

ASSIGNMENT #11
(reissued with additional notes--the one problem is the same)

Electromagnetic Radiation I

(due **Wed November 23, by 12 pm** in the Undergraduate Physics Office)

Reading: Griffiths, Chapter 9, through Section 9.1.1;
Class Notes: *Simple Radiating Systems*

NOTES

I. Fields Of An Infinite Time Varying Planar Current Sheet

I had said in Assignment 10 that I would give you a problem in Assignment 11 to show formally that the heuristic derivation of the electromagnetic fields of an infinite current sheet in Assignment 10 (cf. equations (7) and (8) of that Assignment) were indeed correct. This is Problem 9.38 of Griffiths, page 441.

I decided not to give you that problem, since we have an exam coming up and time is running short before Thanksgiving. So this Assignment has only one problem.

However, since several people have expressed puzzlement about the solution presented in the Notes of Assignment 10, I give below the formal solution to the problem of the electromagnetic fields of an infinite, time varying current sheet, which is the same as the handwaving solution (although of course I think the handwaving solution is more intuitive and illustrative). I also include some additional arguments to convince you that this formal solution makes physical sense.

The way that Griffiths approaches this problem is round about because he starts out in 3-D, and then basically rewrites the general three dimensional solution given in equation 9.8 (page 397 of *Griffiths*) in a form that is appropriate to one dimension. However, the final form he obtains using this approach is not obviously the correct solution when you look at it.

If we are interested in 1-D problems, it is much more straightforward to start out by assuming that the sources (\mathbf{r}, t) and $\mathbf{J}(\mathbf{r}, t)$ have a dependence on time and one spatial dimension, say x . Then the potentials V and \mathbf{A} can only depend on x and t as well, and we need therefore look for solutions to the one dimensional wave equation with a source $s(x, t)$, that is

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \psi(x, t) = -s(x, t) \quad (1)$$

The solution to this equation turns out to be

$$\psi(x, t) = \frac{c}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx s(x, t) \theta\left(t - t - \frac{|x - x'|}{c}\right), \quad \text{where } \theta(\eta) = \begin{cases} 0 & \eta < 0 \\ 1 & \eta > 0 \end{cases} \quad (2)$$

One can show (2) is a solution by direct substitution of (2) into (1). It is easiest to do this if you first show that

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \frac{c}{2} \theta\left(t - \frac{|x|}{c}\right) = -\delta(x)\delta(t) \tag{3}$$

which is reasonably straightforward to show if you remember that

$$\frac{d\theta(\eta)}{d\eta} = \delta(\eta), \quad \frac{d}{dx}|x| = \text{sign}(x) = \begin{cases} 1 & x < 0 \\ -1 & x > 0 \end{cases}, \quad \text{and} \quad \frac{d}{dx} \text{sign}(x) = 2\delta(x) \tag{4}$$

Thus, since the electromagnetic potentials satisfy equations similar in form to (1), i.e.,

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] V(x,t) = -\rho(x,t) / \epsilon_0 \tag{5}$$

and

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \mathbf{A}(x,t) = -\mu_0 \mathbf{J}(x,t) \tag{6}$$

we therefore have the general solutions for the potentials for one dimensional problems:

$$V(x,t) = \frac{c}{2\epsilon_0} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \rho(x,t) \theta\left(t - t - \frac{|x-x|}{c}\right) \tag{7}$$

$$\mathbf{A}(x,t) = \frac{c\mu_0}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \mathbf{J}(x,t) \theta\left(t - t - \frac{|x-x|}{c}\right) \tag{8}$$

The form for $\mathbf{A}(x,t)$ in (8) can be shown to be equivalent to *Griffiths'* expression for $\mathbf{A}(x,t)$ given in Problem 9.38(b), page 441, but it is a little more obvious what is going on in (8), since the function keeps changes in the sources from appearing at the observer until they have had time to propagate (at the speed of light) from source to observer. This is not so obvious in *Griffiths'* version.

We can compute the electric field from these potentials in (7) and (8) by using

$$\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial}{\partial t} \mathbf{A}. \quad \text{This gives}$$

$$\mathbf{E}(x,t) = -\hat{\mathbf{x}} \frac{c}{2\epsilon_0} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \rho(x,t) \frac{\partial}{\partial x} \theta\left(t - t - \frac{|x-x|}{c}\right) - \frac{c\mu_0}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \mathbf{J}(x,t) \frac{\partial}{\partial t} \theta\left(t - t - \frac{|x-x|}{c}\right) \tag{9}$$

Taking the derivatives, and remembering the relations in (4), we obtain

$$\mathbf{E}(x,t) = -\hat{\mathbf{x}} \frac{c}{2\epsilon_0} \int dt \int dx \rho(x,t) \delta\left(t-t - \frac{|x-x'|}{c}\right) \left[-\frac{\text{sign}(x-x')}{c}\right] - \frac{c\mu_0}{2} \int dt \int dx \mathbf{J}(x,t) \delta\left(t-t - \frac{|x-x'|}{c}\right) \quad (10)$$

Finally, we can use the delta functions to do the t' integrations, giving

$$\mathbf{E}(x,t) = \hat{\mathbf{x}} \frac{1}{2\epsilon_0} \int dx \rho(x, t - \frac{|x-x|}{c}) \text{sign}(x-x') - \frac{c\mu_0}{2} \int dx \mathbf{J}(x, t - \frac{|x-x|}{c}) \quad (11)$$

Similarly, using $\mathbf{B}(x,t) = \mathbf{x} \mathbf{A}(x,t)$

$$\mathbf{B}(x,t) = -\left[\frac{\mu_0}{2} \hat{\mathbf{x}}\right] \mathbf{x} \left[\int dx \mathbf{J}(x, t - \frac{|x-x|}{c}) \text{sign}(x-x') \right] \quad (12)$$

The forms given in (11) and (12) for the fields show clearly the effects of retarded time, and are the general solutions for the one-dimensional case.

If we now specialize this to our situation in Assignment 10, where

$$\mathbf{J}(x,t) = \hat{\mathbf{y}} \delta(x) K(t) \quad \text{and} \quad \rho(x,t) = \sigma \delta(x)$$

and insert these source terms into (11) and (12), we easily obtain

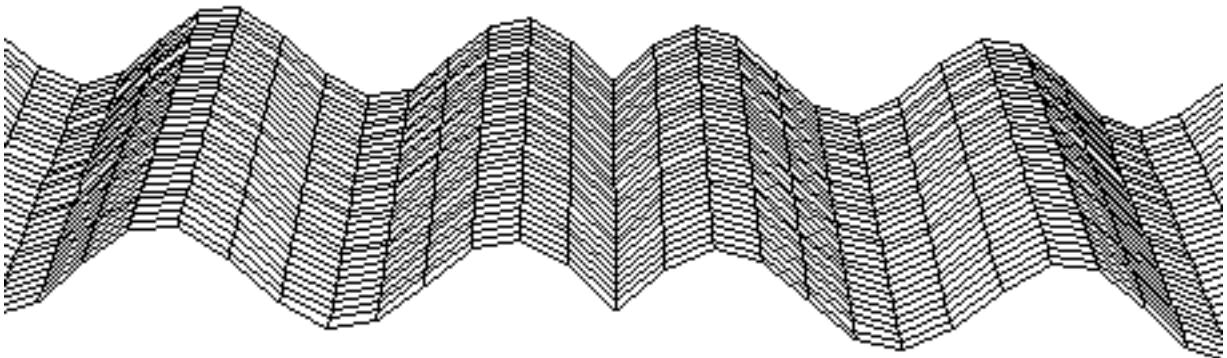
$$\mathbf{E}(x,t) = \hat{\mathbf{x}} \frac{\sigma}{2\epsilon_0} \text{sign}(x) - \hat{\mathbf{y}} \frac{c\mu_0}{2} K\left(t - \frac{|x|}{c}\right) \quad (13)$$

and

$$\mathbf{B}(x,t) = \hat{\mathbf{z}} \text{sign}(x) \frac{\mu_0}{2} K\left(t - \frac{|x|}{c}\right) \quad (14)$$

These are just the solutions given in (7) and (8) of Assignment 10. If $K(t)$ is sinusoidal, we get traveling sine waves. If $K(t)$ is a step function at $t = 0$, we get the "kink" propagating out from $x = 0$ at the speed of light, as shown in the figure on page 2 of Assignment 10.

Incidentally, it is worth pointing out what the sinusoidal solutions look like, because in fact even for the sinusoidal case there is a kink in the electric field at the origin, $x = 0$, as illustrated below.



II. Why A Force At Constant Velocity?

The solutions (13) and (14), while a formal solution, do not offer any insight into the question that has immediately occurred to a number of students in looking at the kink on page 2 of Assignment 10. How can this kink stay around forever? How would you ever get to what you expect for a uniformly moving sheet of charge, that is, an electric field perpendicular to the sheet. Certainly if we walk past a stationary sheet of charge, parallel to the plane of the charge, it appears to be moving to us, yet the electric field we see in this circumstance shows no kink, or component of \mathbf{E} parallel to the sheet, for that matter.

Another way of stating this apparent paradox is, and the key to understanding what the purpose of the kink is, is to ask "*Why is there a force on the sheet when it is moving at constant velocity, after we accelerate it instantaneously from zero to velocity \mathbf{V} ?*". Assignment 10 states that after we get the sheet moving at constant velocity \mathbf{V} , there is still a constant force on an area dA of the sheet given by

$$d\mathbf{F} = +\sigma \delta\mathbf{E} dA = +\sigma \left[-\frac{\sigma}{2\epsilon_0} \frac{\mathbf{V}}{c} \right] dA = -\frac{\sigma^2}{2\epsilon_0 c} \mathbf{V} dA \quad (15)$$

This force is directed opposite to the direction of motion of the charged sheet, and thus serves as a drag force. To keep the sheet moving at constant velocity \mathbf{V} , equation (15) demands that we provide a counterbalancing force which is opposite and equal to this drag force. Thus we must be providing a force opposite and equal to that in (15), and therefore in a time dt , we must be providing a momentum $d\mathbf{P}$ to an area dA of the sheet given by

$$d\mathbf{P}_{\text{supplied by us in time } dt} = -d\mathbf{F} dt = -\sigma \delta\mathbf{E} dA dt = +\frac{\sigma^2}{2\epsilon_0 c} \mathbf{V} dA dt \quad (16)$$

Let us rewrite equation (16) in terms of the electric field inertial mass per unit volume, that is,

$$\rho_{\text{em mass}} = \frac{\frac{1}{2} \epsilon_0 E^2}{c^2} = \frac{\epsilon_0}{2c^2} \frac{\sigma}{2\epsilon_0}^2 = \frac{\sigma^2}{8\epsilon_0 c^2} \quad (17)$$

This quantity has units of mass per unit volume. If we write (16) in terms of this quantity, we have

$$d\mathbf{P}_{\text{supplied by us in time } dt} = +\frac{\sigma^2}{2\epsilon_0 c} \mathbf{V} dA dt = +\frac{\sigma^2}{8\epsilon_0 c^2} 4c\mathbf{V} dA dt = +4\rho_{\text{em mass}} \mathbf{V} dA cdt \quad (18)$$

$$d\mathbf{P}_{\text{supplied by us in time } dt} = +(2\rho_{\text{em mass}} \mathbf{V})(2 dA cdt) \quad (19)$$

What is the meaning of the volume $2 dA cdt$? Well, $c dt$ is the distance the leading edges of our kink have propagated in time dt , so this is just the additional volume with cross-section dA which is swept up by the kinks moving outward at the speed of light. The factor of two in $2 dA cdt$ comes from the fact that we have kinks moving both the left and to the right. The amount of additional electromagnetic inertial mass that is swept up in time dt is thus $\rho_{\text{em mass}} 2 dA cdt$. If we think of this as an additional inertial mass that we must accelerate up to the velocity of the sheet, \mathbf{V} , then to get this additional inertial mass up to speed, we must provide an additional momentum

$$d\mathbf{P}_{\text{supplied by us in time } dt} = +(\rho_{\text{em mass}} \mathbf{V})(2 dA cdt) \quad (20)$$

This is to within a factor of two of the momentum given by (19), the same factor of two that we saw when we accelerated a capacitor in Problem 9-4 of Assignment 9.

Thus we have to continue to provide a force to the charged sheet even after we have gotten it up to velocity \mathbf{V} , because although we have initially gotten the charges in the sheet itself up to speed, we have not gotten the electromagnetic field they produce up to speed--we can only do that at the speed of light. As the kink propagates out, it carries out momentum, continually provided by us (back at the sheet), at a rate sufficient to get the static fields it sweeps up in unit time up to speed. And we must continually provide a momentum at a rate sufficient to do that, and this will go on forever if the field produced by the sheet goes on forever. Thus the kink in that circumstance is always there, because we can never provide the infinite amount of momentum it takes to get an infinite amount of electromagnetic inertia up to speed.

If however we only have a finite amount of electromagnetic inertia, say we have negatively charged sheets somewhere where the field lines end, we will eventually get all of this finite amount of electromagnetic inertia up to speed, and the kink will disappear. The time it will take for that to happen is the distance between the sheets divided by the speed of light. Then we will get to the static situation we expect to see when we walk at finite speed past a stationary sheet.

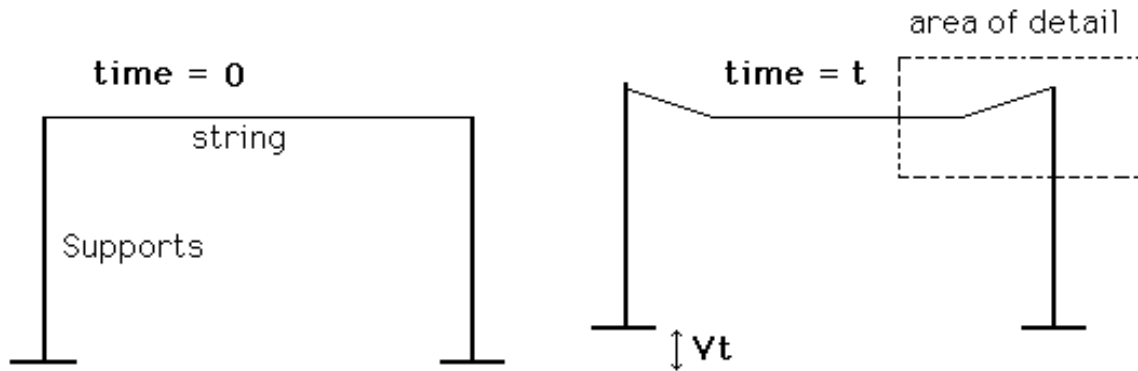
III. It Happens With Strings, Too

Finally, consider the string analogy to this process, which is an exact analogy. Suppose one has a string with tension T and mass per unit length m suspended between two supports (see sketch). Let d be the distance between supports, and c the speed at which waves travel on the string. We know that the speed c for a string is given by

$$c = \sqrt{\frac{T}{m}} \quad (21)$$

Now, we grab the supports at $t = 0$ and instantaneously accelerate them up to constant velocity \mathbf{V} , which is upward in this case. "Instantaneously" means we do this in a time very short compared to d/c . We also assume that the speed V is such that $V \ll c$.

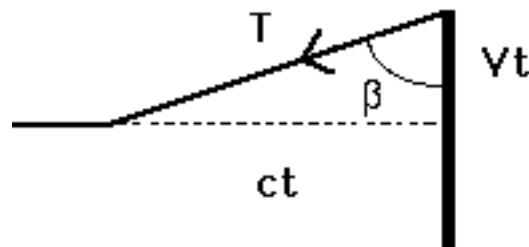
What do things look like at time $t > 0$? Well, they look like the diagram on the right below. The supports have moved upward a distance Vt , and parts of the string close the supports are also moving at the same speed V . However, parts of the string further away have not moved at all, because the information that the supports are moving has not yet reached them.



If we blow up the area around the right support, we have the diagram below. There will be a downward component of the tension of the string given in magnitude by

$$T \cos \beta = T \frac{Vt}{\sqrt{(Vt)^2 + (ct)^2}} = T \frac{V}{c} \tag{22}$$

If we are to have the supports continue to move upward at constant speed \mathbf{V} , we have to provide an upward force to offset the downward component of the tension. In terms of the momentum we must supply in a time dt , we must provide to the right support an amount



$$d\mathbf{P}_{\text{supplied by us to the right support}} = +\hat{\mathbf{z}} T \frac{V}{c} dt \tag{23}$$

If we use equation (21) for c in terms of T and the mass per unit length m , we have

$$d\mathbf{P}_{\text{us}} = TV \sqrt{\frac{m}{T}} dt = mV \sqrt{\frac{T}{m}} dt = mV (c dt) \tag{24}$$

This equation has an obvious physical interpretation. In a time dt , an additional length $c dt$ of the string to the right is brought from rest up to speed V upwards, by the kink in the string as it propagates along. This means that a mass $m c dt$ has acquired velocity V , and we must provide momentum $mV c dt$ to the right support for this to happen. We supply it at the support, and the kink in the string carries it out to accelerate the length of the string it sweeps up in time dt . This is an exact parallel to the electromagnetic situation.

Thus even though the supports are moving at constant velocity, we have to continue to apply a force to keep them moving at that constant speed, because we are continuing to accelerate parts of the string up to that speed. If the string is infinitely long, it has an infinite inertia, and we are never done with this task--the kink never disappears. If it has a finite length, and inertia, eventually (in a time d/c) we will get it all up to speed.

IV. Light Wave In A Dielectric Medium

The solutions in equations (11) and (12) above are more general than for just a planar current sheet, and since we have gone to the trouble to derive them, let us apply them to one more physical problem. This relates to why the speed of light in a dielectric is less than that of light.

Suppose you have an electric field in a dielectric propagating in the positive x direction with wavenumber k and frequency ω , i.e.,

$$\mathbf{E}(x,t) = \hat{y} E_o e^{i(kx - \omega t)} \quad (25)$$

We are NOT assuming that ω/k is c in this expression--for now this parameter can have any value.

Since this is a dielectric, we will get a polarization current as a result of this electric field, as in Problem 10-4 of Assignment 10 (see also *Griffiths* equation 7.55, page 311), with that current given by

$$\mathbf{J}_p = \frac{\partial}{\partial t} \mathbf{P} = \epsilon_o (K_e - 1) \frac{\partial}{\partial t} \mathbf{E} \quad (26)$$

So with the electric field in (25), we will have a polarization current given by

$$\mathbf{J}_p(x,t) = -i\omega \epsilon_o (K_e - 1) \hat{y} E_o e^{i(kx - \omega t)} \quad (27)$$

Now this current will generate electromagnetic waves. It must. No choice. The electric fields of the waves it generates will be given by equation (11) with this form of the current (the charge density will be zero here). Thus we have that the electric field \mathbf{E}_p produced by the polarization currents is given by

$$\mathbf{E}_p(x,t) = -\frac{c\mu_o}{2} \int_{-\infty}^{\infty} dx \mathbf{J}_p(x, t - \frac{|x-x'|}{c}) \quad (28)$$

Lets break this integral into two parts, so that we can resolve the absolute value into terms which can be handled when we come to the integration

$$\mathbf{E}_p(x,t) = -\frac{c\mu_o}{2} \int_{-\infty}^x dx \mathbf{J}_p(x, t - \frac{x-x'}{c}) + \int_x^{\infty} dx \mathbf{J}_p(x, t - \frac{x'-x}{c}) \quad (29)$$

If we insert (28) into (29), we get

$$\mathbf{E}_p(x,t) = \hat{\mathbf{y}} i\omega \epsilon_o (K_e - 1) E_o \frac{c \mu_o}{2} \int_{-\infty}^x dx e^{i[kx - \omega(t - \frac{x-x}{c})]} + \int_x^{\infty} dx e^{i[kx - \omega(t - \frac{x-x}{c})]} \quad (30)$$

which we can rewrite as

$$\mathbf{E}_p(x,t) = \hat{\mathbf{y}} i\omega \epsilon_o (K_e - 1) E_o \frac{c \mu_o}{2} e^{-i\omega(t - \frac{x}{c})} \int_{-\infty}^x dx e^{ix(k - \frac{\omega}{c})} + e^{-i\omega(t + \frac{x}{c})} \int_x^{\infty} dx e^{ix(k + \frac{\omega}{c})} \quad (31)$$

The first term on the right in equation (31) represents plane waves generated to the left of the observer at (x,t), and propagating *at the speed of light* in the positive x direction to the observer. The second term in equation (31) represents plane waves generated to the right of the observer at (x,t), and propagating *at the speed of light* in the negative x direction to the observer. This is what we would expect. Our currents generate electromagnetic waves propagating both to the left and the right, at the speed of light. Just so.

Now, lets do the integrations, adding all these waves arriving at (x,t), generated all up and down the x axis, propagating both left and right at the speed of light to reach the observer.

$$\mathbf{E}_p(x,t) = \hat{\mathbf{y}} i\omega \epsilon_o (K_e - 1) E_o \frac{c \mu_o}{2} \left[\frac{e^{-i\omega(t - \frac{x}{c})}}{i(k - \frac{\omega}{c})} e^{ix(k - \frac{\omega}{c})} \right]_{-\infty}^x + \left[\frac{e^{-i\omega(t + \frac{x}{c})}}{i(k + \frac{\omega}{c})} e^{ix(k + \frac{\omega}{c})} \right]_x^{\infty} \quad (32)$$

We assume that the currents vanish at infinity, and throw away the evaluation of the limits there, so that we end up with

$$\mathbf{E}_p(x,t) = \hat{\mathbf{y}} i\omega \epsilon_o (K_e - 1) E_o \frac{c \mu_o}{2} e^{i(kx - \omega t)} \left[\frac{1}{i(k - \frac{\omega}{c})} - \frac{1}{i(k + \frac{\omega}{c})} \right] \quad (33)$$

Amazingly enough, we have ended up with an electric field which is propagating along toward positive x values, with speed ω/k , *not necessarily the speed of light*, even though we constructed this wave out of many waves going in both directions, all moving at the speed of light. That is the wonder of adding things up with a definite phase--you get all sorts of things you wouldn't expect. If we combine terms in (33), we get

$$\mathbf{E}_p(x,t) = \hat{\mathbf{y}} \frac{(K_e - 1)}{\frac{k^2 c^2}{\omega^2} - 1} E_o e^{i(kx - \omega t)} \quad (34)$$

We get a natural mode of this system when the electric field in (34) agrees with our original electric field in (25). That is, when the electric field that *generates* the currents is the same as the electric field *generated by* the currents. That will happen when

$$\frac{\omega}{k} = \frac{c}{\sqrt{K_e}} \quad (35)$$

and this of course is the speed of light in a dielectric.

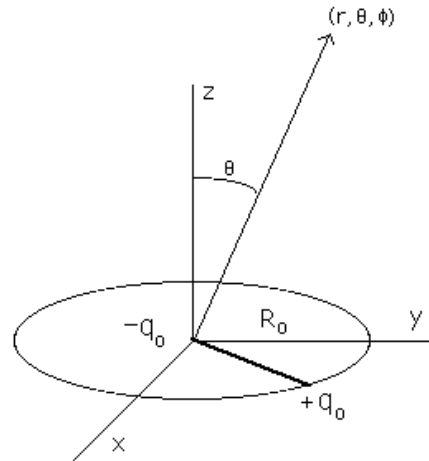
PROBLEMS

Problem 11-1: Electric Dipole Radiation From A Charge In A Circular Orbit

Two particles with equal and opposite charges are arranged as shown in the sketch. One particle sits at rest at the origin. The other moves in a circular orbit with radius R_o about the origin, in the xy plane. It revolves with a period T , at an angular frequency

$\omega = 2\pi/T$. The position $\mathbf{R}(t)$ of this particle as a function of time t is given by

$$\mathbf{R}(t) = R_o [\hat{x} \cos(\omega t) + \hat{y} \sin(\omega t)].$$



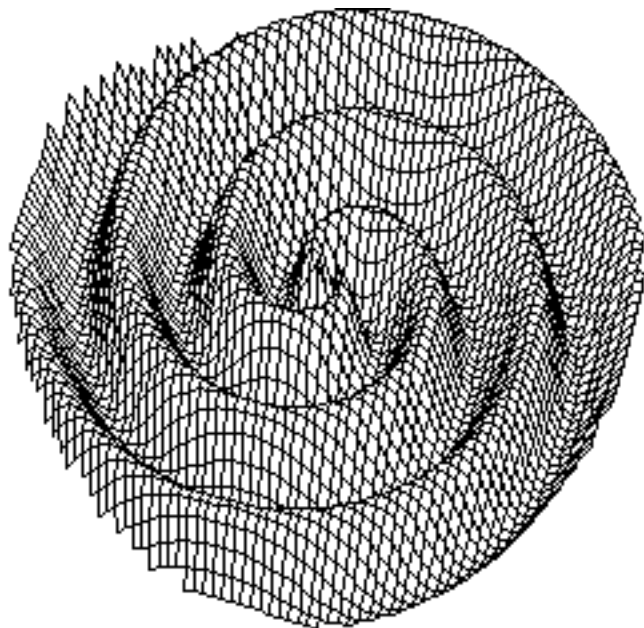
(a) What is the electric dipole moment vector $\mathbf{p}(t)$?

(b) Calculate the electric and magnetic fields seen by an observer at (r, θ, ϕ) . Include both radiation and induction terms in your expressions for the fields (cf. equations (19) and (21) in the Notes

for Simple Radiating Systems), but not the quasi-static ($1/r^3$) terms.

(c) Now consider only the radiation terms in your expressions from (b). What is the polarization of the emitted radiation at the equator ($\theta = \pi/2$)? What is the polarization at the north pole ($\theta = 0$)?

(d) If we plot in the xy plane the square of the electric field times r^2 , we get the pattern shown. What is the relation between r/cT and θ that describes the pattern traced out by the maxima in the xy plane?



(e) Calculate the average rate at which energy is radiated per solid angle, averaging over one period of the orbit $\left\langle \frac{dW_{rad}}{d\Omega dt} \right\rangle$. Plot this as a

function of the polar angle θ . What is the total rate at which energy is radiated, integrating over solid angle?

(e) Calculate the average rate at which angular momentum is radiated per solid angle $\left\langle \frac{d\mathbf{L}_{rad}}{d\Omega dt} \right\rangle$. You must include induction terms in this calculation. Remember, you are

calculating the flux of a vector quantity, so your answer should be a vector. What is the total rate at which angular momentum is radiated, integrating over solid angle?

(f) Show that the magnitude of the angular momentum radiated is the energy radiated divided by c . The energy of a photon is $\hbar\omega$ and its angular momentum is \hbar . Note the dimensional similarities.