}_b \cap \Omega_{h,C}} \mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}) \right] \left[ \begin{array}{c} n_1 \\
\vdots \\
n_b \end{array} \right] + S_0 \left( [n_1 \ldots n_b] \right) \sum_{(n_1 \ldots n_b) \in \mathbb{N}_h \times \mathbb{N}_b \cap \Omega_{h,C}} P \left( \begin{array}{c} n_1 \\
\vdots \\
n_b \end{array} \right| \Omega_{h,C} \right).
\]
(21)

The equality between (19) and (20) comes from the fact that if we know that $B_{h,C}^{t+1}$ has a “special” column, then we do not need to condition on $\Omega_{h,C}$. Eq. (21) uses the fact that
\[
\sum_{n_1 \neq \ldots \neq n_b} P \left( \begin{array}{c} n_1 \\
\vdots \\
n_b \end{array} \right| \Omega_{h,C} \right) = 1.
\]

Note that, given any $(n_1, \ldots, n_b)$ such that $n_i \neq n_j$, we have
\[
\mathbb{E}_{h,1} \left[ S_0 \left( [n_1 \ldots n_b] \right) \right] = \alpha_h,
\]
where the expectation is over a random order of $(n_1, \ldots, n_b)$. This is true regardless of the set $(n_1, \ldots, n_b)$ that is drawn from (see Fact 2). This implies that the second term on the right hand side of Eq. (21) is equal to $\alpha_h$. Therefore, we get Eq. (14).

Now we want to focus on the event $\Omega_{h,C}$. We give an upper bound on $\mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}^{t+1}) \right] \Omega_{h,C}$. For any configuration, in order to retrieve one container, we need at most $P - 1$ relocations (since at most $P - 1$ containers are blocking it), thus for any configuration, the optimal number of relocations is at most $P - 1$ times the number of containers $(h(C + 1))$ which gives us $h(P - 1)(C + 1)$ as an upper bound on the optimal number of relocations. We use this universal bound to get
\[
\mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}^{t+1}) \right] \Omega_{h,C} \leq h(P - 1)(C + 1).
\]  
(22)

Finally using Lemma 4, we have
\[
\mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}^{t+1}) \right] \Omega_{h,C} \leq \mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}^{t+1}) \right] \Omega_{h,C} P(\Omega_{h,C})
\]
\[
+ \mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}^{t+1}) \right] \Omega_{h,C} P(\Omega_{h,C})
\]
\[
+ \mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}^{t+1}) \right] \Omega_{h,C} e^{-\alpha_h(C+1)}
\]
\[
\leq \mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}) \right] + h(P - 1)(C + 1) e^{-\alpha_h(C+1)}
\]
which proves Eq. (10) and hence completes the proof of the theorem.

In the next corollary, we show that the optimal solution of the unrestricted CRP has a similar asymptotic behavior. We remind that the unrestricted CRP refers to the problem where we can also relocate non-blocking containers. The problem is trivial since by definition $S_0(B_{h,C}) \leq \text{zopt}(B_{h,C}) \leq \text{zopt}(B_{h,C})$. 

**Corollary 6.** Let $\text{zopt}(B_{h,C})$ be the optimal number of relocations for the unrestricted CRP. For $C \geq h + 1$, we have
\[
1 \leq \mathbb{E}_{h,C} \left[ z_\text{opt}(B_{h,C}) \right] \leq \alpha_h,
\]
where $\alpha_h$ is the function defined in Theorem 5.
3.1. Experimental results on the efficiency of heuristic H

Theorem 5 gives insights on how the expected optimal solution of the CRP behaves asymptotically on random bays. To give more insights on CRP, we show experimentally that the same result holds for heuristic H, i.e., the ratio of $E_{h,c}[z_{opt}(B_{h,c})]$ and $E_{h,c}[S_0(B_{h,c})]$ converges to 1 as $C$ goes to infinity. We take $h = P - 1 = 4$ and for each size $C$, we compute both expectations over a million instances generated uniformly, take their ratio and plot the result in Fig. 4. Notice that we have

$$1 \leq \frac{E_{h,C}[z_{opt}(B_{h,c})]}{E_{h,C}[S_0(B_{h,c})]} \leq \frac{E_{h,C}[z_{H}(B_{h,c})]}{E_{h,C}[S_0(B_{h,c})]},$$

so Fig. 4 also shows experimentally that Theorem 5 holds.

First, note that Fig. 4 implies that the relative gap between heuristic $H$ and $S_0$ shrinks to 0 as $C$ increases. Moreover we have

$$\frac{E_{h,C}[z_{H}(B_{h,c})]}{E_{h,C}[S_0(B_{h,c})]} = \frac{E_{h,C}[z_{H}(B_{h,c})] - E_{h,C}[S_0(B_{h,c})]}{E_{h,C}[S_0(B_{h,c})]},$$

and thus the relative gap of $H$ with optimality also converges to 0 as $C$ grows to infinity.

In the proof of Theorem 5, we also study the function $g(h) = E_{h,C}[z_{H}(B_{h,c})] - E_{h,C}[S_0(B_{h,c})]$. Note that $g(h)$ is bounded as we proved in Theorem 5. Moreover, the plot implies that heuristic $H$ is on average at most 1.25 away from the optimal solution, so heuristic $H$ is relatively more efficient in the case of large bays. Intuitively, the probability of having a good column converges to 1, as we increase the number of columns; hence the problem tends to become easier as $C$ grows.

Finally, in the proof, we note that the rate of convergence of the minimum $z_{opt}$ to $S_0$ is at least $1/C$. Interestingly, we can infer from Fig. 4 that the rate of convergence of the ratio for heuristic $H$ is also proportional to $1/C$.

4. Conclusion

The Container Relocation Problem (CRP) is known for its computational intractability, so most research studies have designed heuristics to solve the problem, particularly for large bays. The main purpose of this paper is to show a new theoretical result stating that the ratio between the expected minimum number of relocations and a simple lower bound (given by Lemma 1) approaches 1. The main insight of this result is that in large bays each blocking container is relocated at most once with high probability. This leads us to believe that the same theoretical result should hold for heuristic $H$ and we confirm this intuition by simulation.

Furthermore direct extensions of this paper include the formal proof of a similar result for the heuristic $H$, the proof of convergence of the difference between the optimal solution and this lower bound. The study of the CRP with distributions other than the uniform one could also be very interesting theoretically as well as experimentally.

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Appendix. Proof of Lemma 4

Recall that

$$\Omega_{h,C} = \{B_{h,c+1} \text{ has at least one “special” column}\}.$$

We know that each bay of size $C + 1$ can be mapped to a permutation $\pi$ of $\Omega_{h,C+1}$ taken uniformly at random. Let $q(\cdot)$ be the function from $\Omega_{h,C+1}$ to $\mathbb{R}^+$ defined by

$$q : \pi \mapsto \text{number of “special” columns in the resulting bay of $\pi$}.$$

Note that

$$\mathbb{P}(\Omega_{h,C}) = \mathbb{P}(q(\pi) = 0).$$

First we compute the expected value of $q(\cdot)$

$$E_{h,C+1}[q] = E_{h,C+1}\left[\sum_{c=1}^{C+1} \chi(c_i \text{ is a “special” column})\right]$$

$$= (C + 1) \times \mathbb{P}\left(c_1 \text{ is a “special” column}\right),$$

where we use linearity of expectation and the fact that columns are identically distributed.

A simple counting implies that:

$$\mathbb{P}\left(\{c_1 \text{ is a “special” column}\}\right) \leq \frac{(C+1)(C+1) - 1 \ldots (C+1) - h + 1}{h(C+1)(h(C+1) - 1)\ldots[h(C+1) - h + 1]}$$

$$\leq \frac{(C+1) - h + 1}{h(C+1)}.$$
where we use $C + 1 \geq h + 1$ to show the last inequality (notice that when $C \to \infty$, the probability is equivalent to $(1/h)^h$ which would guarantee a faster convergence rate).

Therefore we know that

$$E_{h,C+1}[q] \geq (C + 1) \times \left( \frac{2}{h(h + 1)} \right)^h. \quad (A.1)$$

We claim that $q(.)$ is well concentrated around its mean. In order to do so, we prove that $q(.)$ is 1-Lipschitz.

Define $\rho$ the distance between two permutations $\pi_1, \pi_2 \in \delta_{h(C+1)}$ as $\rho(\pi_1, \pi_2) = |(i \in [h(C+1)] : \pi_1(i) \neq \pi_2(i))|$. We want to prove that $|q(\pi_1) - q(\pi_2)| \leq \rho(\pi_1, \pi_2), \forall (\pi_1, \pi_2) \in \delta_{h(C+1)}$.

Let $\pi_1, \pi_2 \in \delta_{h(C+1)}$. Let us first consider the case where $\rho(\pi_1, \pi_2) = 2$. (Notice that if $\rho(\pi_1, \pi_2) \neq 0$ then $\rho(\pi_1, \pi_2) \geq 2$). In that case, we have $i,j \in \{1,\ldots,n\}$ such that $\pi_1(i) = \pi_2(j)$ and $\pi_1(j) = \pi_2(i)$. Let $B^{(1)}$ and $B^{(2)}$ be the configurations generated by $\pi_1$ and $\pi_2$. Having $\rho(\pi_1, \pi_2) = 2$ corresponds to the fact that if we swap 2 containers in $B^{(1)}$, we get $B^{(2)}$ and we denote those containers $a = \pi_1(i)$ and $b = \pi_1(j)$. We have three cases:

- $a$ and $b$ are both in “special” columns in $B^{(1)}$. In this case, swapping them will not change anything since both their new columns in $B^{(2)}$ will also be “special” and hence $|q(\pi_1) - q(\pi_2)| = 0$.
- $a$ and $b$ are both in columns that are not “special” columns in $B^{(1)}$. If $a, b \geq \omega_h, C$ or $a, b < \omega_h, C$ then we do not create any new special column in $B^{(2)}$. Now suppose that $a \geq \omega_h, C$ and $b < \omega_h, C$ then the column of $a$ in $B^{(2)}$ might be a “special” column, but the column of $b$ in $B^{(2)}$ cannot be “special”. Therefore in that case, $|q(\pi_1) - q(\pi_2)| \leq 1$.
- $a$ is in a “special” column in $B^{(1)}$ but $b$ is not. Now we know that $a \geq \omega_h, C$. If $b < \omega_h, C$ then the column of $b$ in $B^{(2)}$ cannot be “special” but the column of $a$ might be and in that case $|q(\pi_1) - q(\pi_2)| \leq 1$. If $b \geq \omega_h, C$ then the column of $b$ in $B^{(2)}$ is “special” and the column of $a$ in $B^{(2)}$ is not “special” which gives us $|q(\pi_1) - q(\pi_2)| \leq 0$. Note that the proof is identical if $b$ is in a “special” column in $B^{(1)}$ but $a$ is not.

So far we have shown that

If $\rho(\pi_1, \pi_2) = 2$, then $|q(\pi_1) - q(\pi_2)| \leq 1. \quad (A.2)$

Now we suppose that $\rho(\pi_1, \pi_2) = k$ where $2 \leq k \leq h(C + 1)$. Note that we can construct a sequence of permutations $(\pi_1', \pi_2', \ldots, \pi_k')$ such that $\pi'_1 = \pi_1, \pi'_k = \pi_2$ and $\rho(\pi'_i, \pi'_{i+1}) = 2$.

Now using this fact and Eq. (A.2),

$$|q(\pi_1) - q(\pi_2)| \leq \sum_{i=1}^{k-1} |q(\pi_i') - q(\pi_{i+1}')| \leq \sum_{i=1}^{k-1} 1 = k - 1 \leq k = \rho(\pi_1, \pi_2),$$

which proves that $q(.)$ is 1-Lipschitz.

Now we use Theorem 8.3.3 of [4] which states that

$$P(q \leq E_{h,C+1}[q] - t) \leq e^{-\theta_h t^2}$$

and apply it with $t = E_{h,C+1}[q]$ and Eq. (A.1) to get

$$P(q = 0) = P(q \leq E_{h,C+1}[q] - E_{h,C+1}[q]) \leq e^{-\theta_h (E_{h,C+1}[q])^2} \leq e^{-\theta_h (C+1)},$$

where

$$\theta_h = \frac{1}{8h} \left( \frac{2}{h(h + 1)} \right)^h > 0,$$

which concludes the proof.

References


