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Technical Supplement

EC.1. Proof of Theorem 1

Consider a bidder A with type (v, t) in an auction where every other bidder uses strategy $\mathcal{T}[\nu]$, where ν is an arbitrary threshold function. If A is not the first bidder, the first bidder has either placed a bid or exercised the option immediately (following strategy $\mathcal{T}[\nu]$), so that the buyout option is no longer available to bidder A. In that case bidder A's weakly dominant strategy is to bid his true valuation v, as shown in Vickrey (1961). His bid submission time in $[t, T]$ will not affect his utility in any way, so that bidding v at any time in $[t, T]$ constitutes then a best response.

Suppose now that A is the first bidder, so that the buyout option is available to him. We introduce the following notation for the three possible actions he may take at time t:

bid(t): Bid in the auction at time t (in which case it is a dominant strategy for him to bid his valuation v);

buy(t): Buyout at time t;

wait(t, τ): Wait for $\tau - t$ time units before deciding to bid (if the auction is still open) or buy out (if the option is still available).

We define the utility of bidder A with type (v, t) and taking action $a \in \{\text{bid}(t), \text{buy}(t), \text{wait}(t, \tau)\}$ as $U_a(v, t)$. If bidder A chooses bid(t), i.e. bids immediately, the buyout option disappears. Following strategy $[\nu]$, all subsequent bidders will bid their true valuation. Denoting by $N(t, T)$ the random number of bidders arriving in interval $(t, T]$ and $N(t)$ the cumulative number of arrivals up to t, this implies:

$$\mathbf{E}[U_{\text{bid}(t)}(v, t) \mid N_t = 0, N(t, T)] = e^{-\beta(T-t)} \int_{\underline{v}}^{v} F(x)^{N(t,T)}\, dx, \tag{EC1}$$

and using the model assumption that $N(t, T)$ is Poisson with parameter $\lambda(T - t)$, we obtain the expected utility of the first bidders when bidding his valuation v upon his arrival at t:

$$\mathbf{E}[U_{\text{bid}(t)}(v, t) \mid N_t = 0] = e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^{v} e^{\lambda(T-t)F(x)}\, dx \tag{EC2}$$

$$\triangleq B_1(v, t). \tag{EC3}$$

Conditional on A being the first bidder (i.e., $N_t = 0$), the utility from exercising the buyout option immediately is

$$\mathbf{E}[U_{\text{buy}(t)}(v, t) \mid N_t = 0] = v - p. \tag{EC4}$$

The key to the proof of Theorem 1 is the following Lemma, which establishes that bidder A's expected utility from acting immediately upon his arrival (i.e., choosing either *bid*(t) or *buy*(t)) is always as large as that obtained from waiting, i.e., $\mathbf{E}[U_{wait(t,\tau)}(v, t) \mid N_t = 0]$:

Lemma 1. $\mathbf{E}[U_{wait(t,\tau)}(v, t) \mid N_t = 0] \le \max\{B_1(v, t), v - p\}$.

Proof. Let $\mathcal{E} = \{N(t, \tau) = 0\}$ be the event that no bidder arrives in the interval (t, τ). In this case bidder A remains the first bidder so that

$$\begin{aligned}\mathbf{E}[U_{\text{bid}(\tau)}(v, t) \mid N_t = 0, \mathcal{E}] &= e^{-\beta(\tau-t)}\mathbf{E}[U_{\text{bid}(\tau)}(v, \tau) \mid N_\tau = 0] \\ &= e^{-\beta(\tau-t)} B_1(v, \tau),\end{aligned} \tag{EC5}$$

and the buyout option is still available thus

$$\mathbf{E}[U_{wait(t,\tau)}(v,t) \mid N_t = 0, \mathscr{E}] = e^{-\beta(\tau-t)} \max\{B_1(v,\tau), v-p\}.$$

The complementary event $\bar{\mathscr{E}} = \{N(t,\tau) > 0\}$ corresponds to one or more arrivals occurring in the interval (t,τ). In that case the buyout option is no longer available, so that

$$\begin{aligned}\mathbf{E}[U_{\mathrm{wait}(t,\tau)}(v,t) \mid N_t = 0, \bar{\mathscr{E}}] &= \mathbf{E}[U_{bid(\tau)}(v,t) \mid N_t = 0, \bar{\mathscr{E}}] \\ &= e^{-\beta(\tau-t)}\mathbf{E}[U_{bid(\tau)}(v,\tau) \mid N_t = 0, \bar{\mathscr{E}}].\end{aligned} \tag{EC6}$$

Note that the event $\bar{\mathscr{E}}$ includes the event that one of the bidders who arrived during (t,τ) exercised the buyout option, in which case bidder A's utility is zero. The expected utility of the first bidder A if he waits up to time $\tau > t$ is thus

$$\mathbf{E}[U_{\mathrm{wait}(t,\tau)}(v,t) \mid N_t = 0] = e^{-\beta(\tau-t)}(\max\{B_1(v,\tau), v-p\}\cdot P(\mathscr{E}) + \mathbf{E}[U_{\mathrm{bid}(\tau)}(v,\tau) \mid N_t = 0, \bar{\mathscr{E}}]\cdot P(\bar{\mathscr{E}})). \tag{EC7}$$

By the law of conditional expectation, we also have

$$B_1(v,t) = \mathbf{E}[U_{\mathrm{bid}(t)}(v,t) \mid N_t = 0, \mathscr{E}]\cdot P(\mathscr{E}) + \mathbf{E}[U_{\mathrm{bid}(t)}(v,t) \mid N_t = 0, \bar{\mathscr{E}}]\cdot P(\bar{\mathscr{E}}). \tag{EC8}$$

Define $\mathscr{G}$ as the event that the first bidder, say B, arriving in (t,τ) with type (v_B, t_B) (where $t_B \in (t,\tau)$) has valuation $\nu(t_B) < v_B \le v$. Note that $P(\mathscr{G} \mid \bar{\mathscr{E}}) \ge 0$; in particular, $P(\mathscr{G} \mid \bar{\mathscr{E}}) = 0$ if $v \le \nu(t_B)$ for all $t_B \in (t,\tau)$. Then (EC8) can be rewritten as

$$B_1(v,t) = \mathbf{E}[U_{\mathrm{bid}(t)}(v,t) \mid N_t = 0, \mathscr{E}]\cdot P(\mathscr{E}) + \mathbf{E}[U_{\mathrm{bid}(t)}(v,t) \mid N_t = 0, \bar{\mathscr{E}}, \mathscr{G}]\cdot P(\mathscr{G} \mid \bar{\mathscr{E}})\cdot P(\bar{\mathscr{E}}) \tag{EC9}$$

$$+ \mathbf{E}[U_{\mathrm{bid}(t)}(v,t) \mid N_t = 0, \bar{\mathscr{E}}, \bar{\mathscr{G}}]\cdot P(\bar{\mathscr{G}} \mid \bar{\mathscr{E}})\cdot P(\bar{\mathscr{E}}) \tag{EC10}$$

where $\bar{\mathscr{G}}$ is the complementary event.

Conditional on the event $\mathscr{E}$ (i.e., $N(t,\tau) = 0$), the expected utility of bidding is same whether A bids at time t or τ, i.e.,

$$\mathbf{E}[U_{\mathrm{bid}(t)}(v,t) \mid N_t = 0, \mathscr{E}] = \mathbf{E}[U_{\mathrm{bid}(\tau)}(v,t) \mid N_t = 0, \mathscr{E}]. \tag{EC11}$$

Now consider the case when the event $\mathscr{G} \cap \bar{\mathscr{E}}$ occurs, i.e., the bidder B with type (v_B, t_B) has valuation $\nu(t_B) < v_B \le v$. If bidder A bids in the auction at time t then the buyout option disappears and so B also bids in the auction; however if bidder A waits up to τ, then the buyout option is still present at time $t_B \in (t,\tau)$ and bidder B, following strategy $\mathscr{T}[\nu]$, exercises the buyout option. As a result, we have

$$\mathbf{E}[U_{\mathrm{bid}(t)}(v,t) \mid N_t = 0, \bar{\mathscr{E}}, \mathscr{G}] > \mathbf{E}[U_{\mathrm{bid}(\tau)}(v,t) \mid N_t = 0, \bar{\mathscr{E}}, \mathscr{G}] = 0, \tag{EC12}$$

where $\mathbf{E}[U_{bid(t)}(v,t) \mid N_t = 0, \bar{\mathscr{E}}, \mathscr{G}] > 0$ because $v_B \le v$.

Additionally, conditional on the event $\bar{\mathscr{E}} \cap \bar{\mathscr{G}}$ we have

$$\mathbf{E}[U_{\mathrm{bid}(t)}(v,t) \mid N_t = 0, \bar{\mathscr{E}}, \bar{\mathscr{G}}] = \mathbf{E}[U_{\mathrm{bid}(\tau)}(v,t) \mid N_t = 0, \bar{\mathscr{E}}, \bar{\mathscr{G}}]. \tag{EC13}$$

This can be explained as follows: The event $\bar{\mathscr{G}}$ implies that either

(1) $v_B > v$; in this case the expected utility from bidding is zero irrespective of when bidder A bids, or

(2) $v_B \le \nu(t_B)$; in this case bidder B, following strategy $\mathscr{T}[\nu]$, bids in the auction immediately if A waits up to τ and thus the buyout option disappears. If A bids at time t then also the buyout option disappears and thus, irrespective of when A bids, the buyout option is not exercised. Hence the expected utility from bidding for A is same from both actions bid(t) and bid(τ).

Using (EC13), (EC11), and (EC12) in (EC10) we get

$$\begin{aligned}B_1(v,t) &\ge \mathbf{E}[U_{\mathrm{bid}(\tau)}(v,t) \mid N_t = 0, \mathscr{E}]\cdot P(\mathscr{E}) + \mathbf{E}[U_{\mathrm{bid}(\tau)}(v,t) \mid N_t = 0, \bar{\mathscr{E}}, \mathscr{G}]\cdot P(\mathscr{G} \mid \bar{\mathscr{E}})\cdot P(\bar{\mathscr{E}}) \\ &\quad + \mathbf{E}[U_{\mathrm{bid}(\tau)}(v,t) \mid N_t = 0, \bar{\mathscr{E}}, \bar{\mathscr{G}}]\cdot P(\bar{\mathscr{G}} \mid \bar{\mathscr{E}})\cdot P(\bar{\mathscr{E}}) \\ &= \mathbf{E}[U_{\mathrm{bid}(\tau)}(v,t) \mid N_t = 0, \mathscr{E}]\cdot P(\mathscr{E}) + \mathbf{E}[U_{\mathrm{bid}(\tau)}(v,t) \mid N_t = 0, \bar{\mathscr{E}}]\cdot P(\bar{\mathscr{E}}).\end{aligned} \tag{EC14}$$

Furthermore

$$e^{-\beta(\tau-t)}B_1(v,\tau)\geq B_1(v,t), \tag{EC15}$$

because, although both sides of the above inequality have the same time discounting, the right-hand side is conditioned on $N_t=0$ and the left-hand side is conditioned on $N_\tau=0$ (implying fewer competing bidders). Additionally, as indicated earlier in (EC5), we have $e^{-\beta(\tau-t)}B_1(v,\tau)=\mathbf{E}[U_{\mathrm{bid}(\tau)}(v,t)\mid N_t=0,\mathscr{E}]$. Equation (EC14) and inequality (EC15) thus imply together that

$$\mathbf{E}[U_{\mathrm{bid}(\tau)}(v,t)\mid N_t=0,\bar{\mathscr{E}}]\leq B_1(v,t). \tag{EC16}$$

Consider now the following two cases:

- *Case* 1: $v-p\leq B_1(v,\tau)$. Equation (EC7) becomes then

$$\begin{aligned}\mathbf{E}[U_{wait(t,\tau)}(v,t)\mid N_t=0] &= e^{-\beta(\tau-t)}B_1(v,\tau)\cdot P(\mathscr{E})+e^{-\beta(\tau-t)}\mathbf{E}[U_{\mathrm{bid}(\tau)}(v,\tau)\mid N_t=0,\bar{\mathscr{E}}]\cdot P(\bar{\mathscr{E}})\\ &= \mathbf{E}[U_{bid(\tau)}(v,t)\mid N_t=0,\mathscr{E}]\cdot P(\mathscr{E})+\mathbf{E}[U_{\mathrm{bid}(\tau)}(v,t)\mid N_t=0,\bar{\mathscr{E}}]\cdot P(\bar{\mathscr{E}})\\ &\leq B_1(v,t)\leq \max\{B_1(v,t),v-p\},\end{aligned}$$

where the second equality follows from (EC5) and (EC6) and the first inequality follows from (EC14).

- *Case* 2: $v-p> B_1(v,\tau)$. In this case note that

$$\begin{aligned}e^{-\beta(\tau-t)}(v-p) &> e^{-\beta(\tau-t)}B_1(v,\tau)\\ &\geq B_1(v,t)\\ &\geq \mathbf{E}[U_{\mathrm{bid}(\tau)}(v,t)\mid N_t=0,\bar{\mathscr{E}}]\\ &= e^{-\beta(\tau-t)}\mathbf{E}[U_{\mathrm{bid}(\tau)}(v,\tau)\mid N_t=0,\bar{\mathscr{E}}],\end{aligned} \tag{EC17}$$

where the second and third inequalities follow from (EC15) and (EC16) respectively, and the final equality follows from (EC6).

Equation (EC7) thus implies

$$\begin{aligned}\mathbf{E}[U_{\mathrm{wait}(t,\tau)}(v,t)\mid N_t=0] &= e^{-\beta(\tau-t)}((v-p)\cdot P(\mathscr{E})+\mathbf{E}[U_{bid(\tau)}(v,\tau)\mid N_t=0,\bar{\mathscr{E}}]\cdot P(\bar{\mathscr{E}}))\\ &< e^{-\beta(\tau-t)}(v-p)\cdot P(\mathscr{E})+e^{-\beta(\tau-t)}(v-p)\cdot P(\bar{\mathscr{E}})\\ &= e^{-\beta(\tau-t)}(v-p)\\ &< (v-p)\leq \max\{B_1(v,t),v-p\},\end{aligned}$$

where the first inequality follows from (EC17) and the second inequality from the law of total probability.

Because Cases 1 and 2 above are exhaustive, the proof is complete. □

We have thus established that the best response for bidder A, if she sees the buyout option, is to act immediately upon her arrival. Defining now $\delta(v,t)\triangleq\mathbf{E}[U_{\mathrm{buy}(t)}(v,t)\mid N_t=0]-\mathbf{E}[U_{\mathrm{bid}(t)}(v,t)\mid N_t=0]$ as the expected utility difference from exercising the buyout option and placing a bid immediately for the first bidder, Equations (EC1) and (EC4) imply

$$\delta(v,t)=v-p-e^{-(\lambda+\beta)(T-t)}\int_{\underline{v}}^{v}e^{\lambda(T-t)F(x)}\,dx. \tag{EC18}$$

Note that $\delta(v,t)$ is continuous and differentiable on $[\underline{v},+\infty)\times[0,T]$, and it is increasing in v for all $t\in[0,T]$ because

$$\frac{\partial\delta(v,t)}{\partial v}=1-e^{(-\beta-\lambda(1-F(v)))(T-t)}>0. \tag{EC19}$$

Assuming without loss of generality that $p\geq\underline{v}$ implies that $\delta(\underline{v},t)\leq 0$ for all $t\in[0,T]$, which combined with (EC19) proves the existence of a unique $\hat{v}(t)\in[\underline{v},+\infty)$ such that $\delta(\hat{v}(t),t)=0$. Defining $\nu_{\mathrm{tmp}}(t)\triangleq\min(\hat{v}(t),\bar{v})$ and denoting $\mathscr{R}(\mathscr{T}[\nu])$ the set of best-response strategies to the symmetric profile $\mathscr{T}[\nu]$, we have thus proven that $\mathscr{T}[\nu_{\mathrm{tmp}}]\in\mathscr{R}(\mathscr{T}[\nu])$. But because the characterization of ν_{tmp} provided by $\delta(\hat{v}(t),t)=0$ does not depend on the choice of ν as can be seen from Equation (EC18), we also have $\mathscr{T}[\nu_{\mathrm{tmp}}]\in\mathscr{R}(\mathscr{T}[\nu_{\mathrm{tmp}}])$, completing the proof of Theorem 1.

EC.2. Proof of Proposition 1

Let $\tilde{v}(t)$ be the solution of

$$\tilde{v}(t) - p = e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^{\tilde{v}(t)} e^{\lambda(T-t)((x-\underline{v})/m)}\, dx$$
$$= \frac{m e^{-(\lambda+\beta)(T-t)}}{\lambda(T-t)} \left(e^{\lambda(T-t)((\tilde{v}(t)-\underline{v})/m)} - 1\right), \tag{EC20}$$

which is the same equation as (2) except that $F(x)$ has been replaced by $(x-\underline{v})/m$. The solution of (EC20) is given as

$$\tilde{v}(t) = p - \frac{m}{\lambda(T-t)} \left(W\left(-e^{-e^{-(\lambda+\beta)(T-t)} + ((p-\underline{v})\lambda(T-t)/m) - (\lambda+\beta)(T-t)}\right) + e^{-(\lambda+\beta)(T-t)}\right).$$

If $\hat{v}(t) \le \bar{v}$ for some t then (2) and (EC20) are equivalent because $F(x) = (x-\underline{v})/m$, $\forall\, x \in [\underline{v}, \bar{v}]$, and $\hat{v}(t) = \tilde{v}(t)$. It thus follows that $\nu_{\text{tmp}}(t) = \hat{v}(t) = \min(\tilde{v}(t), \bar{v})$.

If $\hat{v}(t) > \bar{v}$ for some t then we have

$$\hat{v}(t) - e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^{\hat{v}(t)} e^{\lambda(T-t)((x-\underline{v})/m)}\, dx \le \hat{v}(t) - e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^{\hat{v}(t)} e^{\lambda(T-t)F(x)}\, dx = p,$$

because $F(x) \le (x-\underline{v})/m$, $\forall\, x \in [\underline{v}, +\infty)$. Furthermore, note that $v - e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^{v} e^{\lambda(T-t)((x-\underline{v})/m)}\, dx$ is increasing in v for all t and this combined with the fact that $\tilde{v}(t)$ is the solution of (EC20) implies that $\tilde{v}(t) \ge \hat{v}(t) > \bar{v}$. Thus $\nu_{\text{tmp}}(t) = \bar{v} = \min(\tilde{v}(t), \bar{v})$.

EC.3. Proof of Theorem 2

Consider a bidder A with type (v, t) arriving in the auction when the buyout option is present. If A is desperate then his strictly dominant strategy is to exercise the buyout option immediately and thus any strategy where the bidder waits cannot be an equilibrium of the game G^{ϵ}.

Suppose now that A is not desperate. We next show that the utility from bidding immediately is strictly greater than the utility from waiting for w units of time (where $0 < w < T - t$) and then bidding, i.e., in the notation of Theorem 1, $\mathbf{E}[U_{\text{bid}(t)}(v, t) \mid \mathfrak{B}] > \mathbf{E}[U_{\text{bid}(t+w)}(v, t) \mid \mathfrak{B}]$ for all $w \in (0, T-t]$, where $\mathfrak{B}$ denotes the event that the buyout option is present when A first arrives (at time t). The utility from bidding is calculated assuming the subsequent nondesperate bidders follow any arbitrary strategy whereas the desperate bidders follow their dominant strategy, which is to exercise the buyout option immediately, if available, and to not participate in the auction otherwise. Note that, as before, bidders are assumed to be rational; therefore, if and when they choose to bid in the auction, they will bid their true valuation (which is their weakly dominant strategy because this is a second-price auction).

As in the proof of Theorem 1, the law of conditional expectation implies

$$\mathbf{E}[U_{\text{bid}(t)}(v, t) \mid \mathfrak{B}] = \mathbf{E}[U_{\text{bid}(t)}(v, t) \mid \mathfrak{B}, \mathscr{D}] \cdot P(\mathscr{D} \mid \mathfrak{B}) + \mathbf{E}[U_{\text{bid}(t)}(v, t) \mid \mathfrak{B}, \bar{\mathscr{D}}] \cdot P(\bar{\mathscr{D}} \mid \mathfrak{B}), \tag{EC21}$$

where $\mathscr{D}$ denotes the event that the buyout option is exercised by a desperate bidder if bidder A waits up to time $t + w$. Note that the event {first bidder arriving in $(t, t+w)$ is desperate} $\subset \{\mathscr{D} \mid \mathfrak{B}\}$ and thus we have

$$P(\mathscr{D} \mid \mathfrak{B}) \ge P(\text{first bidder arriving in } (t, t+w) \text{ is desperate}) > 0. \tag{EC22}$$

Because desperate bidders do not participate in the auction if the buyout option is not present, we have

$$\mathbf{E}[U_{\text{bid}(t)}(v, t) \mid \mathfrak{B}, \mathscr{D}] > \mathbf{E}[U_{\text{bid}(t+w)}(v, t) \mid \mathfrak{B}, \mathscr{D}] = 0. \tag{EC23}$$

Bidder A bids in the auction at $t + w$, if it is still open, and hence the buyout option disappears at $t+w$. Thus no desperate bidder arriving in the interval $(t+w, T]$ participates in the auction. In addition if the event $\mathscr{D}$ does not occur then no desperate bidder arriving in the interval $(t, t+w)$ participates in the auction. The presence of desperate bidders therefore does not affect the expected utility of bidder A if the event $\bar{D}$ occurs and hence the analysis used to obtain (EC14) can be essentially repeated, with minor modifications to incorporate the fact that subsequent bidders follow some arbitrary strategy to get

$$\mathbf{E}[U_{\text{bid}(t)}(v, t) \mid \mathfrak{B}, \bar{\mathscr{D}}] \ge \mathbf{E}[U_{\text{bid}(t+w)}(v, t) \mid \mathfrak{B}, \bar{\mathscr{D}}]. \tag{EC24}$$

Using (EC24), (EC23), and (EC22) in (EC21), we obtain

$$\mathbf{E}[U_{\text{bid}(t)}(v,t)\mid \mathfrak{B}] > \mathbf{E}[U_{\text{bid}(t+w)}(v,t)\mid \mathfrak{B},\mathcal{D}]\cdot P(\mathcal{D}\mid \mathfrak{B}) + \mathbf{E}[U_{\text{bid}(t+w)}(v,t)\mid \mathfrak{B},\bar{\mathcal{D}}]\cdot P(\bar{\mathcal{D}}\mid \mathfrak{B}) \quad \text{(EC25)}$$

$$= \mathbf{E}[U_{\text{bid}(t+w)}(v,t)\mid \mathfrak{B}], \quad \text{(EC26)}$$

which proves that bidder A is strictly better off bidding immediately. Therefore, any strategy whereby a bidder, who arrives when the buyout option is present, waits for w units of time ($w > 0$) is not a Bayesian Nash equilibrium of $G^{(\epsilon)}$.

The second result of the theorem involves characterizing a threshold function ν such that the strategy whereby nondesperate bidders play $\mathcal{T}[\nu]$ is a Bayesian Nash equilibrium of the game $G^{(\epsilon)}$.

If the first bidder arriving in the auction is desperate then his dominant strategy is to exercise the buyout option immediately and the auction ends. Otherwise if first bidder A with type (v,t) is nondesperate, then the analysis of Theorem 1 can be exactly repeated to show that the best response strategy of A is to bid immediately if his valuation $v \le \nu_{\text{tmp}}^{(\epsilon)}(t)$ and to exercise the buyout option immediately otherwise. The threshold valuation $\nu_{\text{tmp}}^{(\epsilon)} = \min(\hat{v}(t), \bar{v})$ where $\hat{v}(t)$ is the unique solution in $[\underline{v}, +\infty)$ of

$$\hat{v}(t) - p = e^{-(\lambda(1-\epsilon)+\beta)(T-t)} \int_{\underline{v}}^{\hat{v}(t)} e^{\lambda(1-\epsilon)(T-t)F(x)}\,dx, \quad \text{(EC27)}$$

which is the same as (EC2) except that λ is replaced by $\lambda(1-\epsilon)$ because the arrival rate of nondesperate bidders in the game $G^{(\epsilon)}$ is $\lambda(1-\epsilon)$. Thus the strategy $\mathcal{T}[\nu_{\text{tmp}}^{(\epsilon)}]$ is a Bayesian Nash equilibrium of $G^{(\epsilon)}$ and, in addition, as $\epsilon \to 0$, the right-hand side of (EC27) converges to the right-hand side of (EC2) and it follows that $\lim_{\epsilon\to 0} \nu_{\text{tmp}}^{(\epsilon)} = \nu_{\text{tmp}}$.

We have thus shown that in any equilibrium of $G^{(\epsilon)}$ a bidder, who arrives when the buyout option is present, acts (bids/buyouts) immediately, and her behavior is characterized by the threshold function $\nu_{\text{tmp}}^{(\epsilon)}$. Once a bid has been placed in the auction, the buyout option disappears and, consequently, for all subsequent bidders a weakly dominant strategy is to bid their true valuation. This proves that for any $\epsilon > 0$, $\mathcal{T}[\nu_{\text{tmp}}^{(\epsilon)}]$ is the unique (see §4.1 for a discussion of what uniqueness means for this analysis) Bayesian Nash equilibrium of the game $G^{(\epsilon)}$, thus implying that $\mathcal{T}[\nu_{\text{tmp}}]$ is the unique equilibrium of a temporary buyout-price auction that is robust to the payoff perturbation discussed above.

EC.4. Proof of Theorem 3

Consider a bidder A with type (v,t) and information I_t in an auction where all other bidders play strategy $\mathcal{P}[\nu]$, where ν: $[0,T]\times[\underline{v},\bar{v}]\cup\{0\}\to[\underline{v},\bar{v}]$ is a continuous function such that $\nu(t,0)$ is nondecreasing in t and $\nu(t,I)$ is nonincreasing in I for any t.

We first derive bidder A's utility if he bids his true valuation at time T. Following strategy $\mathcal{P}[\nu]$ all other bidders also bid at T and thus $I_\tau = 0$, $\forall\,\tau\in[0,T)$. Now bidder A wins the auction if no bidder exercises the buyout option and if every bidder has a valuation less than bidder A's valuation, i.e., the event $\{A \text{ wins}\} = \{v_\tau \le \min(\nu(\tau,0), v)$ for every bidder $(v_\tau,\tau,0)\}$ where the notation v_τ indicates the valuation of a bidder arriving at time τ. Also, because the auction is open at t, it can be inferred that all bidders $(v_\tau,\tau,0)$ with $\tau\in(0,t))$ have valuation $v_\tau \le \nu(\tau,0)$ and thus the arrival rate of bidders at any $\tau\in(0,t)$ is $\lambda F(\nu(\tau,0))$. Then probability that A wins the auction is

$$\Pr\big(A \text{ wins} \mid \{t_i\}_{i=1}^{N(t)}, \{\hat{t}_i\}_{i=1}^{N(t,T)}\big) = \prod_{i=1}^{N(t)} \frac{F(\min(\nu(t_i,0),v))}{F(\nu(t_i,0))} \prod_{i=1}^{N(t,T)} F(\min(\nu(\hat{t}_i,0),v)), \quad \text{(EC28)}$$

where $N(t)$ is a counting process denoting the number of bidder arrivals in $(0,t)$ in a nonhomogeneous Poisson process with arrival rate $\lambda(\tau) = \lambda F(\nu(\tau,0))$ and $\forall\,\tau\in(0,t)$; $\{t_i\}_{i=1}^{N(t)}$ are the corresponding arrival epochs. $N(t,T)$ is a counting process denoting the number of arrivals in $(t,T]$ in a Poisson process with arrival rate λ and $\{\hat{t}_i\}_{i=1}^{N(t,T)}$ are the arrival epochs. The first term of the product in (EC28) is the probability that the event $\{v_{t_i} \le \min(\nu(t_i,0),v) \mid v_{t_i} \le \nu(t_i,0)\}$ occurs for all bidders arriving in $(0,t)$. The second term is the probability that the valuation $v_{\hat{t}_i}$ of every bidder arriving in $(t,T]$ satisfies $v_{\hat{t}_i} \le \min(\nu(\hat{t}_i,0),v)$. Here, and in the remainder of the paper, we assume that if $k=0$ then $\prod_{i=1}^{k}(\cdot) = 1$.

Conditional on bidder A winning, the distribution of the highest bid, $v^{(1)}$, among the other $N(t)+N(t,T)$ bidders arriving at epochs $\{t_i\}_{i=1}^{N(t)}$ and $\{\hat{t}_i\}_{i=1}^{N(t,T)}$ is

$$F_{v^{(1)}\mid A\text{ wins},\,\{t_i\}_{i=1}^{N(t)},\,\{\hat{t}_i\}_{i=1}^{N(t,T)}}(x)=\prod_{i=1}^{N(t)}\frac{F(\min(\nu(t_i,0),x))}{F(\min(\nu(t_j,0),v))}\prod_{j=1}^{N(t,T)}\frac{F(\min(\nu(\hat{t}_j,0),x))}{F(\min(\nu(\hat{t}_j,0),v))}\quad\forall\,x\in[0,v].$$

Using the above distribution function, we have

$$\begin{aligned}\mathbf{E}\big[v^{(1)}\mid A\text{ wins},\{t_i\}_{i=1}^{N(t)},\{\hat{t}_i\}_{i=1}^{N(t,T)}\big]&=\int_0^v\big(1-F_{v^{(1)}\mid A\text{ wins},\,\{t_i\}_{i=1}^{N(t)},\,\{\hat{t}_i\}_{i=1}^{N(t,T)}}(x)\big)\,dx\\&=v-\int_{\underline{v}}^{v}F_{v^{(1)}\mid A\text{ wins},\,\{t_i\}_{i=1}^{N(t)},\,\{\hat{t}_i\}_{i=1}^{N(t,T)}}(x)\,dx,\end{aligned}$$

and thus the expected discounted utility from bidding at T for bidder A is

$$\mathbf{E}[U_{\text{bid}(T)}(v,t,0)]=\mathbf{E}\bigg[e^{-\beta(T-t)}\bigg(\int_{\underline{v}}^{v}\prod_{i=1}^{N(t)}\frac{F(\min(\nu(t_i,0),x))}{F(\nu(t_i,0))}\times\prod_{j=1}^{N(t,T)}F(\min(\nu(\hat{t}_j,0),x))\,dx\bigg)\bigg]\tag{EC29}$$

where the expectation on the right-hand side is over $\{t_i\}_{i=1}^{N(t)}$ and $\{\hat{t}_i\}_{i=1}^{N(t,T)}$.

The rest of the proof proceeds as follows: We first show using Lemma 2 that if bidder A bids in the auction he must do so at time T, and next prove that the bidder is weakly better off making a decision immediately in Lemma 3. Consequently we derive in Lemma 4 the best-response strategy $\mathcal{R}(\mathcal{P}[\nu])$ of a bidder when all other bidders play $\mathcal{P}[\nu]$ and then characterize a threshold function ν_{prm} such that $\mathcal{P}[\nu_{\text{prm}}]\in\mathcal{R}(\mathcal{P}[\nu_{\text{prm}}])$ thus establishing that $\mathcal{P}[\nu_{\text{prm}}]$ is an equilibrium strategy.

Lemma 2. $\mathbf{E}[U_{\text{bid}(\tau)}(v,t,0)]\le\mathbf{E}[U_{\text{bid}(T)}(v,t,0)]$ *for all* $t\le\tau\le T$.

Proof. This result is a direct implication of the assumption that $\nu_{\text{prm}}(t,I_t)$ is nonincreasing in I_t and admits the following justification: Although bidding earlier does not increase the utility of a bidder, it reveals information about his valuation to other bidders, who can use this information to their advantage. More formally, suppose that the bidder A bids immediately, i.e., $\tau=t$ while all other bidders, following strategy $\mathcal{P}[\nu]$, bid at T and hence $I_\omega=\underline{v}$, $\forall\,\omega\in(t,T)$. Then the probability that bidder A wins, from (EC28), is

$$\Pr\big(A\text{ wins}\mid\{t_i\}_{i=1}^{N(t)},\{\hat{t}_i\}_{i=1}^{N(t,T)}\big)=\prod_{i=1}^{N(t)}\frac{F(\min(\nu_{\text{prm}}(t_i,0),v))}{F(\nu_{\text{prm}}(t_i,0))}\prod_{i=1}^{N(t,T)}F(\min(\nu_{\text{prm}}(\hat{t}_i,\underline{v}),v)),$$

and the corresponding expected utility from bidding for bidder A is

$$\mathbf{E}[U_{\text{bid}(t)}(v,t,0)]=\mathbf{E}\bigg[e^{-\beta(T-t)}\bigg(\int_{\underline{v}}^{v}\prod_{i=1}^{N(t)}\frac{F(\min(\nu_{\text{prm}}(t_i,0),x))}{F(\nu_{\text{prm}}(t_i,0))}\times\prod_{j=1}^{N(t,T)}F(\min(\nu_{\text{prm}}(\hat{t}_j,\underline{v}),x))\,dx\bigg)\bigg].\tag{EC30}$$

Because threshold $\nu(t,I)$ is nonincreasing in I, we have

$$F(\min(\nu_{\text{prm}}(\hat{t}_j,\underline{v}),x))\le F(\min(\nu_{\text{prm}}(\hat{t}_j,0),x))\quad\forall\,j=1,2,\ldots,N(t,T).\tag{EC31}$$

Comparing (EC30) with (EC29) using (EC31) we obtain

$$\mathbf{E}[U_{\text{bid}(t)}(v,t,0)]\le\mathbf{E}[U_{\text{bid}(T)}(v,t,0)].$$

If the first bidder bids at some $t<\tau<T$ then $I_\omega=\underline{v}$, $\forall\,\omega\in(\tau,T)$ and then the above analysis can be repeated to obtain

$$\mathbf{E}[U_{\text{bid}(\tau)}(v,t,0)]\le\mathbf{E}[U_{\text{bid}(T)}(v,t,0)].\quad\square$$

More generally, if bidder A places any bid in the auction (not necessarily his true valuation) at time τ ($\tau<T$) then $I_\tau>0$ and the threshold valuation in (τ,T) is lower than if A bids at time T. Thus the probability that the buyout option is exercised by a bidder in (τ,T) is higher, and because bidder A gets zero utility in such an event he will not bid in the auction at any time $\tau<T$. Moreover if a bidder is bidding at time T then his weakly dominant strategy is to bid his true valuation.

We next establish that bidder A is weakly better off making a decision immediately upon his arrival, i.e., he instantaneously decides either to exercise the buyout option immediately or place a bid in the auction at time T.

Lemma 3. *When facing bidders who follow strategy $\mathcal{PS}$, bidder A is weakly better off making a decision immediately i.e.,* $\mathbf{E}[U_{\text{wait}(t,\tau)}(v,t,0)] \le \max\{\mathbf{E}[U_{\text{buy}(t)}(v,t,0)], \mathbf{E}[U_{\text{bid}(T)}(v,t,0)]\}$.

Proof. Here we give an intuitive argument: a formal proof can be constructed on the lines of the proof of Lemma 1. We have already proven in Lemma 2 that if a bidder decides to bid in the auction, she must do so at time T and thus her expected utility from bidding is independent of when she makes the decision to bid. Indeed, if we let $\mathbf{E}[U^{(\tau)}_{\text{bid}(T)}(v,t,0)]$ denote the utility of a bidder of type $(v,t,0)$ who waits up to time τ ($\tau > t$) and then decides to place a bid in the auction at time T, then it can shown that

$$\mathbf{E}[U^{(\tau)}_{\text{bid}(T)}(v,t,0)] = \mathbf{E}[U_{\text{bid}(T)}(v,t,0)],$$

where we recall that $\mathbf{E}[U_{\text{bid}(T)}(v,t,0)]$ denotes the utility of a bidder $(v,t,0)$ who decides immediately to place a bid at time T.

Additionally, although the buyout price remains constant throughout the auction, waiting decreases the bidder's utility from exercising the buyout option because of his time-sensitivity. Thus if a bidder waits before making a decision, his expected utility from bidding remains constant but the utility from exercising the buying option decreases and so he is weakly better off making a decision immediately. □

Thus we have shown, in Lemma 2 and Lemma 3, that bidder A must choose, at time t, one of the two actions $\{bid(T), buy(t)\}$. We now show that the best response strategy $\mathcal{R}(\mathcal{P}[\nu])$ to $\mathcal{P}[\nu]$ is indeed a threshold strategy.

Lemma 4. *The best response strategy to* $\mathcal{P}[\nu]$ *is*

$$\mathcal{R}(\mathcal{P}[\nu])(v,t,0)\colon \begin{cases} \textit{Buyout at } p \textit{ immediately} & \textit{if } v > \min(\tilde{v}_{[\nu]}(t,0), \bar{v}) \\ \textit{Bid } v \textit{ at time } T & \textit{if } v \le \min(\tilde{v}_{[\nu]}(t,0), \bar{v}), \end{cases}$$

where $\tilde{v}_{[\nu]}(t,0)$ *is such that*

$$\mathbf{E}[U_{\text{buy}(t)}(\tilde{v}_{[\nu]}(t,0),t,0)] = \mathbf{E}[U_{\text{bid}(T)}(\tilde{v}_{[\nu]}(t,0),t,0)]. \tag{EC32}$$

Note that Lemma 4 only specifies the equilibrium path behavior of bidders. The best response strategy is completely specified by choosing a continuous threshold function $\tilde{v}_{[\nu]}(t,I)$, which is nonincreasing in I for all t and such that $\tilde{v}_{[\nu]}(t,0)$ satisfies (EC32).

Proof. The derivative of the expected utility from bidding $\mathbf{E}[U_{\text{bid}(T)}(v,t,0)]$ with respect to v is

$$\frac{\partial}{\partial v}\mathbf{E}[U_{\text{bid}(T)}(v,t,0)] = \mathbf{E}\left[e^{-\beta(T-t)}\left(\prod_{i=1}^{N(t)} \frac{F(\min(\nu(t_i,0),v))}{F(\nu(t_i,0))} \times \prod_{j=1}^{N(t,T)} F(\min(\nu(\hat{t}_j,0),v))dx\right)\right]. \tag{EC33}$$

For every realization of $N(t)$, $N(t,T)$, $\{t_i\}_{i=1}^{N(t)}$, and $\{\hat{t}_i\}_{i=1}^{N(t,T)}$, the term inside the expectation on the right-hand side of (EC33) is nonnegative and less than 1. Thus $0 \le (\partial/\partial v)\mathbf{E}[U_{\text{bid}(T)}(v,t,0)] < 1$ for all $v \in [\underline{v}, +\infty)$ and $t \in (0,T)$.

The utility from exercising the buyout option $\mathbf{E}[U_{\text{buy}(t)}(v,t,0)] = v - p$ and thus

$$0 \le \frac{\partial}{\partial v}\mathbf{E}[U_{\text{bid}(T)}(v,t,0)] < 1 = \frac{\partial}{\partial v}\mathbf{E}[U_{\text{buy}(t)}(v,t,0)]. \tag{EC34}$$

Assuming, without loss of generality, that $p \ge \underline{v}$ we get $\mathbf{E}[U_{\text{buy}(t)}(\underline{v},t,0)] = \underline{v} - p \le 0 = \mathbf{E}[U_{\text{bid}(T)}(v,t,0)]$. This together with (EC34) implies that there exists a unique valuation $\tilde{v}_{[\nu]}(t,0) \ge \underline{v}$ such that

$$\mathbf{E}[U_{\text{buy}(t)}(\tilde{v}_{[\nu]}(t,0),t,0)] = \mathbf{E}[U_{\text{bid}(T)}(\tilde{v}_{[\nu]}(t,0),t,0)], \tag{EC35}$$

where the notation $\tilde{v}_{[\nu]}(t,0)$ indicates the dependence of this valuation on strategy $\mathcal{P}[\nu]$ and the fact that this corresponds to the case when $I = 0$.

Thus a bidder $(v,t,0)$ with $v \le \tilde{v}_{[\nu]}(t,0)$ bids in the auction at time T because $\mathbf{E}[U_{\text{buy}(t)}(v,t,0)] \le \mathbf{E}[U_{\text{bid}(T)}(v,t,0)]$. On the other hand, if $v > \tilde{v}_{[\nu]}(t,0)$ then $\mathbf{E}[U_{\text{buy}(t)}(v,t,0)] > \mathbf{E}[U_{\text{bid}(T)}(v,t,0)]$ and thus it is profitable for bidder A to exercise the buyout option. □

Next we characterize a continuous threshold function $\nu_{\text{prm}}(t, I)$ such that $\nu_{\text{prm}}(t, 0)$ is nondecreasing in t, $\nu_{\text{prm}}(t, I)$ is nonincreasing in I for all t and is such that

$$\mathcal{P}[\nu_{\text{prm}}] \in \mathcal{R}(\mathcal{P}[\nu_{\text{prm}}]). \tag{EC36}$$

Now notice that $\mathcal{R}(\mathcal{P}[\nu_{\text{prm}}])$ is also a threshold strategy and indeed (EC36) holds, if $\nu_{\text{prm}}(t, I) = \min(\tilde{v}_{[\nu_{\text{prm}}]}(t, I), \bar{v})$ for all t, I.

For $I = 0$ substituting $\nu_{\text{prm}}(t, 0) = \min(\tilde{v}_{[\nu_{\text{prm}}]}(t, 0), \bar{v})$ in (EC32) implies that $\tilde{v}_{[\nu_{\text{prm}}]}(t, 0) = \tilde{v}(t)$ where $\tilde{v}(t)$ must satisfy

$$\tilde{v}(t) - p = \mathbf{E}_t\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\tilde{v}(t)} \prod_{i=1}^{N(t)} \frac{F(\min(\tilde{v}(t_i), x))}{F(\tilde{v}(t_i))} \prod_{j=1}^{N(t,T)} F(\min(\tilde{v}(\hat{t}_j), x))\, dx\right)\right]. \tag{EC37}$$

for all $t \in [0, T]$.

For $I > 0$, choosing any $\tilde{v}_{[\nu_{\text{prm}}]}(t, I)$, which is nonincreasing in I for all t, suffices and so we set $\tilde{v}_{[\nu_{\text{prm}}]}(t, I) = \nu_{\text{prm}}(t, I)$.

We thus obtain that if a threshold function $\nu_{\text{prm}}(t, I)$ is nonincreasing in I for all t and $\nu_{\text{prm}}(t, 0) = \min(\tilde{v}(t), \bar{v})$ is nondecreasing in t where $\tilde{v}(\cdot)$ is the solution of (EC37), then the corresponding strategy $\mathcal{P}[\nu_{\text{prm}}]$ defines an equilibrium. We next prove the existence of ν_{prm}.

First, consider the following equation obtained by substituting $F(\min(\tilde{v}(\hat{t}_j), x))$ with $F(x)$ in (EC37) for all bidders arriving in the interval $(t, T]$:

$$\tilde{v}(t) - p = \mathbf{E}_t\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\tilde{v}(t)} \prod_{i=1}^{N(t)} \frac{F(\min(\tilde{v}(t_i), x))}{F(\tilde{v}(t_i))} \prod_{j=1}^{N(t,T)} F(x)\, dx\right)\right]. \tag{EC38}$$

Note that the right hand side of (EC38) at any time t depends only on $[\tilde{v}(\tau)]_{\tau \in [0, t)}$ whereas the right-hand side of (EC37) is the function of $[\tilde{v}(\tau)]_{\tau \in [0, T]}$. However if the solution $\tilde{v}(t)$ to (EC38) is nondecreasing in t then $\min(\tilde{v}(\hat{t}_j), x) = x$ for all $x \in [\underline{v}, \tilde{v}(t)]$, $\hat{t}_j \in (t, T]$ and thus $\tilde{v}(t)$ also solves (EC37).

Lemma 5. *For any $\varepsilon > 0$, there exists a solution to* (EC38) in the interval $[0, T - \varepsilon]$.

Proof. Define

$$G(\phi)(t) = \mathbf{E}_t\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\phi(t)} \prod_{i=1}^{N(t)} \frac{F(\min(\phi(t_i), x))}{F(\phi(t_i))} \prod_{j=1}^{N(t,T)} F(x)\, dx\right)\right] + p.$$

Using this notation, (EC38) becomes

$$\phi(t) = G(\phi)(t), \tag{EC39}$$

which we seek to solve on the interval $[0, T]$.

To prove the existence of a solution of the above equation, we use the following theorem (Theorem 4.1 in Smart 1974), which is a slight generalization of Schauder's (1930) fixed point theorem.

Theorem EC.1. *Let $\mathcal{M}$ be a nonempty convex subset of a normed space $\mathcal{B}$. Let T be a continuous mapping of $\mathcal{M}$ into a compact set $\mathcal{K} \subset \mathcal{M}$. Then T has a fixed point.*

Using the above theorem, we show that (EC39) has a solution on the interval $[0, T - \varepsilon]$ for any $\varepsilon > 0$. For $M > 0$ let $\mathcal{F}$ be the set of continuous functions with domain $[0, T - \varepsilon]$ and range $[\underline{v}, q]$ $(q > \underline{v})$ that satisfy the following Lipschitz condition

$$|\phi(t') - \phi(t)| \le M|t' - t|; \quad t', t \in [0, T - \varepsilon]. \tag{EC40}$$

Let $\mathcal{K} = \{G(\phi) \mid \phi \in \mathcal{F}\}$, i.e., G maps the set $\mathcal{F}$ to $\mathcal{K}$. We first prove the following lemma.

Lemma 6. *If $q > (p - e^{-\beta\varepsilon}\underline{v})/(1 - e^{-\beta\varepsilon})$ and $M > e^{-\beta\varepsilon}(2\lambda + \beta)q/(1 - e^{-\beta\varepsilon})$ then $\mathcal{K} \subset \mathcal{F}$.*

Proof. For any $\phi \in \mathcal{F}$, $t \in [0, T-\varepsilon]$ we have

$$\underline{v} \le p \le G(\phi)(t) \le p + e^{-\beta\varepsilon}(q - \underline{v}). \tag{EC41}$$

Clearly, if $q > (p - e^{-\beta\varepsilon}\underline{v})/(1-e^{-\beta\varepsilon})$ then $G(\phi)(t) \in [\underline{v}, q]$ for all $\phi \in \mathcal{F}$, $t \in [0, T-\varepsilon]$. Next we show that for any $\phi \in \mathcal{F}$

$$|G(\phi)(t) - G(\phi)(t')| \le M|t-t'| \quad \forall\, t, t' \in [0, T-\varepsilon]. \tag{EC42}$$

Indeed for any $\phi \in \mathcal{F}$ and $t \in [0, T-\varepsilon)$ consider

$$G(\phi)(t) = \mathbf{E}_t\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\phi(t)} \prod_{i=1}^{N(t)} \frac{F(\min(\phi(t_i), x))}{F(\phi(t_i))} \prod_{j=1}^{N(t,T)} F(x)\,dx\right)\right] + p, \tag{EC43}$$

where the expectation $\mathbf{E}_t$ is with respect to the number $N(t)$ and epochs $t_1, \ldots, t_{N(t)}$ of arrivals in $[0, t)$ of a nonhomogeneous Poisson process with rate $\lambda F(\tilde{v}(t))$, and number $N(t, T)$ of arrivals in $(t, T]$ of a Poisson process with rate λ.

Let

$$\mathbf{E}[H(\phi(t), t)] = G(\phi)(t) - p = \mathbf{E}_t\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\phi(t)} \prod_{i=1}^{N(t)} \frac{F(\min(\phi(t_i), x))}{F(\phi(t_i))} \prod_{j=1}^{N(t,T)} F(x)\,dx\right)\right].$$

Now to calculate $\mathbf{E}[H(\phi(t), t)]$, we condition on the number of arrivals in the interval $(t, t+\Delta t)$ where $\Delta t > 0$ and small, and is such that $t + \Delta t \le T - \varepsilon$.

For the sake of brevity, let

$$\Gamma\left(\{t_i\}_{i=1}^{N(t)}\right) = \prod_{i=1}^{N(t)} \frac{F(\min(\phi(t_i), x))}{F(\phi(t_i))}. \tag{EC44}$$

First suppose that there was arrival in $(t, t+\Delta t)$; an event that has probability $\lambda\Delta t$ because the arrival process is Poisson with rate λ. Then the conditional expectation is

$$\begin{aligned}&\mathbf{E}[H(\phi(t), t) \mid N(t, t+\Delta t) = 1]\\ &\quad = \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\phi(t)} \Gamma\left(\{t_i\}_{i=1}^{N(t)}\right) \times F(x) \times F(x)^l\,dx\right)\right] P(N(t+\Delta t, T) = l).\end{aligned} \tag{EC45}$$

Note that here we first calculate the expected utility given $N(t+\Delta t, T) = l$ and then sum over all possible l. The expectation $\mathbf{E}_t$ on the right-hand side is over $N(t)$ and $\{t_i\}_{i=1}^{N(t)}$.

Now if there was no arrival in $(t, t+\Delta t)$, an event that has probability $(1 - \lambda\Delta t + o(\Delta t))$ where $o(\Delta t)$ indicates any function $f(\Delta t)$ such that $\lim_{\Delta t\to 0} f(\Delta t)/\Delta t = 0$, we get

$$\mathbf{E}[H(\phi(t), t) \mid N(t, t+\Delta t) = 0] = \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\phi(t)} \Gamma\left(\{t_i\}_{i=1}^{N(t)}\right) \times F(x)^l\,dx\right)\right] P(N(t+\Delta t, T) = l). \tag{EC46}$$

The probability of more than one arrival in an interval of length Δt is $o(\Delta t)$ and thus the unconditional expectation of $H(\phi(t), t)$ becomes:

$$\begin{aligned}\mathbf{E}[H(\phi(t), t)] &= \mathbf{E}[H(\phi(t), t) \mid N(t, t+\Delta t) = 1] \times (\lambda\Delta t)\\ &\quad + \mathbf{E}[H(\phi(t), t) \mid N(t, t+\Delta t) = 0] \times (1 - \lambda\Delta t) + o(\Delta t).\end{aligned} \tag{EC47}$$

Substituting for the terms, we get

$$\begin{aligned}\mathbf{E}[H(\phi(t), t)] &= \left(\sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\phi(t)} \Gamma\left(\{t_i\}_{i=1}^{N(t)}\right) \times F(x) \times F(x)^l\,dx\right)\right] P(N(t+\Delta t, T) = l)\right)(\lambda\Delta t)\\ &\quad + \left(\sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\phi(t)} \Gamma\left(\{t_i\}_{i=1}^{N(t)}\right) \times F(x)^l\,dx\right)\right] P(N(t+\Delta t, T) = l)\right)(1 - \lambda\Delta t) + o(\Delta t).\end{aligned} \tag{EC48}$$

Similar to the above analysis, we next condition on the number of arrivals in (t, t') to evaluate $\mathbf{E}[H(\phi(t'))]$, where for ease of exposition the notation $t' = t + \Delta t$ is used.

Now suppose that there is arrival in the interval (t, t'). For small Δt, the probability of this event is $\lambda F(\phi(t))\Delta t$, because the arrival process at t is nonhomogeneous Poisson with rate $\lambda F(\phi(t))$. We then have

$$\begin{aligned}&\mathbf{E}[H(\phi(t'), t') \mid N(t, t') = 1]\\&\quad = \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\underline{v}}^{\phi(t')} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \frac{F(\min(\phi(t), x))}{F(\phi(t))} F(x)^l \, dx\right)\right] P(N(t', T) = l).\end{aligned} \tag{EC49}$$

Now if there was no arrival in (t, t'), an event with probability $(1 - \lambda F(\phi(t))\Delta t + o(\Delta t))$, we get

$$\mathbf{E}[H(\phi(t'), t') \mid N(t, t') = 0)] = \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\underline{v}}^{\phi(t')} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \times F(x)^l \, dx\right)\right] P(N(t', T) = l). \tag{EC50}$$

The unconditional expectation of $H(\phi(t'), t')$ is

$$\begin{aligned}\mathbf{E}[H(\phi(t'), t')] = {} & \mathbf{E}[H(\phi(t'), t') \mid N(t, t') = 1)](\lambda F(\phi(t))\Delta t)\\& + \mathbf{E}[H(\phi(t'), t') \mid N(t, t') = 0](1 - \lambda F(\phi(t))\Delta t) + o(\Delta t).\end{aligned} \tag{EC51}$$

Substituting for the terms we get

$$\begin{aligned}&\mathbf{E}[H(\phi(t'), t')]\\&\quad = \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\underline{v}}^{\phi(t')} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \times \frac{F(\min(\phi(t), x))}{F(\phi(t))} \times F(x)^l \, dx\right)\right] P(N(t', T) = l)(\lambda F(\phi(t))\Delta t)\\&\qquad + \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\underline{v}}^{\phi(t')} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \times F(x)^l \, dx\right)\right] P(N(t', T) = l)(1 - \lambda F(\phi(t))\Delta t) + o(\Delta t).\end{aligned} \tag{EC52}$$

By the definition of the function G we have

$$G(\phi)(t') - G(\phi)(t) = \mathbf{E}[H(\phi(t'), t')] - \mathbf{E}[H(\phi(t), t)].$$

Substituting (EC48) and (EC52) in the above expression we obtain

$$\begin{aligned}&G(\phi)(t') - G(\phi)(t)\\&\quad = \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\underline{v}}^{\phi(t')} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \times \frac{F(\min(\phi(t), x))}{F(\phi(t))} \times F(x)^l \, dx\right)\right] \times P(N(t', T) = l)(\lambda F(\phi(t))\Delta t)\\&\qquad + \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\underline{v}}^{\phi(t')} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \times F(x)^l \, dx\right)\right] \times P(N(t', T) = l)(1 - \lambda F(\phi(t))\Delta t) + o(\Delta t)\\&\qquad - \left(\sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\phi(t)} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \times F(x) \times F(x)^l \, dx\right)\right] \times P(N(t', T) = l)\right)(\lambda \Delta t)\\&\qquad - \left(\sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\phi(t)} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \times F(x)^l \, dx\right)\right] \times P(N(t', T) = l)\right)(1 - \lambda \Delta t) + o(\Delta t).\end{aligned} \tag{EC53}$$

Simplifying the above expression we get the following bound

$$\begin{aligned}&|G(\phi)(t') - G(\phi)(t)|\\&\quad \le \left|\sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\underline{v}}^{\phi(t')} (\Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \times F(\min(\phi(t), x))F(x)^l - F(x)^{l+1}) \, dx\right)\right] P(N(t', T) = l)(\lambda \Delta t)\right|\\&\qquad + \left|\sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\phi(t)}^{\phi(t')} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \times F(\min(\phi(\tau), x))F(x)^l \, dx\right)\right] P(N(t', T) = l)(\lambda \Delta t)\right|\\&\qquad + \left|\left(\sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\phi(t)}^{\phi(t')} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \times F(x)^l \, dx\right)\right] P(N(t', T) = l)\right)\right|\end{aligned}$$

$$+\left|\lambda\Delta t(1-F(\phi(t)))\sum_{0}^{\infty}\mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\underline{v}}^{\phi(t)}\Gamma\left(\{t_i\}_{i=1}^{N(t)}\right)\times F(x)^l\,dx\right)\right]P(N(t',T)=l)\right|$$
$$+\left|\lambda\Delta t\sum_{0}^{\infty}\mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\phi(t)}^{\phi(t')}\Gamma\left(\{t_i\}_{i=1}^{N(t)}\right)\times F(x)^l\,dx\right)\right]P(N(t',T)=l)\right|$$
$$+\left|\left(e^{-\beta(T-t')}-e^{-\beta(T-t)}\right)\mathbf{E}[H(\phi(t),t)]\right|+o(\Delta t).$$

Now noting that $t, t' \le (T-\varepsilon)$ and $H(\phi(t),t)\le q\ \forall\, t\in[0,T-\varepsilon]$, the above bound can be further simplified to obtain

$$|G(\phi)(t')-G(\phi)(t)|\le e^{-\beta\varepsilon}(2q\lambda\Delta t+2|\phi(t')-\phi(t)|\lambda\Delta t+|\phi(t')-\phi(t)|+(e^{\beta\Delta t}-1)q)+o(\Delta t). \quad \text{(EC54)}$$

Because $\phi\in\mathcal{F}$ we have $|\phi(t')-\phi(t)|\le M|t'-t|$, $\forall\, t,\, t'\in[0,T-\varepsilon]$. We use this and rearrange terms to obtain

$$|G(\phi)(t')-G(\phi)(t)|\le e^{-\beta\varepsilon}(q(2\lambda+\beta)+M)\Delta t+o(\Delta t). \quad \text{(EC55)}$$

Thus if we choose $M > e^{-\beta\varepsilon}(q(2\lambda+\beta)+M)$, which can be rearranged to obtain $M>e^{-\beta\varepsilon}(2\lambda+\beta)q/(1-e^{-\beta\varepsilon})$, we get that

$$|G(\phi)(t')-G(\phi)(t)|\le M(t'-t)\quad \forall\, 0\le t'-t\le\delta;\quad t,t'\in[0,T-\varepsilon], \quad \text{(EC56)}$$

where $\delta=\sup\{\delta>0\mid o(\delta)/\delta<M-e^{-\beta\varepsilon}(2\lambda+\beta)q/(1-e^{-\beta\varepsilon})\}$. Note that $\delta>0$ because $\lim_{\delta\to 0}o(\delta)/\delta=0$.

Because δ is independent of t, it follows from (EC56) that

$$|G(\phi)(t')-G(\phi)(t)|\le M|t'-t|\quad \forall\, |t'-t|\le\delta;\quad t,t'\in[0,T-\varepsilon], \quad \text{(EC57)}$$

where δ is as defined above.

Now for any $t, t'\in[0,T-\varepsilon]$, assuming without loss of generality that $t'>t$, we have

$$\begin{aligned}|G(\phi)(t')-G(\phi)(t)| &\le |G(\phi)(t')-G(\phi)(t'-\delta)|+\cdots+|G(\phi)(t'-n\delta)-G(\phi)(t)|\\ &\le M|\delta|+\cdots+M|t'-n\delta-t|=M|t'-t|,\end{aligned}$$

where $n=\lfloor (t'-t)/\delta\rfloor$. The second inequality follows from (EC57).

Thus we have shown that for any $\phi\in\mathcal{F}$

$$|G(\phi)(t')-G(\phi)(t)|\le M|t'-t|\quad \forall\, t,\, t'\in[0,T-\varepsilon]. \quad \text{(EC58)}$$

This combined with the fact that for all $\phi\in\mathcal{F}$

$$G(\phi)(t)\in[\underline{v},q]\quad \forall\, t\in[0,T-\varepsilon] \quad \text{(EC59)}$$

implies that the set $\mathcal{K}=\{G(\phi)\mid \phi\in\mathcal{F}\}\subset\mathcal{F}$. □

Thus if we choose $q>(p-e^{-\beta\varepsilon}\underline{v})/(1-e^{-\beta\varepsilon})$ and $M>e^{-\beta\varepsilon}(2\lambda+\beta)q/(1-e^{-\beta\varepsilon})$ the operator G maps $\mathcal{F}$ to $\mathcal{K}\subset\mathcal{F}$. We next show that the set $\mathcal{K}$ is compact in $\mathbb{C}$ where $\mathbb{C}$ is the space of continuous functions with domain $[0,T-\varepsilon]$.

Lemma 7. *$\mathcal{K}$ is compact in $\mathbb{C}$.*

Proof. First note that because $q>(p-e^{-\beta\varepsilon}\underline{v})/(1-e^{-\beta\varepsilon})$ it follows from Lemma 6 that $G(\phi)(t)\le q$ for all $\phi\in\mathcal{F}$ and $t\in[0,T-\varepsilon]$.

It follows from (EC58) that for any $\kappa>0$ and all $|t-t'|<\kappa/M$

$$|G(\phi)(t')-G(\phi)(t)|\le M|t'-t|<\kappa\quad \forall\,\phi\in\mathcal{F};\quad t,t'\in[0,T-\varepsilon], \quad \text{(EC60)}$$

proving that the set $\mathcal{K}=\{G(\phi)\mid \phi\in\mathcal{F}\}$ is equicontinuous on the interval $[0,T-\varepsilon]$ (§7.22 in Rudin 1976). The compactness of set $\mathcal{K}$ then follows from the Arzelà-Ascoli Theorem (Theorem 3(3.I) in Kantorovich and Akilov 1964). □

We next show that the operator G is continuous on the set $\mathcal{F}$.

LEMMA 8. *G is continuous on the set $\mathcal{F}$.*

PROOF. Let ϱ denote the sup norm, i.e.,

$$\varrho(\phi, \psi) = \max_{0\le t\le T-\varepsilon} |\phi(t) - \psi(t)|. \tag{EC61}$$

Let $\epsilon > 0$ be given. For any $\phi, \psi \in \mathcal{F}$, $t \in [0, T-\varepsilon]$ we have

$$|G(\phi)(t) - G(\psi)(t)| \le \left|\mathbf{E}_t\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\phi(t)}\left(\prod_{i=1}^{N(t)}\frac{F(\min(\phi(t_i), x))}{F(\phi(t_i))} - \prod_{i=1}^{\widetilde{N}(t)}\frac{F(\min(\psi(\tilde{t}_i), x))}{F(\psi(\tilde{t}_i))}\right)\prod_{j=1}^{N(t,T)}F(x)\,dx\right)\right]\right|$$
$$+\left|\mathbf{E}\left[\int_{\phi(t)}^{\psi(t)}\prod_{i=1}^{\widetilde{N}(t)}\frac{F(\min(\psi(\tilde{t}_i), x))}{F(\psi(\tilde{t}_i))}\prod_{j=1}^{N(t,T)}F(x)\,dx\right]\right|, \tag{EC62}$$

where $N(t)$ (respectively $\widetilde{N}(t)$) is the number of arrivals in a nonhomogeneous Poisson process with rate $\lambda F(\phi(\tau))$ (respectively $\lambda F(\psi(\tau))$) for $\tau \in (0, t)$, and the corresponding arrival times are given by $\{t_i\}_{i=1}^{N(t)}$ (respectively $\{\tilde{t}_i\}_{i=1}^{\widetilde{N}(t)}$). $N(t, T)$ is the number of arrivals in the interval $(t, T]$ of a Poisson process with arrival rate λ.

If we assume that all bidders in $(t, T]$ bid in the auction, the first term on the right-hand side of (EC62) can be interpreted as the difference in the expected utility for a bidder with valuation $\phi(t)$ if every bidder of type (v, τ) with $\tau \in (0, t)$ has valuation $v \le \phi(\tau)$ as opposed to having valuation $v \le \psi(\tau)$. This term is positive only if there are one or more arrivals in the interval $(0, t)$ in a nonhomogeneous Poisson process with rate $\lambda|F(\phi(\tau)) - F(\psi(\tau))|$. Now, because F is a given continuous function for any $\kappa > 0$, there exists a $\delta(\kappa) > 0$ such that for all $\phi, \psi \in \mathcal{F}$, $\varrho(\phi, \psi) < \delta(\kappa)$

$$|F(\phi(\tau)) - F(\psi(\tau))| < \kappa \quad \forall\, \tau \in [0, T-\varepsilon].$$

Thus the rate $\lambda|F(\phi(\tau)) - F(\psi(\tau))|$ can be upper bounded by $\lambda\kappa$ for all $\tau \in [0, T-\varepsilon]$ if $\varrho(\phi, \psi) < \delta(\kappa)$. Now the probability that there is one or more arrival in the interval $(0, t)$ in a Poisson process with rate $\lambda\kappa$ is $\lambda\kappa t + o(\kappa)$. Thus probability of at least one arrival in the interval $(0, t)$ in a nonhomogeneous Poisson process with rate $\lambda|F(\phi(\tau)) - F(\psi(\tau))|$ can be upper bounded by $\lambda\kappa t + o(\kappa)$. In addition an arrival in the above mentioned process can lead to a difference in expected utility, which is bounded above by q. Hence the first term in (EC62) can be upper bounded by $q(\lambda\kappa t + o(\kappa))$. In addition because $F(\min(\psi(\tilde{t}_i), x)) \le F(\psi(\hat{t}_i))$ and $F(\cdot) \le 1$, the second term of (EC62) is bounded above by $|\psi(t) - \phi(t)|$. We thus get

$$|G(\phi)(t) - G(\psi)(t)| \le q(\lambda\kappa t + o(\kappa)) + |\phi(t) - \psi(t)|$$
$$\le q(\lambda\kappa T + o(\kappa)) + \varrho(\phi, \psi).$$

Define $\kappa_1 = \epsilon/3q\lambda T$ and $\kappa_2 = \sup(\kappa \mid o(\kappa) < \epsilon/3q)$ ($\kappa_2 > 0$ because $\lim_{\kappa\to 0} o(\kappa)/\kappa = 0$) and let $\kappa = \min(\kappa_1, \kappa_2)$. Thus for $\delta = \min(\delta(\kappa), \epsilon/3)$, we have for all $\phi, \psi \in \mathcal{F}$ such that $\varrho(\phi, \psi) < \delta$

$$\varrho(G(\phi), G(\psi)) = \sup_{t\in[0,T-\varepsilon]} |G(\phi)(t) - G(\psi)(t)| \le q(\lambda\kappa T + o(\kappa)) + \varrho(\phi, \psi)$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,$$

and hence the operator G is continuous on the set $\mathcal{F}$. □

Thus G is a continuous mapping of $\mathcal{F}$ (which is nonempty and convex) into a compact set $\mathcal{K} \subset \mathcal{F}$ and hence, by Theorem EC.1, G has a fixed point, i.e., there exists a solution to the equation (EC39) on the interval $[0, T-\varepsilon]$. □

Because Lemma 5 holds for any $\varepsilon > 0$, it proves that a solution to (EC38) exists on the interval $[0, T)$. We next show that $\tilde{v}(t)$ satisfying (EC38) on the interval $[0, T)$, is nondecreasing in t. For that we first need the following result. Let $\mathbf{E}[H(v, t)]$ be the right-hand side of (EC38), i.e.,

$$\mathbf{E}[H(v, t)] = \mathbf{E}_t\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{v}\prod_{i=1}^{N(t)}\frac{F(\min(\tilde{v}(t_i), x))}{F(\tilde{v}(t_i))}\prod_{j=1}^{N(t,T)}F(x)\,dx\right)\right]. \tag{EC63}$$

Lemma 9. $\mathbf{E}_t[H(v,t')] > \mathbf{E}_t[H(v,t)]$ *for* $t' > t$; $t, t' \in [0,T)$.

Proof. Let $t' > t$ for some $t, t' \in [0,T)$. Suppose that there are k bidders in $(0,t)$ and l bidders in $(t,T]$ and they arrive at time $0 < t_1 < t_2 < \cdots < t_k < t$ and $t < \hat{t}_1 < \hat{t}_2 < \cdots < \hat{t}_l < T$ respectively. Suppose also that j of the l bidders ($0 \le j \le l$) arrive in the interval (t,t').

Then from (EC63), we have

$$\mathbf{E}[H(v,t') \mid k, l, j, \{t_i\}_{i=1}^k, \{\hat{t}_j\}_{i=1}^l] = e^{-\beta(T-t')}\left(\int_{\underline{v}}^{v} \frac{\prod_{i=1}^k F(\min(\tilde{v}(t_i),x))}{\prod_{i=1}^k F(\tilde{v}(t_i))} \times \frac{\prod_{i=1}^j F(\min(\tilde{v}(\hat{t}_i),x))}{\prod_{i=1}^j F(\tilde{v}(\hat{t}_i))} \prod_{i=j+1}^{l} F(x)\,dx\right).$$

For this arrival stream of bidders, $H(v,t)$ is

$$\mathbf{E}[H(v,t) \mid k, l, \{t_i\}_{i=1}^k, \{\hat{t}_j\}_{i=1}^l] = e^{-\beta(T-t')}\left(\int_{\underline{v}}^{v} \frac{\prod_{i=1}^k F(\min(\tilde{v}(t_i),x))}{\prod_{i=1}^k F(\tilde{v}(t_i))} \prod_{i=1}^{l} F(x)\,dx\right).$$

Because $F(\min(\tilde{v}(\hat{t}_i),x))/F(\tilde{v}(\hat{t}_i)) \ge F(x)$ for $i = 1,\ldots,j$ and $e^{-\beta(T-t')} > e^{-\beta(T-t)}$, we have

$$\mathbf{E}[H(v,t') \mid k, l, j, \{t_i\}_{i=1}^k, \{\hat{t}_j\}_{i=1}^l] > \mathbf{E}[H(v,t) \mid k, l, \{t_i\}_{i=1}^k, \{\hat{t}_j\}_{i=1}^l]. \tag{EC64}$$

The inequality (EC64) is true for any realization of the random bidder arrival process and hence the inequality holds if we take the expectation over the arrival process. Thus, we have

$$\mathbf{E}[H(v,t')] > \mathbf{E}[H(v,t)]. \quad \square$$

We next use Lemma 9 to prove $\tilde{v}(t)$ is nondecreasing in t.

Lemma 10. *The solution* $\tilde{v}(t)$ *of* (EC38) *is nondecreasing in* t.

Proof. By definition of $\tilde{v}(t)$, we have

$$\tilde{v}(t,0) - p = \mathbf{E}[H(\tilde{v}(t),t)]. \tag{EC65}$$

Assume for contradiction that $\tilde{v}(t) < \tilde{v}(t-dt)$ for some $t \in [0,T)$ and $dt > 0$ (and such that $t - dt \in [0,T)$). Then there exists $dv > 0$, such that

$$\tilde{v}(t,0) = \tilde{v}(t-dt,0) - dv. \tag{EC66}$$

Using Lemma 9, we have

$$\mathbf{E}[H(\tilde{v}(t),t)] > \mathbf{E}[H(\tilde{v}(t),t-dt)]. \tag{EC67}$$

Substituting (EC66) in (EC65) and using (EC67), we get

$$\begin{aligned}\tilde{v}(t-dt) - dv - p &= \mathbf{E}[H(\tilde{v}(t-dt)-dv,t)] \\ &> \mathbf{E}[H(\tilde{v}(t-dt)-dv,t-dt)].\end{aligned} \tag{EC68}$$

By the definition of $\tilde{v}(t-dt)$, we have

$$\tilde{v}(t-dt) - p = \mathbf{E}[H(\tilde{v}(t-dt),t-dt)]. \tag{EC69}$$

Using (EC69) in (EC68), we get

$$\mathbf{E}[H(\tilde{v}(t-dt),t-dt)] - \mathbf{E}[H(\tilde{v}(t-dt)-dv,t-dt)] > dv.$$

However, by arguments similar to those in the proof of Lemma 4, it can be shown that $(\partial/\partial v)\cdot \mathbf{E}[H(v,t)] \le 1$ for $t \in [0,T)$, which contradicts the above result. Hence $\tilde{v}(t)$ is nondecreasing in t. $\square$

Thus we have shown that $\tilde{v}(t)$, satisfying (EC38) on the interval $[0,T)$, is nondecreasing in t. If we ignore the zero probability event of a bidder arriving first to the auction at time T then, as argued before, $\tilde{v}(t)$ also satisfies (EC37).

Next, consider any function $\nu_{\text{prm}}(t,I)$ that is nonincreasing in I and satisfies $\nu_{\text{prm}}(t,0) = \min(\tilde{v}(t),\bar{v})$ where $\tilde{v}(t)$ is the solution of (EC37) (and (EC38)) on the interval $[0,T)$, which exists by Lemma 5. Then Lemma 10 shows that $\nu_{\text{prm}}(t,0)$ is nondecreasing in t and thus, as argued before, the corresponding strategy $\mathcal{P}[\nu_{\text{prm}}]$ satisfies $\mathcal{P}[\nu_{\text{prm}}] \in \mathcal{R}(\mathcal{P}[\nu_{\text{prm}}])$ and hence defines a Bayesian Nash equilibrium for the online auction game with a permanent buyout price p.

EC.5. Proof of Theorem 4

For any continuous function ν: $[0, T] \times [\underline{v}, \bar{v}] \cup \{0\} \to [\underline{v}, \bar{v}]$ such that $\nu(t, I)$ is nonincreasing in I for all t, suppose that normal (*noncommon value*) bidders follow the strategy

$$\mathscr{P}'[\nu](v, t, I_t) \begin{cases} \text{Buyout at time } \tau \geq t & \text{if } v > \nu(\tau, I_\tau) \\ \text{Bid } v \text{ at time } T & \text{if } p < v \leq \nu(\tau, I_\tau) \text{ for all } t \leq \tau \leq T \\ \text{Bid } v \text{ at any time in } [t, T] & \text{if } v \leq p. \end{cases}$$

The *common value* bidders are assumed to play the following strategy:

$$\mathscr{P}_{(c)}[\nu_c](v, t, I) = \begin{cases} \text{Exercise buyout option at } \tau \in [t, T) & \text{if } v > \nu_c(\tau, I_\tau) \text{ at any } \tau \in [t, T) \\ \text{Bid true valuation at } T & \text{otherwise,} \end{cases}$$

where $\nu_c(t, 0) = \nu(t, 0)$ and $\nu_c(t, I) = \underline{v}$, $\forall I > 0, t$. That is, such a bidder continuously monitors the auction and exercises the buyout option at the first time $\tau \in [t, T)$ when $v > \nu_c(\tau, I_\tau)$; otherwise he bids his true valuation at T.

We next show that if normal bidders play $\mathscr{P}'[\nu]$ then there exists at least one bidder who has an incentive to deviate thus proving that the strategy $\mathscr{P}'[\nu]$ does not define a Bayesian Nash equilibrium of the game $G^{(\epsilon)}$.

Indeed, consider a bidder, e.g. A, of type $(v, t, 0)$ with valuation $\underline{v} < v < p$. We prove that

$$\mathbf{E}[U_{\text{bid}(T)}(v, t)] \geq \mathbf{E}[U_{\text{bid}(\tau)}(v, t)] \quad \forall \tau \in [t, T],$$

where the equality holds only at $\tau = T$. Thus bidder A has an incentive to deviate from the strategy $\mathscr{P}'[\nu]$ and bid at time T.

Using the law of conditional expectation, we have

$$\mathbf{E}[U_{\text{bid}(t)}(v, t)] = \mathbf{E}[U_{\text{bid}(t)}(v, t) \mid \mathscr{C}]P(\mathscr{C}) + \mathbf{E}[U_{\text{bid}(t)}(v, t) \mid \bar{\mathscr{C}}]P(\bar{\mathscr{C}}), \tag{EC70}$$

where $\mathscr{C}$ is the event that at least one common value bidder arrives in the auction (notice that $P(\mathscr{C}) > 0$). Now notice that if bidder A bids at time t then the threshold ν_c for a common value bidder becomes $\underline{v}$ for the remaining duration of the auction. As a consequence, a common value bidder will exercise the buyout option with probability 1 and thus $\mathbf{E}[U_{\text{bid}(t)}(v, t) \mid \mathscr{C}] = 0$ because the bidder A does not get the product in this case.

Similarly, we have

$$\mathbf{E}[U_{\text{bid}(T)}(v, t)] = \mathbf{E}[U_{\text{bid}(T)}(v, t) \mid \mathscr{C}]P(\mathscr{C}) + \mathbf{E}[U_{\text{bid}(T)}(v, t) \mid \bar{\mathscr{C}}]P(\bar{\mathscr{C}}). \tag{EC71}$$

If bidder A bids at time T then $\mathbf{E}[U_{\text{bid}(T)}(v, t) \mid \mathscr{C}] > 0$. To see this, consider the event $\mathscr{E}$ that no other normal bidders arrive in the auction and all common value bidders arriving in the auction have valuation less than v. If event $\mathscr{E}$ occurs the information $I_t = 0$ for all $t < T$ and consequently all common value bidders use the threshold function $\nu(t, 0)$. Because $v < p \leq \nu(t, 0)$, this implies that all bidders will bid in the auction and then, because bidder A has the highest valuation, she wins the auction and gains positive utility. Also note that $P(\mathscr{E} \mid \mathscr{C}) > 0$ and thus $\mathbf{E}[U_{\text{bid}(T)}(v, t) \mid \mathscr{C}] > 0$.

If event $\bar{\mathscr{C}}$ occurs then no common value bidders arrive in the auction and thus the auction only has normal bidders. However, we have shown in Lemma 2 that in the presence of normal bidders $\mathbf{E}[U_{\text{bid}(\tau)}(v, t, 0)] \leq \mathbf{E}[U_{\text{bid}(T)}(v, t, 0)]$ for all $t \leq \tau \leq T$ and thus we have that

$$\mathbf{E}[U_{\text{bid}(T)}(v, t) \mid \bar{\mathscr{C}}] \geq \mathbf{E}[U_{\text{bid}(t)}(v, t) \mid \bar{\mathscr{C}}].$$

This combined with the above arguments and using (EC70) and (EC71) yields that

$$\mathbf{E}[U_{\text{bid}(T)}(v, t)] > \mathbf{E}[U_{\text{bid}(t)}(v, t)].$$

The same analysis can be repeated to show that for any $\tau < T$

$$\mathbf{E}[U_{\text{bid}(T)}(v, t)] > \mathbf{E}[U_{\text{bid}(\tau)}(v, t)],$$

and thus bidder A is strictly better off bidding at time T. Hence for any threshold ν the game $G^{(\epsilon)}$ does not have an equilibrium of the form $\mathscr{P}'[\nu]$. The above analysis can be extended to show that indeed any strategy where a bidder places a bid before time T does not form a Bayesian Nash equilibrium of $G^{(\epsilon)}$. On the other hand, the proof of Theorem 3 still holds for this case and thus a strategy where normal bidders play strategy $\mathscr{P}[\nu_{\text{prm}}]$ still defines a Bayesian Nash equilibrium of $G^{(\epsilon)}$ where ν_{prm} is as defined in Theorem 3.

EC.6. Proof of Proposition 2

Let $\tilde{v}(t)$ be the solution of (EC38), i.e.,

$$\tilde{v}(t) - p = \mathbf{E}_t\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\tilde{v}(t)} \prod_{i=1}^{N(t)} \frac{F(\min(\tilde{v}(t_i), x))}{F(\tilde{v}(t_i))} \prod_{j=1}^{N(t,T)} F(x)\,dx\right)\right], \tag{EC72}$$

for all $t \in [0, T-\varepsilon]$ for some $\varepsilon > 0$ and small.

As discussed in the proof of Theorem 3, $\tilde{v}(t)$ is nondecreasing in t and thus it follows from (EC29) that the expected utility from bidding for a bidder with type $(v, t, 0)$, assuming other bidders follow strategy $\mathscr{P}[\tilde{v}]$, is

$$\mathbf{E}_t[U_{\text{bid}(T)}(v, t, 0)] = \mathbf{E}\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{v} \prod_{i=1}^{N(t)} \frac{F(\min(\tilde{v}(t_i), x))}{F(\tilde{v}(t_i))} \prod_{j=1}^{N(t,T)} F(x)\,dx\right)\right].$$

Therefore, (EC72) can be restated as

$$\tilde{v}(t) - p = \mathbf{E}_t[U_{\text{bid}(T)}(\tilde{v}(t), t, 0)], \tag{EC73}$$

for all $t \in [0, T-\varepsilon]$.

For any $t, (t+\Delta t) \in [0, T-\varepsilon]$, $\tilde{v}(\cdot)$ must satisfy

$$\frac{\tilde{v}(t+\Delta t) - \tilde{v}(t)}{\Delta t} = \frac{\mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t+\Delta t), t+\Delta t, 0)] - \mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t), t, 0)]}{\Delta t}. \tag{EC74}$$

Consider a bidder A with type $(\tilde{v}(t), t)$ and information $I_t = 0$. Because the auction is running at time t, all bidders of type $(v_i, t_i, 0)$ arriving in the interval $(0, t)$ have valuation $v_i \le \tilde{v}(t_i)$. Thus for bidder A the arrival process of other bidders is

1. nonhomogeneous Poisson process in $(0, t)$ with arrival rate $\lambda(\tau) = \lambda F(v_{\text{prm}}(\tau))$ for $\tau \in (0, t)$,
2. homogeneous Poisson process in $(t, T]$ with arrival rate λ.

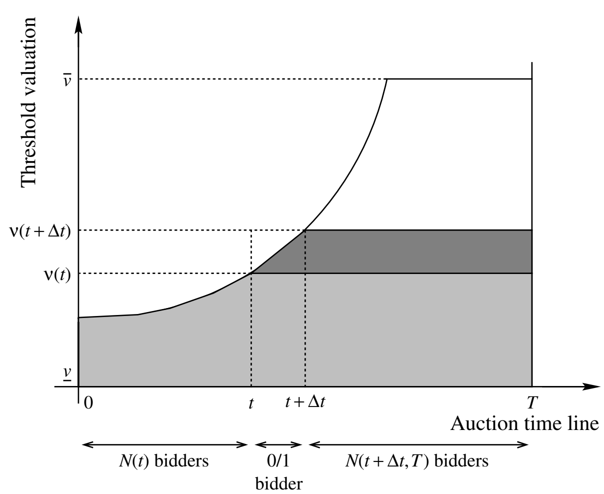
Now to calculate $\mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t), t, 0)]$, we condition on the number of arrivals in the interval $(t, t+\Delta t)$ (Figure EC.1). To reduce notational complexity, let

$$\Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) = \prod_{i=1}^{N(t)} \frac{F(\min(\tilde{v}(t_i), x))}{F(\tilde{v}(t_i))}. \tag{EC75}$$

Suppose that there was arrival in $(t, t+\Delta t)$; this event has probability $\lambda\Delta t + o(\Delta t)$, where $o(\Delta t)$ indicates any function $f(\Delta t)$ such that $\lim_{\Delta t\to 0}(f(\Delta t)/\Delta t) = 0$, because the arrival process is Poisson with rate λ. Then the conditional expected utility is

$$\begin{aligned}&\mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t), t, 0) \mid N(t, t+\Delta t) = 1]\\ &\quad = \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\tilde{v}(t)} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \times F(x) \times F(x)^l\,dx\right)\right] P(N(t+\Delta t, T) = l).\end{aligned} \tag{EC76}$$

Figure EC.1 Threshold Valuation v_{prm}



Note that here we first calculate the expected utility given $N(t+\Delta t, T) = l$ and then sum over all possible l. The expectation $\mathbf{E}_t$ on the right-hand side is over $N(t)$ and $\{t_i\}_{i=1}^{N(t)}$.

Now, if there was no arrival in $(t, t+\Delta t)$, an event which has probability $1-\lambda\Delta t + o(\Delta t)$, we get

$$\begin{aligned}\mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t), t, 0) \mid N(t, t+\Delta t) = 0] \\ = \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\tilde{v}(t)} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \times F(x)^l\, dx\right)\right] P(N(t+\Delta t, T) = l).\end{aligned} \tag{EC77}$$

The probability of more than one arrival in an interval of length Δt is $o(\Delta t)$ and thus the unconditional expected utility from bidding becomes

$$\begin{aligned}\mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t), t, 0)] &= \mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t), t, 0) \mid N(t, t+\Delta t) = 1] \times (\lambda\Delta t) \\ &\quad + \mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t), t, 0) \mid N(t, t+\Delta t) = 0] \times (1-\lambda\Delta t) + o(\Delta t).\end{aligned} \tag{EC78}$$

Substituting for the terms, we get

$$\begin{aligned}&\mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t), t, 0)] \\ &\quad = \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\tilde{v}(t)} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) F(x)^{l+1}\, dx\right)\right] P(N(t+\Delta t, T) = l) \times (\lambda\Delta t) \\ &\quad\quad + \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t)}\left(\int_{\underline{v}}^{\tilde{v}(t)} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) F(x)^{l}\, dx\right)\right] P(N(t+\Delta t, T) = l) \times (1-\lambda\Delta t) + o(\Delta t).\end{aligned} \tag{EC79}$$

Similar to the above analysis, we next condition on the number of arrivals in (t, t') to calculate $\mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t'), t', 0)]$, where to reduce notational complexity the notation $t' = t+\Delta t$ is used.

Consider a bidder B with type $(\tilde{v}(t'), t')$ and information $I_{t'} = 0$. Because the auction is running at time t', all bidders of type $(v_i, t_i, 0)$ arriving in the interval $(0, t')$ have valuation $v_i \le \tilde{v}(t_i)$. Thus for bidder B the arrival process of other bidders is

1. nonhomogeneous Poisson process in $(0, t')$ with arrival rate $\lambda(\tau) = \lambda F(\tilde{v}(\tau))$ for $\tau \in (0, t)$,
2. homogeneous Poisson process in $(t', T]$ with arrival rate λ.

Now suppose that there is arrival in the interval (t, t'). For small Δt, the probability of this event is $\lambda F(\tilde{v}(t))\Delta t + o(\Delta t)$, because the arrival rate at t is $\lambda F(\tilde{v}(t))$. Therefore,

$$\begin{aligned}\mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t'), t', 0) \mid N(t, t') = 1] \\ = \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\underline{v}}^{\tilde{v}(t')} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \frac{F(\min(\tilde{v}(t), x))}{F(\tilde{v}(t))} F(x)^l\, dx\right)\right] P(N(t', T) = l).\end{aligned} \tag{EC80}$$

If there was no arrival in (t, t'), an event with probability $(1-\lambda F(\tilde{v})\Delta t + o(\Delta t))$, we get

$$\mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t'), t', 0) \mid N(t, t') = 0)] = \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\underline{v}}^{\tilde{v}(t')} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) F(x)^l\, dx\right)\right] P(N(t', T) = l). \tag{EC81}$$

Because the probability of more than one arrival in an interval of length Δt is $o(\Delta t)$, the unconditional expected utility is

$$\begin{aligned}\mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t'), t', 0)] &= \mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t'), t', 0) \mid N(t, t') = 1)](\lambda F(\tilde{v}(t))\Delta t) \\ &\quad + \mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t'), t', 0) \mid N(t, t') = 0](1-\lambda F(\tilde{v}(t))\Delta t) + o(\Delta t).\end{aligned}$$

Substituting we get

$$\begin{aligned}&\mathbf{E}[U_{\text{bid}(T)}(\tilde{v}(t'), t', 0)] \\ &\quad = \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\underline{v}}^{\tilde{v}(t')} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) \frac{F(\min(\tilde{v}(t), x))}{F(\tilde{v}(t))} F(x)^l\, dx\right)\right] P(N(t', T) = l)(\lambda F(\tilde{v}(t))\Delta t) \\ &\quad\quad + \sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\underline{v}}^{\tilde{v}(t')} \Gamma\big(\{t_i\}_{i=1}^{N(t)}\big) F(x)^l\, dx\right)\right] P(N(t', T) = l)(1-\lambda F(\tilde{v}(t))\Delta t) + o(\Delta t).\end{aligned} \tag{EC82}$$

Consider now the error associated with substituting $F(\min(\tilde{v}(\tau), x))$ with $F(x)$ in the first term of the right-hand side of (EC82). The substitution effectively assumes that one bidder has valuation in the interval $[\underline{v}, \tilde{v}(t')]$ instead of $[\underline{v}, \tilde{v}(t)]$. When calculating the maximum valuation among the bidders, this assumption can lead to a difference $d(\tilde{v}(t), \tilde{v}(t'))$, which is bounded as follows:

$$0 \le d(\tilde{v}(t), \tilde{v}(t')) \le \tilde{v}(t') - \tilde{v}(t) \le M(t' - t)$$

where we have used the fact that $\tilde{v}(\cdot)$ satisfies the Lipschitz condition for constant M. Thus, if we let T_1 be the first term in Equation (EC82) and T_1^a be the corresponding approximate expression, then we have

$$\begin{aligned} 0 \le T_1^a - T_1 &\le \sum_{l=0}^{\infty} (\mathbf{E}[e^{-\beta(T-t')}(\tilde{v}(t') - (\tilde{t}))] \times \Pr(N(t', T) = l) * \lambda F(v_{\text{prm}}(\tau))\Delta t) \\ &\le e^{-\beta(T-t')} M \lambda F(\tilde{v}(t))(\Delta t)^2 \end{aligned}$$

and the error due to the approximation is thus $o(\Delta t)$.

Therefore, (EC82) can be rewritten as

$$\begin{aligned} \mathbf{E}[U_{\text{bid}}(\tilde{v}(t'), t')] = &\sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\underline{v}}^{\tilde{v}(t')} \prod F(x)^{l+1}\,dx\right)\right] \Pr(N(t', T) = l)(\lambda \Delta t) \\ &+ \sum_{l=0}^{\infty}\left(\mathbf{E}\left[e^{-\beta(T-t')}\left(\int_{\underline{v}}^{\tilde{v}(t')} \prod F(x)^{l}\,dx\right)\right] \Pr(N(t', T) = l)(1 - \lambda F(\tilde{v}(\tau))\Delta t)\right) + o(\Delta t). \quad \text{(EC83)} \end{aligned}$$

Subtracting Equation (EC79) from Equation (EC83), dividing by Δt and taking the limit $\Delta t \to 0$, we get, after simplification,

$$\frac{d\tilde{v}(t)}{dt} = \frac{(\beta + \lambda(1 - F(\tilde{v}(t))))(\tilde{v}(t) - P)}{1 - e^{-(\beta+\lambda(1-F(\tilde{v}(t))))(T-t)}}.$$

Recall that bidder valuations are assumed to be uniformly distributed and thus $F(x) = (x - \underline{v})/m$ for $x \in [\underline{v}, \bar{v}]$.

Substituting $t = 0$ in (EC72), we get the initial value for the above differential equation

$$\tilde{v}(0) = P - \frac{m}{\lambda T}\left(W\left(-e^{-(\beta+\lambda)T} - \frac{-(P - \underline{v})\lambda T + m e^{-(\beta+\lambda)T}}{m}\right) + e^{-(\beta+\lambda)T}\right),$$

where, as before, W is Lambert's W function.

EC.7. Empirical Analysis of Bid Times

The outcome predictions in Theorem 1 and Theorem 3, for temporary and permanent buyout-price auctions respectively, suggest the following two verifiable hypotheses about the timing of bids placed in the corresponding buyout-price auction, which we test using data from eBay and Yahoo! auctions:

Hypothesis 1. *The first activity* (*bid/buyout*) *in a temporary buyout-price auction occurs earlier than in a standard auction* (*without a buyout price*).

Hypothesis 2. *The average bid time in a permanent buyout price auction is higher than in a standard auction* (*without a buyout price*).

In practice, bidding in online auctions is affected by several factors including the number of competing auctions, presence of a reserve price, seller's feedback ratings, level of bidders' rationality and experience and their different incentives and, in all likelihood, the bidding data we collect from auction websites depends on some or all of these factors. However, for the purpose of this analysis we assume that the difference in bid times (as observed below) is a consequence primarily of the presence of a buyout price.

For testing the hypotheses we collect bidding data for auctions belonging to the "Consumer Electronics" category, which have the string "iPod"[EC1] in their title. This criterion gives data on auctions

[EC1] The search is not case-sensitive.

for iPods and its accessories (including headphones, iPod skins, chargers, cases); this market was chosen for several reasons. First, the volume transacted is high as compared to other products. Second, most of these items are available through fixed-price mechanisms and hence bidder valuations should have well-defined upper and lower bounds. Furthermore, these items will most likely have little or no common value and so the independent private valuation assumption should hold for bidders participating in such auctions. Further details of the data collection process are provided in Gupta (2006).

EC.7.1. eBay Data. For all auctions where at least one bid is placed, the time elapsed in the auction (expressed as a fraction of the auction length) before the first activity occurs (bid/buyout) is recorded. Note that for auctions where the buyout option is exercised we use the scheduled auction ending time, as opposed to the actual ending time (which is the time of buyout exercise), to calculate the auction duration. We ignore all auctions where the buyout price (bp) is too close to the starting price (sp) (bp/sp < 1.25), because in such a case the auction effectively becomes a fixed-price mechanism. We also filter out auctions where the buyout price is much higher than the winning bid wb, in particular wb/bp < 0.8, because in such cases the buyout price may be set too high for any bidder to ever exercise it. Note that, although the winning bid is a random variable depending on the bidder arrival process and bidder valuations, we assume that for auctions of the above category the variance of the winning bid is low enough so that it serves as a good indicator of the "actual price" of the auctioned item.

From the data it is observed that the time of the first bid/buyout decreases as the winning bid increases for auctions with and without a buyout price. Furthermore, because in our data set almost 90% of the buyout-price auctions have a winning bid of less than \$30, for the purpose of this analysis we only consider auctions where the winning bid is less than \$30. Table EC.1 summarizes the descriptive statistics of the data. The number in parentheses is the standard deviation associated with the corresponding mean.

Observe that the average time of the first activity in auctions without a buyout option is higher than in auctions with a buyout option. Furthermore, the average winning bid in buyout-price auctions is lower and, because the first bid time tends to decrease as the winning bid increases, adjusting the data to ensure that the average winning bid is the same for both cases would further lower the average first activity time in buyout-price auctions (or alternately increase the average first bid time in auctions without a buyout price). The two sample t-test gives a p-value of 2.75×10^{-27} implying that the hypothesis—that the average first activity time in buyout price auctions is lower than the first bid time in auctions without a buyout price—can be accepted.

In a separate analysis we find that in our data set, the buyout option is exercised in about 53% of the auctions in which it is present with a mean exercise time, expressed as a fraction of the auction duration, of 0.7617 (standard deviation $= 0.0064$). The presence of a buyout option thus decreases the waiting time of the auction participants to about three-quarters of what they would have experienced if the buyout option was not present.

EC.7.2. Yahoo! Data. Collecting bid timing data for auctions where the buyout price is exercised requires obtaining the scheduled auction ending time, as opposed to the actual ending time (the time the buyout price is exercised), which would necessitate tracking auctions that have not closed. This would, however, take a prohibitively long time on Yahoo! because the number of open auctions in the market segment of our interest that satisfy certain conditions on the buyout price, starting price, and winning bid are very small. For buyout-price auctions, we thus restrict our analysis to auctions where the buyout price is not exercised. Note that this biases the data because we are more likely to get auctions where the buyout price is higher than what bidders expect to pay for getting the product. We partially correct this bias by only considering, as in the earlier case, auctions where the ratio of

Table EC.1 Summary of Auction Data with bp/sp ≥ 1.25, and wb/bp ≥ 0.8

	No buyout price	Buyout price
Number of auctions	3,414	1,954
Mean winning bid	\$5.56	\$4.93
Mean first activity time	0.866 (0.0041)	0.781 (0.0067)

Table EC.2 Summary of Auction Sample with wb $>$ \$100, bp/sp $\geq$ 1.25, and wb/bp $\geq$ 0.8

	No buyout price	Buyout price (not exercised)
Number of auctions	194	47
Mean winning bid	\$180.39	\$266.07
Mean bid time	0.7843 (0.01348)	0.8146 (0.0187)

the winning bid to the buyout price is high enough ($wb/sp \geq 0.8$); and as before, we ignore auctions where the buyout price is too close to the starting price—in particular we require bp/sp $\geq$ 1.25.

The median winning bid in the complete Yahoo! data (including auctions where the buyout option is exercised) is \$152.51, as compared to \$5.95 for eBay data, which presumably happens because the search for the string "iPod" on Yahoo! leads to more auctions where iPod's are sold as opposed to some of its cheaper accessories. Indeed it turns out that almost 95% of buyout price auctions satisfying the above criterion (buyout option not exercised, wb/sp $\geq$ 0.8, bp/sp $\geq$ 1.25) have a winning bid greater than \$100 and hence we only consider auctions where the winning bid is greater than \$100. The important statistics of the restricted data set are summarized in Table EC.2.

Observe that the average bid time in buyout-price auctions is higher than in auctions without a buyout price. However a two sample t-test returns a p-value of 0.0963, which suggests that although Hypothesis 2 in §EC.7 can be accepted based on this data set; it is with a much lower confidence level than the earlier case. A separate analysis reveals that similar to eBay data the average bid time usually decreases as the winning bid increases. Also observe that for this data set, the average winning bid is higher in buyout-price auctions, and hence the average bid time would further increase for buyout price auctions (or alternately, decrease for auctions without a buyout price) if the data were adjusted to ensure that the average winning bid is same for both cases.

One of the main reasons the result in this case is not as conclusive as in the temporary case is because some Yahoo! auctions use a slightly different auction mechanism than the one assumed in this paper; in particular, whereas we assume a fixed auction end time, some Yahoo! auctions have a floating deadline that extends if a bid is placed near the end of the auction. The presence of a floating deadline may alter bidding strategy—in particular, the strategy "bid just near the end of the auction" cannot be played because whenever a bid is placed in the auction, the deadline is automatically extended; see also Roth and Ockenfels (2002) who empirically test the effect of a floating deadline on bid times. The primary motivation of bidding late in an auction with a buyout price is to prevent bidders from utilizing the information provided by one's bid; however, in an auction where the deadline extends when a bid is placed, other bidders always get some time to respond to a bid irrespective of when it is placed, thus decreasing the incentive of bidding late in the auction. Furthermore, as discussed before, our data is biased towards auctions with a buyout price that is higher than what bidders expect to pay for getting the product. As mentioned above, an important reason why bidders bid late in a permanent buyout-price auction is that information about a bidder's valuation may cause another bidder to exercise the buyout option; however a very highly priced buyout option has a small probability of exercise and hence decreases the incentive of bidding late.

In summary, bidding data from eBay and Yahoo! suggests that the hypotheses advocated by equilibrium strategies $\mathcal{T}[\nu_{\text{tmp}}]$ and $\mathcal{P}[\nu_{\text{prm}}]$ for the temporary and permanent case respectively can be accepted, albeit with a lower degree of confidence for a permanent option than a temporary option. This implies that bidders do seem to follow strategies $\mathcal{T}[\nu_{\text{tmp}}]$ and $\mathcal{P}[\nu_{\text{prm}}]$ in temporary and permanent buyout-price auctions respectively, although, as mentioned before, this simple empirical analysis does ignore other factors that impact bidder behavior. An extensive study of these factors, however, is beyond the scope of this paper.

EC.8. Derivation of Results Shown in Table 2

We will use the following lemma to derive the results in Table 2.

Lemma 11. *Consider a continuous function* $g(p,\lambda)$: $[\underline{v},\bar{v}]\times[0,\infty)\to[0,\infty)$ *and let* $p^*(\lambda)=\arg\max_{p\in X} g(p,\lambda)$. *Suppose* $g(p,\lambda)$ *is such that* $\lim_{\lambda\to\infty} g(\lambda,p)=g(p)$ *(in the sup norm) where* $g(p)$ *is also continuous on the set* $[\underline{v},\bar{v}]$. *Then if* $g(\cdot)$ *has a unique maximizer* $p^*=\arg\max_{p\in X} g(p)$, $\lim_{\lambda\to\infty} p^*(\lambda)=p^*$.

Proof. We show that for any $\epsilon > 0$ there exists Λ, δ such that for all $\lambda > \Lambda$,

$$g(p^*, \lambda) - g(p, \lambda) > \delta \quad \forall p \notin (p^* - \epsilon, p^* + \epsilon),$$

hence proving that $|p^*(\lambda) - p^*| < \epsilon$ for all $\lambda > \Lambda$.

Consider an $\epsilon > 0$. We have

$$\begin{aligned} g(p^*, \lambda) - g(p, \lambda) &= g(p^*, \lambda) - g(p^*) + g(p) - g(p, \lambda) + g(p^*) - g(p) \\ &\geq -|g(p^*, \lambda) - g(p^*)| - |g(p, \lambda) - g(p)| + g(p^*) - g(p). \end{aligned} \tag{EC84}$$

Now because $g(p)$ is continuous and p^* is the unique maximizer, there exists $\delta_1 > 0$ such that

$$g(p^*) - g(p) > \delta_1 \quad \forall p \notin (p^* - \epsilon, p^* + \epsilon). \tag{EC85}$$

Also because $\lim_{\lambda\to\infty} g(\lambda, p) \to g(p)$ in the sup norm, $\exists$ a $\Lambda > 0$ such that $\forall \lambda > \Lambda$

$$|g(p^*, \lambda) - g(p)| < \delta_1/3 \quad \forall p \in [\underline{v}, \bar{v}]. \tag{EC86}$$

Substituting (EC85) and (EC86) in (EC84), we obtain that for any $p \notin (p^* - \epsilon, p^* + \epsilon)$

$$g(p^*, \lambda) - g(p, \lambda) > -\delta_1/3 - \delta_1/3 + \delta_1 = \delta_1/3.$$

Setting $\delta = \delta_1/3$ completes the proof. □

We first analyze the temporary case. Consider a market environment where the bidder arrival rate is high ($\lambda \to \infty$). We derive the result for the case when seller sensitivity is high ($\alpha \to \infty$). The result for the other case ($\alpha \to 0$) is similarly obtained.

The case ($\lambda \to \infty$, $\alpha \to 0$) corresponds to a regime where $\alpha = f(\lambda)$ and where f is any function such that $\lim_{\lambda\to\infty} f(\lambda) = \infty$. First, note that we have

$$\begin{aligned} \lim_{\lambda\to\infty} \nu_{\text{tmp}}(p, t) &= p, \\ \lim_{\lambda\to\infty} \mathbf{E}_t\big[\max\big(\underline{v}, v^{(2)}_{N(t,T)+1}\big) \mid v_1 \leq \nu_{\text{tmp}}(p, t)\big] &= \bar{v}. \end{aligned}$$

Using the above and the fact that $\alpha = f(\lambda)$, we get that

$$\begin{aligned} \lim_{\lambda\to\infty} \frac{\mathbf{E}[U^S_{\text{tmp}}(p)]}{\mathbf{E}[U^S_{\text{tmp}}(p_1)]} &= \lim_{\lambda\to\infty} \frac{\Big(\int_0^T e^{-f(\lambda)T} \bar{v} F(p) \lambda e^{-\lambda t}\, dt + \int_0^T e^{-f(\lambda)t} p(1 - F(p)) \lambda e^{-\lambda t}\, dt\Big)}{\Big(\int_0^T e^{-f(\lambda)T} \bar{v} F(p_1) \lambda e^{-\lambda t}\, dt + \int_0^T e^{-f(\lambda)t} p_1(1 - F(p_1)) \lambda e^{-\lambda t}\, dt\Big)} \\ &= \frac{p(1 - F(p)}{p_1(1 - F(p_1)}. \end{aligned}$$

By definition of p_1 the function $g_1(p) = p(1 - F(p))/p_1(1 - F(p_1))$ is uniquely maximized at p_1 and this implies, by Lemma 11, that in the case $\alpha = f(\lambda)$, $\lim_{\lambda\to\infty} p^*_{\text{tmp}} = p_1$.

Next, consider a market environment where the bidder arrival rate is small and bidders have high time sensitivity ($\lambda \to 0$, $\beta \to \infty$). We will derive the result for the case when $\alpha \to 0$; the optimal buyout price when the seller has high time sensitivity ($\alpha \to \infty$) follows similarly. The case ($\lambda \to 0$, $\beta \to \infty$, $\alpha \to 0$) corresponds to a regime where $\beta = 1/f_2(\lambda)$, $\alpha = f_1(\lambda)$ and $\lambda \to 0$; the functions f_1 and f_2 are such that $\lim_{\lambda\to 0} f_i = 0$, $i = 1, 2$. Note that because $\beta \to \infty$, the threshold valuation $\nu_{\text{tmp}}(p, t) \to p$, and in addition because the bidder arrival rate $\lambda \to 0$, we have $\mathbf{E}_t[\max(\underline{v}, v^{(2)}_{N(t,T)+1}) \mid v_1 \leq \nu_{\text{tmp}}(p, t)] \to \underline{v}$. Consider the following ratio

$$\begin{aligned} \lim_{\lambda\to 0} \frac{\mathbf{E}[U^S_{\text{tmp}}(p)]}{\mathbf{E}[U^S_{\text{tmp}}(p_2)]} &= \lim_{\lambda\to 0} \frac{\int_0^T (e^{-f_1(\lambda)T} \underline{v} F(p) + e^{-f_1(\lambda)t} p(1 - F(p))) \lambda e^{-\lambda t}\, dt}{\int_0^T (e^{-f_1(\lambda)T} \underline{v} F(p_2) + e^{-f_1(\lambda)t} p_2(1 - F(p_2))) \lambda e^{-\lambda t}\, dt} \\ &= \frac{\underline{v} F(p) + p(1 - F(p))}{\underline{v} F(p_2) + p_2(1 - F(p_2))}. \end{aligned}$$

By definition p_2 uniquely maximizes the function $g_2(p) = p(1 - F(p)) + \underline{v}F(p)$ and hence it is also the unique maximizer of the function $g_2(p)/g_2(p_2)$. Thus, using Lemma 11, in the case $\beta = 1/f_2(\lambda)$ and $\alpha = f_1(\lambda)$, the optimal buyout price converges to p_2 in the limit $\lambda \to 0$.

Now, consider the regime $\alpha = f_1(\lambda)$, $\beta = f_2(\lambda)$ and $\lambda \to 0$, i.e., the case when $\alpha \to 0$, $\beta \to 0$, $\lambda \to 0$. Note that $\lim_{\lambda\to 0} \mathbf{E}_t[\max(\underline{v}, v^{(2)}_{N(t,T)+1}) \mid v_1 \le \nu_{\text{tmp}}(p,t)] = \underline{v}$; in addition for any $p > \underline{v}$ the threshold valuation $\lim_{\lambda\to 0} \nu_{\text{tmp}}(p,t) = \bar{v}$ for $t \in [0,T]$ while $\lim_{\lambda\to 0} \nu_{\text{tmp}}(p,t) = \underline{v}$ for $t \in [0,T]$ if $p = \underline{v}$. This implies that for any $p > \underline{v}$

$$\begin{aligned}
\lim_{\lambda\to 0} \frac{\mathbf{E}[U^S_{\text{tmp}}(p,\lambda)]}{\mathbf{E}[U^S_{\text{tmp}}(\underline{v},\lambda)]} &= \lim_{\lambda\to 0} \frac{\int_0^T e^{-f_1(\lambda)T}\underline{v}\lambda e^{-\lambda t}\,dt}{\int_0^T e^{-f_1(\lambda)t}2\bar{v}\lambda e^{-\lambda t}\,dt} \\
&= \lim_{\lambda\to 0} e^{-f_1(\lambda)T}\frac{(1-e^{-\lambda T})}{\lambda}\frac{\lambda + f_1(\lambda)}{(1-e^{-(\lambda+f_1(\lambda))t})} \\
&= 1, \qquad \text{(EC87)}
\end{aligned}$$

where the dependence of the seller utility on the bidder arrival rate λ is explicitly shown. Let $g_\lambda(p) = \mathbf{E}[U^S_{\text{tmp}}(p,\lambda)]/\mathbf{E}[U^S_{\text{tmp}}(\underline{v},\lambda)]$ and consider the family of functions $\mathcal{F} = \{g_\lambda;\ \lambda \ge 0\}$ defined on the set $E = [\underline{v}, \bar{v}]$. The above equality (EC87) shows that the sequence of functions g_λ converges pointwise to 1 or, more formally, to the constant function, which takes the value 1 over the entire set E. We next show that $\mathcal{F}$ is equicontinuous. Indeed for any $p', p'' \in E$, we have from (8)

$$\begin{aligned}
|g_\lambda(p') - g_\lambda(p'')| &= \Bigg| \frac{\int_0^T e^{-\alpha T}(\mathbf{E}_t[\cdot \mid v_1 \le \nu_{\text{tmp}}(p',t)]F(\nu_{\text{tmp}}(p',t)) - \mathbf{E}_t[\cdot \mid v_1 \le \nu_{\text{tmp}}(p'',t)]F(\nu_{\text{tmp}}(p'',t)))\lambda e^{-\lambda t}\,dt}{\underline{v}\int_0^T \lambda e^{-(\lambda+\alpha)t}\,dt} \\
&\quad + \frac{\int_0^T e^{-\alpha t}(p'(1-F(\nu_{\text{tmp}}(p',t))) - p''(1-F(\nu_{\text{tmp}}(p'',t))))\lambda e^{-\lambda t}\,dt}{\underline{v}\int_0^T \lambda e^{-(\lambda+\alpha)t}\,dt} \Bigg| \\
&\le \frac{|p'-p''| + \bar{v}|F(p') - F(p'')|}{\underline{v}} + \frac{2|p'-p''| + \bar{v}|F(p') - F(p'')|}{\underline{v}} \\
&= \frac{3|p'-p''| + 2\bar{v}|F(p') - F(p'')|}{\underline{v}},
\end{aligned}$$

where for the sake of brevity we use $\mathbf{E}_t[\cdot \mid v_1 \le \nu_{\text{tmp}}(p',t)]$ to denote $\mathbf{E}_t[\max(\underline{v}, v^{(2)}_{N(t,T)+1}) \mid v_1 \le \nu_{\text{tmp}}(p,t)]$. The inequality in the above expression follows because

$$|\mathbf{E}_t[\cdot \mid v_1 \le \nu_{\text{tmp}}(p',t)] - \mathbf{E}_t[\cdot \mid v_1 \le \nu_{\text{tmp}}(p'',t)]| \le |p' - p''|.$$

Now because $F(\cdot)$ is assumed to be continuous there exists a $\delta_1 > 0$ such that $|F(p') - F(p'')| < \epsilon\underline{v}/(4\bar{v})$ if $|p' - p''| < \delta_1$. We thus have that for any $\epsilon > 0$ if we choose $\delta = \min(\delta_1, \epsilon\underline{v}/6)$ then

$$|g_\lambda(p') - g_\lambda(p'')| < \epsilon$$

whenever $|p' - p''| < \delta$, $p', p'' \in E$, and $g_\lambda \in \mathcal{F}$ thus proving that $\mathcal{F} = \{g_\lambda;\ \lambda \ge 0\}$ is equicontinuous. We next use the following result.

Lemma 12 (Rudin 1976). *An equicontinuous sequence of functions* $\{f_n\}$ *on a compact set K that converges pointwise on K also converge uniformly on K.*

Because $\mathcal{F} = \{g_\lambda;\ \lambda \ge 0\}$ is equicontinuous and the sequence g_λ converges pointwise on E to 1 the above result implies that the sequence of functions g_λ also converges uniformly to 1 on the set E. Thus for any $\epsilon > 0$ there exists a $\Lambda > 0$ such that for $\lambda < \Lambda$

$$|g_\lambda(p) - 1| < \epsilon$$

for all $p \in E$. This implies that for $\lambda < \Lambda$

$$\left|\max_p g_\lambda(p) - 1\right| \le \max_p |g_\lambda(p) - 1| < \epsilon,$$

which proves that $\lim_{\lambda\to 0}(\max_p \mathbf{E}[U^S_{\text{tmp}}(p,\lambda)]/\mathbf{E}[U^S_{\text{tmp}}(\underline{v},\lambda)]) = 1$. The same analysis could essentially be repeated to show that $\lim_{\lambda\to 0}(\max_p \mathbf{E}[U^S_{\text{tmp}}(p,\lambda)]/\mathbf{E}[U^S_{\text{tmp}}(p',\lambda)]) = 1$ for any $p' \in [\underline{v}, \bar{v}]$.

Finally, consider the regime when $(\alpha = 1/f_1(\lambda),\ \beta = f_2(\lambda),\ \lambda \to 0)$ corresponding to the case $(\alpha\to\infty,\ \beta\to 0,\ \lambda\to 0)$. Lemma 11 is not applicable in this case and so we prove the convergence of the optimal solution using basic principles. Indeed, we show that for any $\epsilon > 0$ there exists a $\Lambda > 0$ such that for all $\lambda < 1/\Lambda$, $|p^*_{\text{tmp}}(\lambda) - \underline{v}| < \epsilon$.

Consider an $\epsilon > 0$. Recall that $\nu_{\text{tmp}}(p,t) = \min(\hat{v}(p,t), \bar{v})$ where $\hat{v}(p,t)$ is the solution of (2). Note that when $\beta = f_2(\lambda)$, $\lim_{\lambda\to 0}\hat{v}(p,t) = \infty$, $\forall\, t\in[0,T)$ if $p > \underline{v}$. Thus, there exists a $\Lambda_1 > 0$ such that for all $\lambda < 1/\Lambda_1$ we have $\hat{v}(p,t) \geq \bar{v}$, $\forall\, t\in[0,T)$, $p \geq \underline{v}+\epsilon$. This implies that $\nu_{\text{tmp}}(p,t) = \bar{v}$, $\forall\, t\in[0,T)$, $p\geq \underline{v}+\epsilon$.

For $\lambda < 1/\Lambda_1$, and any $p \geq \underline{v}+\epsilon$, consider the ratio

$$\frac{\mathbf{E}^S_{\text{tmp}}(p)}{\mathbf{E}^S_{\text{tmp}}(\underline{v})} = \frac{\int_0^T e^{-\alpha T}\mathbf{E}_t[\max(\underline{v}, v^{(2)}_{N(t,T)+1}) \mid v_1 \leq \nu_{\text{tmp}}(p,t)]\lambda e^{-\lambda t}\,dt}{\int_0^T e^{-\alpha t}\underline{v}\lambda e^{-\lambda t}\,dt}$$

$$< \frac{\bar{v}\int_0^T e^{-\alpha T}e^{-\lambda t}\,dt}{\underline{v}\int_0^T e^{-\alpha t}e^{-\lambda t}\,dt} = \frac{\bar{v}}{\underline{v}}e^{-\alpha T} + \frac{\bar{v}}{\underline{v}\lambda}\alpha e^{-\alpha T}.$$

Now because $\alpha = 1/f_1(\lambda)$ and $\lim_{\lambda\to 0} f_1(\lambda) = 0$, we obtain that $\alpha\to\infty$ as $\lambda\to 0$. Also note that $\lim_{x\to\infty} xe^{-x} = 0$ and thus $\exists\Lambda_2 > 0$ such that for all $\lambda < 1/\Lambda_2$

$$\frac{\bar{v}}{\underline{v}}e^{-\alpha T} < 1/2 \qquad \text{and} \qquad \frac{\bar{v}}{\underline{v}\lambda}\alpha e^{-\alpha T} < 1/2.$$

Thus for all $\lambda < \min(1/\Lambda_1, 1/\Lambda_2)$ we obtain that

$$\frac{\mathbf{E}^S_{\text{tmp}}(p)}{\mathbf{E}^S_{\text{tmp}}(\underline{v})} < 1 \quad \forall\, p \geq \underline{v}+\epsilon$$

hence proving that $p^*_{\text{tmp}}(\lambda) \in [\underline{v}, \underline{v}+\epsilon)$. Setting $\Lambda = 1/\min(1/\Lambda_1, 1/\Lambda_2)$ completes the proof.

We next consider a permanent buyout option. Consider for any $\mu > 0$, the regime $\lambda\to\infty$ and $\alpha = f(\lambda)$ where f is such that $\lim_{\lambda\to\infty} f(\lambda)/\lambda = \mu$.

We have that

$$\lim_{\lambda\to\infty} \nu_{\text{prm}}(p,t) = p, \quad \forall\, t\in[0,T]$$

$$\lim_{\lambda\to\infty} \mathbf{E}[\mathbf{1}_{\{N_{\text{bid}}(T)>0\}}\max(\underline{v}, v^{(2)}_{N_{\text{bid}}(T)}) \mid v_i \leq p,\ \forall\, i] = p.$$

Using the above, the seller's revenue can be written as

$$\begin{aligned}\lim_{\lambda\to\infty}\mathbf{E}[U^S_{\text{prm}}(p)] &= \lim_{\lambda\to\infty}\int_0^T e^{-f(\lambda)t}p\lambda(1-F(p))e^{-\lambda(1-F(p))t}\\ &= \lim_{\lambda\to\infty} p(1-F(p))\frac{\lambda}{\lambda(1-F(p))+f(\lambda)}(1-e^{-(\lambda(1-F(p))+f(\lambda))T})\\ &= \frac{p(1-F(p))}{1-F(p)+\mu}.\end{aligned}$$

The function $g(p) = p(1-F(p))/(1-F(p)+\mu)$ is uniquely maximized at $p_3(\mu)$, which proves that $\lim_{\lambda\to\infty} p^*_{\text{prm}} = p_3(\mu)$.

Next, consider the case when $\alpha,\beta,\lambda\to 0$. Note that $\lim_{\lambda\to 0}\mathbf{E}[\mathbf{1}_{\{N_{\text{bid}}(T)>0\}}\max(\underline{v}, v^{(2)}_{N_{\text{bid}}(T)}) \mid v_i \leq \nu_{\text{prm}}(p,t_i)\ \forall\, i] = \underline{v}\lambda(\int_0^T F(\nu_{\text{prm}}(p,t))\,dt)e^{-\lambda\int_0^T F(\nu_{\text{prm}}(p,t))\,dt}$; in addition for any $p > \underline{v}$ the threshold valuation $\lim_{\lambda\to 0}\nu_{\text{prm}}(p,t) = \bar{v}$ for $t\in[0,T]$ while $\lim_{\lambda\to 0}\nu_{\text{prm}}(p,t) = \underline{v}$ for $t\in[0,T]$ if $p = \underline{v}$. This implies that for any $p > \underline{v}$

$$\begin{aligned}\lim_{\lambda\to 0}\frac{\mathbf{E}[U^S_{\text{prm}}(p,\lambda)]}{\mathbf{E}[U^S_{\text{prm}}(\underline{v},\lambda)]} &= \lim_{\lambda\to 0}\frac{e^{-(f_1(\lambda)+\lambda)T}\underline{v}\lambda T}{\int_0^T e^{-f_1(\lambda)t}\underline{v}\lambda e^{-\lambda t}\,dt}\\ &= \lim_{\lambda\to 0} e^{-(f_1(\lambda)+\lambda)T}\frac{(\lambda+f_1(\lambda))T}{(1-e^{-(\lambda+f_1(\lambda))T})}\\ &= 1.\end{aligned} \tag{EC88}$$

As in the temporary case, let $g_\lambda(p) = \mathbf{E}[U^S_{\text{prm}}(p,\lambda)]/\mathbf{E}[U^S_{\text{prm}}(\underline{v},\lambda)]$ and consider the family of functions $\mathscr{F}_{\text{prm}} = \{g_\lambda;\ \Lambda \ge \lambda \ge 0\}$ defined on the set $E = [\underline{v}, \bar{v}]$ where $\Lambda \gg 0$. Equation (EC88) shows that the sequence of functions g_λ converges pointwise to 1. We next show that $\mathscr{F}_{\text{prm}}$ is equicontinuous. Indeed for any $p', p'' \in E$, we have from (9)

$$\begin{aligned}
&|g_\lambda(p') - g_\lambda(p'')|\\
&= \frac{1}{\int_0^T \underline{v}\lambda e^{-(\lambda+\alpha)t}\,dt}\bigg|\int_0^T e^{-\alpha t} p'\lambda(1-F(\nu_{\text{prm}}(p',t)))e^{-\lambda\int_0^t(1-F(\nu_{\text{prm}}(p',\tau)))\,d\tau}\,dt\\
&\qquad\qquad - \int_0^T e^{-\alpha t} p''\lambda(1-F(\nu_{\text{prm}}(p'',t)))e^{-\lambda\int_0^t(1-F(\nu_{\text{prm}}(p'',\tau)))\,d\tau}\,dt\bigg|\\
&\quad + \frac{1}{\int_0^T \underline{v}\lambda e^{-(\lambda+\alpha)t}\,dt}\Big|e^{-\lambda\int_0^T(1-F(\nu_{\text{prm}}(p',t)))\,dt}e^{-\alpha T}\mathbf{E}[\mathbf{1}_{\{N_{\text{bid}}(T)>0\}}\max(\underline{v}, v^{(2)}_{N_{\text{bid}}(T)}) \mid v_i \le \nu_{\text{prm}}(p',t_i)\ \forall i]\\
&\qquad\qquad - e^{-\lambda\int_0^T(1-F(\nu_{\text{prm}}(p'',t)))\,dt}e^{-\alpha T}\mathbf{E}[\mathbf{1}_{\{N_{\text{bid}}(T)>0\}}\max(\underline{v}, v^{(2)}_{N_{\text{bid}}(T)}) \mid v_i \le \nu_{\text{prm}}(p'',t_i)\ \forall i]\Big|\\
&\le e^{\Lambda T}\frac{|p'-p''| + (2\Lambda T+1)\bar{v}|F(\nu_{\text{prm}}(p',t_{\max})) - F(\nu_{\text{prm}}(p'',t_{\max}))|}{\underline{v}}\\
&\quad + e^{\Lambda T}\frac{|\mathbf{E}[\mathbf{1}_{\{N_{\text{bid}}(T)>0\}}\cdot \mid v_i \le \nu_{\text{prm}}(p',t_i)\ \forall i] - \mathbf{E}[\mathbf{1}_{\{N_{\text{bid}}(T)>0\}}\cdot \mid v_i \le \nu_{\text{prm}}(p'',t_i)\ \forall i]|}{\underline{v}\lambda T},
\end{aligned} \tag{EC89}$$

where

$$t_{\max} = \arg\max_{t\in[0,T]} |F(\nu_{\text{prm}}(p',t)) - F(\nu_{\text{prm}}(p'',t))|;$$

for the sake of brevity, we use the notation $\mathbf{E}[\mathbf{1}_{\{N_{\text{bid}}(T)>0\}}\cdot \mid v_i \le \nu_{\text{prm}}(p',t_i)\ \forall i]$ to denote

$$\mathbf{E}[\mathbf{1}_{\{N_{\text{bid}}(T)>0\}}\max(\underline{v}, v^{(2)}_{N_{\text{bid}}(T)}) \mid v_i \le \nu_{\text{prm}}(p'',t_i)\ \forall i].$$

The last term of (EC89) is nonzero only if a bidder of type (v,t) with valuation $v \in [\nu_{\text{prm}}(p',t), \nu_{\text{prm}}(p'',t)]$ arrives in the auction. Thus the term can be upper bounded as following

$$\begin{aligned}
&|\mathbf{E}[\mathbf{1}_{\{N_{\text{bid}}(T)>0\}}\cdot \mid v_i \le \nu_{\text{prm}}(p',t_i)\ \forall i] - \mathbf{E}[\mathbf{1}_{\{N_{\text{bid}}(T)>0\}}\cdot \mid v_i \le \nu_{\text{prm}}(p'',t_i)\ \forall i]|\\
&\quad \le \bar{v}\big(1 - e^{-\int_0^T \lambda|F(\nu_{\text{prm}}(p',t)) - F(\nu_{\text{prm}}(p'',t))|\,dt}\big)\\
&\quad \le \bar{v}\lambda T|F(\nu_{\text{prm}}(p',t_{\max})) - F(\nu_{\text{prm}}(p'',t_{\max}))|.
\end{aligned}$$

Using the above, we get

$$|g_\lambda(p') - g_\lambda(p'')| \le e^{\Lambda T}\frac{|p'-p''| + (2\Lambda T+2)\bar{v}|F(\nu_{\text{prm}}(p',t_{\max})) - F(\nu_{\text{prm}}(p'',t_{\max}))|}{\underline{v}}.$$

Now since $F(\cdot)$ and $\nu_{\text{prm}}(\cdot)$ are assumed to be continuous, there exists a $\delta_1 > 0$ such that $|F(\nu_{\text{tmp}}(p',t)) - F(\nu_{\text{tmp}}(p'',t))| < e^{-\Lambda T}(\epsilon\underline{v}/(2\bar{v}(2\Lambda T+2)))$ if $|p'-p''| < \delta_1$. We thus have that for any $\epsilon > 0$ if we choose $\delta = \min(\delta_1, e^{-\Lambda T}(\epsilon\underline{v}/2))$ then

$$|g_\lambda(p') - g_\lambda(p'')| < \epsilon$$

whenever $|p'-p''| < \delta$, $p', p'' \in E$ and $g_\lambda \in \mathscr{F}_{\text{prm}}$ thus proving that $\mathscr{F}_{\text{prm}} = \{g_\lambda;\ \Lambda \ge \lambda \ge 0\}$ is equicontinuous. The remaining proof is identical to that in the temporary case and hence omitted.

For all other regimes the limiting optimal permanent and temporary buyout prices are same and the derivation for the permanent case is similar to that in the temporary case. We thus omit that analysis.

EC.9. Lemma 13

Lemma 13. *Define* $p_1 = \arg\max_{p\in E} p(1-F(p))$, $p_2 = \arg\max_{p\in E}(p(1-F(p)) + \underline{v}F(p))$, *and* $p_3(\mu) = \arg\max_{p\in E}(p(1-F(p))/(\mu+1-F(p)))$ *where* $E = [\underline{v}, \bar{v}]$. *Then*

1. $p_2 \ge p_1$, $p_3(\mu) \ge p_1$ *for all* μ;
2. $p_3(\mu)$ *is decreasing in* μ *and* $\lim_{\mu\to 0} p_3(\mu) = \bar{v}$, $\lim_{\mu\to\infty} p_3(\mu) = p_1$.

Proof. 1. By the optimality of p_1 and p_2 we have

$$p_1(1-F(p_1)) \geq p_2(1-F(p_2)) \qquad \text{and} \qquad p_2(1-F(p_2)) + \underline{v}F(p_2) \geq p_1(1-F(p_1)) + \underline{v}F(p_1),$$

which together imply that $F(p_2) \geq F(p_1)$, i.e. $p_2 \geq p_1$. Similarly the optimality of $p_3(\mu)$ and p_1 imply that $p_3(\mu) \geq p_1$ for all μ.

1. Let $\mu_1 > \mu_2 > 0$ and $p_3(\mu_1)$ and $p_3(\mu_2)$ be the corresponding optimal buyout prices. By the optimality of $p_3(\mu_1)$ and $p_3(\mu_2)$ we have that

$$\frac{p_3(\mu_1)(1-F(p_3(\mu_1)))}{\mu_1+1-F(p_3(\mu_1))} > \frac{p_3(\mu_2)(1-F(p_3(\mu_2)))}{\mu_1+1-F(p_3(\mu_2))} \qquad \text{and}$$
$$\frac{p_3(\mu_2)(1-F(p_3(\mu_2)))}{\mu_2+1-F(p_3(\mu_2))} > \frac{p_3(\mu_1)(1-F(p_3(\mu_1)))}{\mu_2+1-F(p_3(\mu_1))}.$$

Multiplying the above two inequalities yields, on simplification, that

$$(1+\mu_1)(F(p_3(\mu_2)) - F(p_3(\mu_1))) > (1+\mu_2)(F(p_3(\mu_2)) - F(p_3(\mu_1))). \tag{EC90}$$

Because $\mu_1 > \mu_2$, (EC90) implies that $F(p_3(\mu_2)) > F(p_3(\mu_1))$ thus proving that $p_3(\mu)$ is decreasing in μ.

Now, the function $p_3(\mu)$ as $\mu \downarrow 0^+$ is a monotonic bounded increasing function with an upper bound of $\bar{v}$. We next show that $\bar{v}$ is indeed the lowest upper bound thus proving that $\lim_{\mu\to 0} p_3(\mu) = \bar{v}$.

Consider an $\epsilon > 0$, we show that there exists a $\delta > 0$ such that for all $\mu < \delta$, $p_3(\mu) > \bar{v} - \epsilon$ thus proving that $\bar{v} - \epsilon$ is not an upper bound of the function $p_3(\mu)$.

Indeed, define $p_3^{(\epsilon)}(\mu) = \arg\max_{[\underline{v},\, \bar{v}-\epsilon/3]}(p(1-F(p))/(\mu+1-F(p)))$ and note that for $p \in [\underline{v}, \bar{v}-\epsilon/3]$,

$$\lim_{\mu\to 0} \frac{p(1-F(p))}{\mu+1-F(p)} = g(p) = p.$$

Clearly, $g(p)$ on the interval $[\underline{v}, \bar{v}-\epsilon/3]$ is maximized at $\bar{v}-\epsilon/3$, which, using Lemma 11, shows that $\lim_{\mu\to 0} p_3^{(\epsilon)}(\mu) \to \bar{v}-\epsilon/3$. This then implies that there exists a $\delta > 0$ such that for $\mu < \delta$, $p_3^{(\epsilon)}(\mu) > \bar{v}-\epsilon/2$. Now, recall that $p_3(\mu) = \arg\max_{p\in[\underline{v},\, \bar{v}]}(p(1-F(p))/(\mu+1-F(p)))$ and thus $p_3(\mu) \geq p_3^{(\epsilon)}(\mu)$. Thus for $\mu < \delta$, $p_3(\mu) > \bar{v}-\epsilon/2$ thus proving that $\bar{v}-\epsilon$ is not an upper bound. This proves that $\lim_{\mu\to 0} p_3(\mu) = \bar{v}$.

Consider next for any $\mu > 0$ the ratio

$$g_\mu(p) = \frac{p(1-F(p))/(\mu+1-F(p))}{p_1(1-F(p_1))/(\mu+1-F(p_1))}.$$

Then $\lim_{mu\to\infty} g_\mu(p) = g(p) = p(1-F(p))/p_1(1-F(p_1))$. It follows from the definition of p_1 that the function $g(p)$ is uniquely maximized at p_1, which, using Lemma 11, proves that $\lim_{\mu\to\infty} p_3(\mu) = p_1$. □

EC.10. Proof of Theorem 5

Consider any bidder A with type (v, t). We show that for any continuous nondecreasing threshold function ν, bidder A's best response strategy to $\mathcal{T}[\nu]$ is same as $\mathcal{T}[\nu]$ if

$$\begin{aligned} p(t) &= \nu(t) - \mathbf{E}[U_{\text{bid}(t)}(\nu(t), t) \mid N_t = 0] \\ &= \nu(t) - e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^{\nu(t)} e^{\lambda(T-t)F(x)}\, dx \end{aligned}$$

for all $t \in [0, T]$. This proves that $\mathcal{T}[\nu]$ is a Bayesian Nash equilibrium for the auction with temporary buyout-price trajectory $[p(t)]_{t\in[0,\,T]}$ defined above.

If A is not the first bidder, the first bidder (following strategy $\mathcal{T}[\nu]$) would have either bid in the auction or exercised the buyout option immediately and hence the buyout option is not available to bidder A. In that case, the auction progresses as a standard second-price auction and A's weakly dominant strategy is to bid his true valuation.

If A is the first bidder and the buyout option is available to him then he can either act immediately—exercise the buyout option or place a bid—or wait in the auction. The following lemma shows that A cannot increase his utility by waiting.

Lemma 14. *When other bidders follow strategy $\mathcal{T}[\nu]$ in an auction with a temporary dynamic buyout price, the first bidder is weakly better off acting immediately; i.e., the utility from acting immediately is at least as much as from waiting, if and only if the threshold valuation $\nu(t)$ is nondecreasing in t for all $t \in [0, T)$.*

Proof. Suppose the first bidder is of type (v, t). If the bidder waits up to time τ ($\tau > t$), his expected utility, using the notation in the proof of Theorem 1, is

$$\mathbf{E}[U_{wait(t,\tau)}(v,t)\mid N_t=0]=e^{-\beta(\tau-t)}(\max\{B_1(v,\tau),v-p(\tau)\}\cdot P(\mathscr{E})+\mathbf{E}[U_{bid(\tau)}(v,\tau)\mid N_t=0,\bar{\mathscr{E}}]\cdot P(\bar{\mathscr{E}})) \quad \text{(EC91)}$$

Then using (EC14), we get

$$\mathbf{E}[U_{wait(t,\tau)}(v,t)\mid N_t=0]\le e^{-\beta(\tau-t)}(\max\{v-p(\tau)-\mathbf{E}[U_{\text{bid}(\tau)}(v,\tau)\mid N_\tau=0],0\}\cdot P(\mathscr{E}))+B_1(v,t). \quad \text{(EC92)}$$

Now the first bidder of type (v, t) makes a decision immediately if she cannot gain by waiting, i.e., if

$$\max\{v-p(t),B_1(v,t)\}\ge \mathbf{E}[U_{\text{wait}(t,\tau)}(v,t)\mid N_t=0] \quad \forall\,\tau>t. \quad \text{(EC93)}$$

Indeed for the result of the lemma to hold in general, this condition must be true for all $v \in [\underline{v}, \bar{v}]$ and $t \in [0, T]$.

We enforce the following constraint, for all $v \in [\underline{v}, \bar{v}]$; $t, \tau \in [0, T]$ and $\tau > t$

$$\max\{v-p(t),B_1(v,t)\}\ge e^{-\beta(\tau-t)}(\max\{v-p(\tau)-\mathbf{E}[U_{\text{bid}(\tau)}(v,\tau)\mid N_\tau=0],0\}\cdot P(\mathscr{E}))+B_1(v,t),$$

which can be rewritten as

$$\max\{v-P(t)-\mathbf{E}[U_{\text{bid}(t)}(v,t)\mid N_t=0],0\}\ge e^{-\beta(\tau-t)}(\max\{v-p(\tau)-\mathbf{E}[U_{\text{bid}(\tau)}(v,\tau)\mid N_\tau=0],0\}P(\mathscr{E})), \quad \text{(EC94)}$$

where by definition $B_1(v,t) = \mathbf{E}[U_{\text{bid}(t)}(v,t) \mid N_t = 0]$. By (EC92), the constraint (EC94) implies the condition (EC93) and thus if (EC94) holds then the first bidder is weakly better off acting immediately.

For an arbitrary t and τ ($\tau > t$), consider the following two cases:

1. $v \le \nu(t)$: In this case bidding is more attractive to the first bidder at time t, i.e., we have $v-p(t)-\mathbf{E}[U_{\text{bid}(t)}(v,t) \mid N_t = 0] \le 0$. Thus the constraint (EC94) becomes

$$\begin{aligned}0&=\max\{v-p(t)-\mathbf{E}[U_{\text{bid}(t)}(v,t)\mid N_t=0],0\}\\&\ge e^{-\beta(\tau-t)}(\max\{v-p(\tau)-\mathbf{E}[U_{\text{bid}(\tau)}(v,\tau)\mid N_\tau=0],0\}P(\mathscr{E})),\end{aligned}$$

which holds if

$$v-p(\tau)-\mathbf{E}[U_{\text{bid}(\tau)}(v,\tau)\mid N_\tau=0]\le 0 \quad \forall\,v\le\nu(t). \quad \text{(EC95)}$$

Now note that $v-p(\tau)-\mathbf{E}[U_{\text{bid}(\tau)}(v,\tau) \mid N_\tau = 0]$ is increasing in valuation v because

$$\frac{\partial}{\partial v}(v-p(\tau)-\mathbf{E}[U_{\text{bid}(\tau)}(v,\tau)\mid N_\tau=0])=1-e^{-(\lambda(1-F(v))+\beta)(T-\tau)}\ge 0$$

and thus it is sufficient to impose the condition (EC95) at $v = \nu(t)$. This gives

$$\nu(t)-p(\tau)-\mathbf{E}[U_{\text{bid}(\tau)}(\nu(t),\tau)\mid N_\tau=0]\le 0.$$

Substituting $\nu(t) = p(t) + \mathbf{E}[U_{\text{bid}(t)}(\nu(t), t) \mid N_t = 0]$, the condition can be rewritten as

$$p(\tau)-p(t)\ge \mathbf{E}[U_{\text{bid}(t)}(\nu(t),t)\mid N_t=0]-\mathbf{E}[U_{\text{bid}(\tau)}(\nu(t),\tau)\mid N_\tau=0] \quad \text{(EC96)}$$

2. $v > \nu(t)$: In this case $v-p(t)-\mathbf{E}[U_{\text{bid}(t)}(v,t) \mid N_t = 0] > 0$. Thus (EC94) becomes

$$v-p(t)-\mathbf{E}[U_{\text{bid}(t)}(v,t)\mid N_t=0]\ge e^{-\beta(T-t)}P(\mathscr{E})(v-p(\tau)-\mathbf{E}[U_{\text{bid}(\tau)}(v,\tau)\mid N_\tau=0])P(\mathscr{E}) \quad \forall\,v>\nu(t).$$

Using the fact that $P(\mathscr{E})$, the probability of no arrival in the interval (t, τ), is $e^{-\lambda(\tau-t)}$ the above condition can be expressed as, for all $v > \nu(t)$,

$$p(\tau)-e^{(\lambda+\beta)(\tau-t)}p(t)\ge e^{(\lambda+\beta)(\tau-t)}\mathbf{E}[U_{\text{bid}(t)}(v,t)\mid N_t=0]-\mathbf{E}[U_{\text{bid}(\tau)}(v,\tau)\mid N_\tau=0]+v(1-e^{(\lambda+\beta)(\tau-t)}),$$

which is equivalent to

$$p(\tau)-e^{(\lambda+\beta)(\tau-t)}p(t) \geq \sup_{v>\nu(t)} (e^{(\lambda+\beta)(\tau-t)}\,\mathbf{E}[U_{\text{bid}(t)}(v,t)\mid N_t=0]-\mathbf{E}[U_{\text{bid}(\tau)}(v,\tau)\mid N_\tau=0]+v(1-e^{(\lambda+\beta)(\tau-t)}))$$
$$= e^{(\lambda+\beta)(\tau-t)}\,\mathbf{E}[U_{Bid}(\nu(t),t)]-\mathbf{E}[U_{Bid}(\nu(t),\tau)]+\nu(t)(1-e^{(\lambda+\beta)(\tau-t)}), \qquad \text{(EC97)}$$

where the equality follows from the fact that the supremum of the above expression occurs at $v=\nu(t)$. To see this notice that for all $v\in[\underline{v},\bar{v}]$

$$\begin{aligned}&\frac{\partial}{\partial v}(e^{(\lambda+\beta)(T-t)}\,\mathbf{E}[U_{\text{bid}(t)}(v,t)\mid N_t=0]-\mathbf{E}[U_{\text{bid}(\tau)}(v,\tau)\mid N_\tau=0]+v(1-e^{(\lambda+\beta)(\tau-t)}))\\
&\quad= e^{-(\lambda+\beta)(T-\tau)}(e^{\lambda(T-t)F(v)}-e^{\lambda(T-\tau)F(v)})+1-e^{(\lambda+\beta)(\tau-t)}\\
&\quad= e^{-(\lambda+\beta)(T-\tau)}e^{\lambda(T-\tau)F(v)}(e^{\lambda(\tau-t)F(v)}-1)+1-e^{(\lambda+\beta)(\tau-t)}\\
&\quad\leq (e^{(\lambda+\beta)(\tau-t)}-1)(e^{-(\lambda(1-F(v))+\beta)(T-\tau)}-1)\\
&\quad\leq 0,\end{aligned}$$

where the first inequality follows because $\lambda F(v)<\lambda+\beta$.

Substituting $\nu(t)=p(t)+\mathbf{E}[U_{\text{bid}(t)}(\nu(t),t)\mid N_t=0]$ in (EC97) we get

$$p(\tau)-p(t)\geq \mathbf{E}[U_{\text{bid}(t)}(\nu(t),t)\mid N_t=0]-\mathbf{E}[U_{\text{bid}(\tau)}(\nu(t),\tau)\mid N_\tau=0],$$

which is same as the condition (EC96) obtained in Case 1.

Therefore, the first bidder is weakly better off acting immediately if for all $t,\tau\in[0,T]$, $\tau>t$

$$p(\tau)-p(t)\geq \mathbf{E}[U_{\text{bid}(t)}(\nu(t),t)\mid N_t=0]-\mathbf{E}[U_{\text{bid}(\tau)}(\nu(t),\tau)\mid N_\tau=0]. \qquad \text{(EC98)}$$

Substituting $p(t)=\nu(t)-\mathbf{E}[U_{\text{bid}(t)}(\nu(t),t)\mid N_t=0]$ in (EC98) gives the condition

$$\nu(\tau)-\nu(t)-\mathbf{E}[U_{\text{bid}(\tau)}(\nu(\tau),\tau)\mid N_\tau=0]+\mathbf{E}[U_{\text{bid}(\tau)}(\nu(t),\tau)\mid N_\tau=0]\geq 0. \qquad \text{(EC99)}$$

for all $\tau,t\in[0,T]$, $\tau>t$. By setting $\tau=t+\Delta t$ $(\Delta t>0)$ we get that (EC99) holds if and only if for all $t\in[0,T)$

$$\lim_{\Delta t\to 0}(\nu(t+\Delta t)-\nu(t))\left(1-\frac{\partial}{\partial v}\mathbf{E}[U_{\text{bid}(t+\Delta t)}(v,t+\Delta t)\mid N_{t+\Delta t}=0]\Big|_{v=\nu(t)}\right)\geq 0. \qquad \text{(EC100)}$$

It can be easily shown that $(\partial/\partial v)\,\mathbf{E}[U_{Bid}(v,t)\mid N_t=0]|_{v=\nu(t)}<1$, $\forall\, v\in[\underline{v},+\infty)$, $t\in[0,T)$ and thus (EC100) holds if and only if $\nu(\cdot)$ is nondecreasing in t for all $t\in[0,T)$.

We have thus shown that if $\nu(t)$ is nondecreasing in t then the first bidder does not gain by waiting if the other bidders play the strategy $\mathcal{T}[\nu]$. We now prove the other direction, i.e., if the first bidder is weakly better off acting immediately then the threshold valuation $\nu(t)$ is nondecreasing in t.

Assume, for contradiction, that $\nu(t)$ is not nondecreasing in t and indeed there exists an interval $[t_1,t_2]\subset[0,T]$ such that $\nu(t)$ is decreasing over $[t_1,t_2]$. We now show that under this assumption there exists a case when the first bidder is strictly better off waiting in the auction. Indeed, suppose that the first bidder, say A, with type $(\nu(t_1),t_1)$ waits up to time $\tau=t_1+\epsilon$ (where $t_1<\tau<t_2$). Then his utility from the auction, as derived in (EC7), is

$$\begin{aligned}&\mathbf{E}[U_{wait(t_1,\tau)}(\nu(t_1),t)\mid N_{t_1}=0]\\
&\quad= e^{-\beta(\tau-t_1)}(\max\{B_1(\nu(t_1),\tau),\nu(t_1)-p(\tau)\}\cdot P(\mathcal{E})+\mathbf{E}[U_{bid(\tau)}(\nu(t_1),\tau)\mid N_{t_1}=0,\bar{\mathcal{E}}]\cdot P(\bar{\mathcal{E}})),\end{aligned} \qquad \text{(EC101)}$$

where, recall that the event $\mathcal{E}=\{N(t_1,\tau)=0\}$, $\bar{\mathcal{E}}$ denotes the complimentary event.

If the event $\mathscr{E}$ occurs then the buyout option is still available at time τ. Furthermore, because $\nu(t_1) > \nu(\tau)$, i.e., $\nu(t_1) - p(\tau) > B_1(\nu(t_1), \tau)$, bidder A will choose to exercise the buyout option at time τ.

Next, by defining $\mathscr{G}$ as the event that the first bidder, say B, arriving in (t_1, τ) with type (v_B, t_B) (where $t_B \in (t_1, \tau)$) has valuation $\nu(t_B) < v_B \le \nu(t_1)$, (EC101) can be rewritten as

$$\mathbf{E}[U_{wait(t_1,\tau)}(\nu(t_1), t) \mid N_{t_1} = 0] = e^{-\beta(\tau - t_1)}((\nu(t_1) - p(\tau)) \cdot P(\mathscr{E}) + \mathbf{E}[U_{bid(\tau)}(\nu(t_1), \tau) \mid N_{t_1} = 0, \bar{\mathscr{E}}, \mathscr{G}] \cdot P(\mathscr{G} \mid \bar{\mathscr{E}}) \cdot P(\bar{\mathscr{E}})$$
$$+ \mathbf{E}[U_{bid(\tau)}(\nu(t_1), \tau) \mid N_{t_1} = 0, \bar{\mathscr{E}}, \bar{\mathscr{G}}] \cdot P(\bar{\mathscr{G}} \mid \bar{\mathscr{E}}) \cdot P(\bar{\mathscr{E}})). \qquad \text{(EC102)}$$

where $\bar{\mathscr{G}}$ denotes the complimentary event.

Bidder A's utility from the auction if he acts immediately is $B_1(\nu(t_1), t_1)$, which can be rewritten as

$$B_1(\nu(t_1), t_1) = \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1) \mid N_{t_1} = 0, \mathscr{E}]P(\mathscr{E}) + \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1) \mid N_{t_1} = 0, \bar{\mathscr{E}}, \mathscr{G}]P(\mathscr{G} \mid \bar{\mathscr{E}})P(\bar{\mathscr{E}})$$
$$+ \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1) \mid N_{t_1} = 0, \bar{\mathscr{E}}, \bar{\mathscr{G}}]P(\bar{\mathscr{G}} \mid \bar{\mathscr{E}})P(\bar{\mathscr{E}}). \qquad \text{(EC103)}$$

Let Δ denote the difference in utility of bidder A if he waits up to time τ as opposed to acting immediately at t_1, i.e., $\Delta = \mathbf{E}[U_{wait(t_1,\tau)}(\nu(t_1), t_1) \mid N_{t_1} = 0] - B_1(\nu(t_1), t_1)$, then subtracting (EC103) from (EC102) we get, using (EC12) and (EC13)

$$\Delta = (e^{-\beta(\tau - t_1)}(\nu(t_1) - p(\tau)) - \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1) \mid N_{t_1} = 0, \mathscr{E}]) \cdot P(\mathscr{E})$$
$$- \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1) \mid N_{t_1} = 0, \bar{\mathscr{E}}, \mathscr{G}] \cdot P(\mathscr{G} \mid \bar{\mathscr{E}}) \cdot P(\bar{\mathscr{E}}). \qquad \text{(EC104)}$$

Now substituting $p(\tau) = \nu(\tau) - \mathbf{E}[U_{\mathrm{bid}(\tau)}(\nu(\tau), \tau) \mid N_\tau = 0]$, the first term of (EC104) becomes

$$e^{-\beta(\tau - t_1)}(\nu(t_1) - p(\tau)) - \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1) \mid N_{t_1} = 0, \mathscr{E}]$$
$$= e^{-\beta(\tau - t_1)}(\nu(t_1) - \nu(\tau)) + e^{-\beta(\tau - t_1)}\mathbf{E}[U_{\mathrm{bid}(\tau)}(\nu(\tau), \tau) \mid N_\tau = 0] - \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1) \mid N_{t_1} = 0, \mathscr{E}]$$
$$= e^{-\beta(\tau - t_1)}(\nu(t_1) - \nu(\tau) + \mathbf{E}[U_{\mathrm{bid}(\tau)}(\nu(\tau), \tau) \mid N_\tau = 0] - \mathbf{E}[U_{bid(\tau)}(\nu(t_1), \tau) \mid N_\tau = 0]), \qquad \text{(EC105)}$$

where the second equality follows because (EC12) and (EC6) imply that

$$\mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1) \mid N_{t_1} = 0, \mathscr{E}] = \mathbf{E}[U_{bid(\tau)}(\nu(t_1), t_1) \mid N_\tau = 0]$$
$$= e^{-\beta(\tau - t_1)}\mathbf{E}[U_{bid(\tau)}(\nu(t_1), \tau) \mid N_\tau = 0].$$

Thus using (EC105), for small ϵ, Δ can be written as

$$\Delta = \left(e^{-\beta(\tau - t_1)}\frac{\partial}{\partial v}(v - \mathbf{E}[U_{\mathrm{bid}(t_1)}(v, t_1) \mid N_{t_1} = 0])\Big|_{v = \nu(t_1)}(\nu(t_1) - \nu(\tau))\right) \cdot P(\mathscr{E})$$
$$- \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1) \mid N_{t_1} = 0, \bar{\mathscr{E}}, \mathscr{G}] \cdot P(\mathscr{G} \mid \bar{\mathscr{E}}) \cdot P(\bar{\mathscr{E}}).$$

Now, because $P(\mathscr{E}) = 1 - \lambda\epsilon + o(\epsilon)$ and $P(\bar{\mathscr{E}}) = \lambda\epsilon + o(\epsilon)$, and $P(\mathscr{G} \mid \bar{\mathscr{E}}) \le F(\nu(t_1)) - F(\nu(\tau))$, we obtain that

$$\Delta \ge e^{-\beta(\tau - t_1)}(\nu(t_1) - \nu(\tau))(1 - e^{-(\lambda(1 - F(\nu(t_1))) + \beta)(T - t)}) \cdot P(\mathscr{E}) - \bar{v}(F(\nu(t_1)) - F(\nu(\tau))) \cdot P(\bar{\mathscr{E}})$$
$$\ge (\nu(t_1) - \nu(\tau))(e^{-\beta(\tau - t_1)}(1 - e^{-(\lambda(1 - F(\nu(t_1))) + \beta)(T - t)})(1 - \lambda\epsilon + o(\epsilon)) - \bar{v}K(\lambda\epsilon + o(\epsilon))), \qquad \text{(EC106)}$$

where the second inequality follows because F is Lipschitz continuous and K is the Lipschitz constant. Now note that $(\nu(t_1) - \nu(\tau)) > 0$ by assumption and that $e^{-\beta(\tau - t_1)}(1 - e^{-(\lambda(1 - F(\nu(t_1))) + \beta)(T - t)}) > 0$ and increasing with ϵ. This together with (EC106) implies that there exists an $\epsilon > 0$ and small such that

$$\Delta = \mathbf{E}[U_{wait(t_1,\tau)}(\nu(t_1), t_1) \mid N_{t_1} = 0] - B_1(\nu(t_1), t_1) > 0$$

where $\tau = t + \epsilon$. Thus bidder A is strictly better off waiting for $\epsilon > 0$ units of time, which is a contradiction, thus proving that if the first bidder is weakly better off acting immediately then the threshold valuation $\nu(t)$ is nondecreasing in t. □

Because $\nu(t)$ is assumed to be nondecreasing in t, the above lemma implies that bidder A will either exercise the buyout option immediately or place a bid in the auction immediately. Now notice that the buyout price $p(t)=\nu(t)-\mathbf{E}[U_{\text{bid}(T)}(\nu(t),t)\mid N_t=0]$ is such that

$$U_{\text{buy}(t)}(\nu(t),t)=\mathbf{E}[U_{\text{bid}(t)}(\nu(t),t)\mid N_t=0]. \tag{EC107}$$

Additionally, as in the static buyout price case, the excess utility function $\delta(v,p(t),t)=v-p(t)-\mathbf{E}[U_{\text{bid}(t)}(v,t)\mid N_t=0]$ is increasing in valuation v. Combining this with (EC107), bidder A's best response strategy is to exercise the buyout option immediately if $v>\nu(t)$ and bid his true valuation immediately otherwise.

Thus bidder A's best response strategy to $\mathcal{T}[\nu]$ is to himself play $\mathcal{T}[\nu]$ and because the choice of A is arbitrary, it proves that $\mathcal{T}[\nu]$ is a Bayesian Nash equilibrium of an auction game with temporary buyout price $p(t)=\nu(t)-\mathbf{E}[U_{\text{bid}(T)}(\nu(t),t)\mid N_t=0]$ for $t\in[0,T]$.

EC.11. Proof of Theorem 6

As in the proof of Theorem 5, we prove that $\mathcal{P}[\nu]$ is a Nash equilibrium by showing that the best response strategy of an arbitrarily chosen bidder to $\mathcal{P}[\nu]$ is to herself play $\mathcal{P}[\nu]$ if the buyout price $p(t)$ is set to be

$$p(t)=\nu(t)-e^{-\beta(T-t)}\mathbf{E}_t\left[\int_{\underline{v}}^{\nu(t)}\frac{\prod_{i=1}^{N(t)}F(\min(\nu(t_i),x))}{\prod_{i=1}^{N(t)}F(\nu(t_i))}(F(x))^{N(t,T)}\,dx\right] \tag{EC108}$$

for $t\in[0,T]$.

Consider any bidder A with type (v,t) and information $I_t=0$. Because the threshold function $\nu(t,I)$ is assumed to be decreasing in I for all t, Lemma 2 shows that bidder A is weakly better off bidding at T.

We next show that bidder A cannot increase her utility by waiting before making a decision.

Lemma 15. *When other bidders follow strategy $\mathcal{P}[\nu]$ in an auction with a permanent dynamic buyout price, a bidder is weakly better off acting immediately, i.e., utility from acting immediately is at least as much as from waiting, if and only if the threshold valuation trajectory $\nu(t)$ is nondecreasing in t for all $t\in[0,T)$.*

Proof. Consider the bidder A who has type $(v,t,0)$. To ensure that he makes a decision immediately, we enforce the constraint that his utility from acting immediately must be at least as much as the utility he obtains from waiting in the auction. Recall that we have already shown, in Lemma 2, that if a bidder decides to bid in the auction he must place a bid at time T and thus bidder A's utility from the auction if he makes a decision immediately is $\max\{U_{\text{buy}(t)}(v,t),\mathbf{E}[U_{\text{bid}(T)}(v,t,0)]\}$.

Suppose A waits up to time τ ($\tau>t$) and define $\mathcal{E}$ as the event that the buyout option is not exercised in (t,τ), i.e., every bidder $(\hat{v},\hat{t},0)$ arriving in the interval (t,τ) has valuation $\hat{v}\le\nu(\hat{t})$. Then, if $\mathbf{E}[U^{(\tau)}_{\text{bid}(T)}(v,t,0)]$ denotes the expected utility from bidding for a bidder who arrives at time t, waits up to time τ ($\tau>t$) and then decides to place a bid in the auction at time T, we have

$$\mathbf{E}[U^{(\tau)}_{\text{bid}(T)}(v,t,0)\mid\mathcal{E}]=e^{-\beta(\tau-t)}\,\mathbf{E}[U_{\text{bid}(T)}(v,\tau,0)] \tag{EC109}$$

because if the event $\mathcal{E}$ occurs, the buyout option is still present at time τ. Furthermore, because no bids are placed in the auction, the information bidder A receives at time τ, $I_\tau=0$. Thus apart from the waiting cost incurred by bidder A, the situation is equivalent to a case where bidder A arrives to the auction at time τ. Another consequence of this argument is that

$$\mathbf{E}[U_{\text{bid}(T)}(v,t,0)\mid\mathcal{E}]=e^{-\beta(\tau-t)}\,\mathbf{E}[U_{\text{bid}(T)}(v,\tau,0)]. \tag{EC110}$$

The complementary event $\bar{\mathcal{E}}$ corresponds to the arrival of a bidder $(\hat{v},\hat{t},0)$ with $\hat{t}\in(t,\tau)$ and valuation $\hat{v}>\nu(\hat{t})$. Such a bidder, following strategy $\mathcal{P}[\nu]$, exercises the buyout option and so $\mathbf{E}[U^{(\tau)}_{\text{bid}(T)}(v,t,0)\mid\bar{\mathcal{E}}]=0$.

Thus the utility from waiting up to time τ is

$$\mathbf{E}[U_{\text{wait}(t,\tau)}(v,t,0)]=e^{-\beta(\tau-t)}\max\{v-p(\tau),\mathbf{E}[U_{\text{bid}(T)}(v,\tau,0)]\}\cdot P(\mathcal{E}). \tag{EC111}$$

Using the law of conditional expectation, we have

$$\mathbf{E}[U_{\mathrm{bid}(T)}(v,t,0)] = \mathbf{E}[U_{\mathrm{bid}(T)}(v,t,0) \mid \mathscr{E}] \cdot P(\mathscr{E}), \tag{EC112}$$

where again $\mathbf{E}[U_{\mathrm{bid}(T)}(v,t,0) \mid \bar{\mathscr{E}}] = 0$.

Using (EC112) and (EC110), the utility from waiting up to τ can be rewritten as

$$\mathbf{E}[U_{\mathrm{wait}(t,\tau)}(v,t,0)] = e^{-\beta(\tau-t)} \max\{v - p(\tau) - \mathbf{E}[U_{\mathrm{bid}(T)}(v,\tau,0), 0]\} \cdot P(\mathscr{E}) + \mathbf{E}[U_{\mathrm{bid}(T)}(v,t,0)].$$

Thus a bidder of type $(v,t,0)$ makes a decision immediately if and only if

$$\max\{U_{\mathrm{buy}(t)}(v,t), \mathbf{E}[U_{\mathrm{bid}(T)}(v,t,0)]\} \geq \mathbf{E}[U_{\mathrm{wait}(t,\tau)}(v,t,0)] \quad \forall \tau > t,$$

which can be expressed as

$$\max\{v - p(t) - \mathbf{E}[U_{\mathrm{bid}(T)}(v,t,0)], 0\} \geq e^{-\beta(\tau-t)} \max\{v - p(\tau) - \mathbf{E}[U_{\mathrm{bid}(T)}(v,\tau,0)], 0\} P(\mathscr{E}). \tag{EC113}$$

Indeed, no bidder in the auction has an incentive to wait, if and only if the condition (EC113) holds for all $t, \tau \in [0, T]$, and $v \in [\underline{v}, \bar{v}]$.

For some $\tau > t$, consider the following two cases:

1. $v \leq \nu(t)$: In this case bidding is more attractive to the bidder at t, i.e., we have $v - p(t) - \mathbf{E}[U_{\mathrm{bid}(T)}(v,t,0)] \leq 0$. Thus the condition (EC113) becomes

$$0 = \max\{v - p(t) - \mathbf{E}[U_{\mathrm{bid}(T)}(v,t,0)], 0\} \geq e^{-\beta(\tau-t)} \max\{v - p(\tau) - \mathbf{E}[U_{\mathrm{bid}(T)}(v,\tau,0)], 0\} P(\mathscr{E}),$$

which holds if and only if

$$v - p(\tau) - \mathbf{E}[U_{\mathrm{bid}(T)}(v,\tau,0)] \leq 0 \quad \forall v \leq \nu(t). \tag{EC114}$$

Now note that

$$\frac{\partial}{\partial v}(v - p(\tau) - \mathbf{E}[U_{\mathrm{bid}(T)}(v,\tau,0)]) = 1 - \mathbf{E}\left[e^{-\beta(T-t)} \frac{\prod_{i=1}^{N(t)} F(\min(\nu(t_i), v))}{\prod_{i=1}^{k} F(\nu(t_i))} \prod_{j=1}^{N(t,T)} F(\min(\nu(\hat{t}_j), v))\right] \geq 0,$$

i.e., $v - p(\tau) - \mathbf{E}[U_{\mathrm{bid}(T)}(v,\tau,0)]$ is nondecreasing in valuation v and thus it is sufficient to impose (EC114) at $v = \nu(t)$. This gives the condition

$$\nu(t) - p(\tau) - \mathbf{E}[U_{\mathrm{bid}(T)}(\nu(t),\tau,0)] \leq 0,$$

which, on substituting $\nu(t) = p(t) + \mathbf{E}[U_{\mathrm{bid}(T)}(\nu(t),t,0)]$, becomes

$$p(\tau) - p(t) \geq \mathbf{E}[U_{\mathrm{bid}(T)}(\nu(t),t,0)] - \mathbf{E}[U_{\mathrm{bid}(T)}(\nu(t),\tau,0)]. \tag{EC115}$$

2. $v > \nu(t)$: In this case $v - p(t) - \mathbf{E}[U_{\mathrm{bid}(T)}(v,t,0)] > 0$. Thus we need

$$v - p(t) - \mathbf{E}[U_{\mathrm{bid}(T)}(v,t,0)] \geq e^{-\beta(T-t)}(v - p(\tau) - \mathbf{E}[U_{\mathrm{bid}(T)}(v,\tau,0)]) P(\mathscr{E}) \quad \forall v > \nu(t).$$

Using (EC112) and (EC110), the above condition can be expressed as

$$v - p(t) \geq e^{-\beta(T-t)}(v - p(\tau)) P(\mathscr{E}) \quad \forall v > \nu(t), \tag{EC116}$$

which can be rewritten as

$$p(\tau) - \frac{1}{e^{-\beta(\tau-t)} P(\mathscr{E})} p(t) \geq v\left(1 - \frac{1}{e^{-\beta(\tau-t)} P(\mathscr{E})}\right) \quad \forall v > \nu(t).$$

Now because $e^{-\beta(\tau-t)} P(\mathscr{E}) < 1$ the right-hand side decreases with v and hence it is sufficient to impose (EC116) at $v = \nu(t)$. This gives

$$p(\tau) - \frac{1}{e^{-\beta(\tau-t)} p(\mathscr{E})} p(t) \geq \nu(t)\left(1 - \frac{1}{e^{\beta(\tau-t)} P(\mathscr{E})}\right).$$

Substituting $\nu(t) = p(t) + \mathbf{E}[U_{\mathrm{bid}(T)}(\nu(t),t,0)]$ and using (EC112) and (EC110) in the above condition we get

$$p(\tau) - p(t) \geq \mathbf{E}[U_{\mathrm{bid}(T)}(\nu(t),t,0)] - \mathbf{E}[U_{\mathrm{bid}(T)}(\nu(t),\tau,0)],$$

which is the same as the condition (EC115) obtained in Case 1.

Thus, no bidder waits before making a decision if and only if for all $\tau, t \in [0, T]$, $\tau > t$

$$p(\tau) - p(t) \geq \mathbf{E}[U_{\text{bid}(T)}(\nu(t), t, 0)] - \mathbf{E}[U_{\text{bid}(T)}(\nu(t), \tau, 0)]. \tag{EC117}$$

Substituting $p(t) = \nu(t) - \mathbf{E}[U_{\text{bid}(T)}(\nu(t), t, 0)]$ in (EC117) gives the condition

$$\nu(\tau) - \nu(t) - \mathbf{E}[U_{\text{bid}(T)}(\nu(\tau), \tau, 0)] + \mathbf{E}[U_{\text{bid}(T)}(\nu(t), \tau, 0)] \geq 0 \tag{EC118}$$

for all $\tau, t \in [0, T]$, $\tau > t$. By setting $\tau = t + \Delta t$ $(\Delta t > 0)$ in the above condition we get that (EC118) holds if and only if for all $t \in [0, T)$

$$\lim_{\Delta t \to 0} (\nu(t + \Delta t) - \nu(t)) \left(1 - \frac{\partial}{\partial v} \mathbf{E}[U_{\text{bid}(T)}(v, t + \Delta t, 0)]\Big|_{v = \nu(t)}\right) \geq 0. \tag{EC119}$$

We have shown earlier that $(\partial/\partial v)\,\mathbf{E}[U_{\text{bid}(T)}(v, t, 0)] < 1$, $\forall\, v \in [\underline{v}, +\infty)$, $t \in [0, T)$ and thus (EC119) (and hence (EC118)) holds if and only if $\nu(\cdot)$ is nondecreasing in t for all $t \in [0, T)$. □

Therefore bidder A immediately decides whether to bid or exercise the buyout option. Now the buyout price is chosen such that

$$U_{\text{buy}(t)}(\nu(t), t) = \mathbf{E}[U_{\text{bid}(T)}(\nu(t), t, 0)]. \tag{EC120}$$

This combined with the fact that the excess utility function $\delta(v, p(t), t) = U_{\text{buy}(t)}(v, t) - \mathbf{E}[U_{\text{bid}(T)}(v, t, 0)]$ is increasing in valuation v implies that a bidder with valuation $v > \nu(t)$ exercises the buyout option immediately whereas a bidder with $v \leq \nu(t)$ will choose to bid his true valuation at time T.

Thus bidder A's best response strategy to $\mathcal{P}[\nu]$ is to himself play $\mathcal{P}[\nu]$ and because the choice of the bidder was arbitrary this proves that $\mathcal{P}[\nu]$ is a Bayesian Nash equilibrium of an auction game with permanent buyout price $p(t) = \nu(t) - \mathbf{E}[U_{\text{bid}(T)}(\nu(t), t, 0)]$.

EC.12. Proof of Proposition 3

Consider the following problem

$$\hat{Z}_{\text{tmp}} \triangleq \sup_{\nu \in \mathscr{C}_0^+} \mathbf{E}[U^S_{\text{tmp}}(\nu)] = \sup_{\nu \in \mathscr{C}_0^+} \int_0^T u_{\text{tmp}}(\nu(t), t) \lambda e^{-\lambda t}\, dt, \tag{EC121}$$

where C_0^+ denotes the set of all nondecreasing functions ν: $[0, T] \to [\underline{v}, \overline{v}]$.

Clearly $C^+ \subset C_0^+$ and thus $\hat{Z}_{\text{tmp}} \geq Z^*_{\text{tmp}}$. The compactness of set C_0^+ follows from the Helly compactness theorem (§7.9 in Ewing 1985). In addition it can be shown that the objective function of (EC121) is continuous over C_0^+ and thus there exists a $\nu^* \in C_0^+$ that achieves the optimal utility $\hat{Z}_{\text{tmp}}$. We first prove the following result.

Lemma 16. *The function $u_{\text{tmp}}(\nu^*(t), t)$ is decreasing in t for $t \in [0, T]$, where $\nu^*(t)$ is the solution of* (EC121).

Proof. We proceed by first proving that $u_{\text{tmp}}(v, t)$ is decreasing in t for $t \in [0, T]$. Indeed, consider the partial derivative of $u_{\text{tmp}}(v, t)$ with respect to t

$$\begin{aligned}\frac{\partial}{\partial t} u_{\text{tmp}}(v, t) = {}& -\alpha(v - \mathbf{E}[U_{\text{bid}(t)}(v, t) \mid N_t = 0])(1 - F(v)) + e^{-\alpha t}\left(-\frac{\partial}{\partial t}\mathbf{E}[U_{\text{bid}(t)}(v, t) \mid N_t = 0]\right)(1 - F(v)) \\ & + e^{-\alpha T} F(v) \frac{\partial}{\partial t} \mathbf{E}_t[\max(\underline{v}, v^{(2)}_{N(t, T)+1}) \mid v_1 \leq v].\end{aligned}$$

Now note that $\mathbf{E}[U_{\text{bid}(t)}(v, t) \mid N_t = 0]$ is increasing in t whereas $\mathbf{E}_t[\max(\underline{v}, v^{(2)}_{N(t, T)+1}) \mid v_1 \leq v]$ is decreasing in t. Using this in the above expression yields that $(\partial/\partial t) u_{\text{tmp}}(v, t) < 0$ and thus $u_{\text{tmp}}(v, t)$ is decreasing in t.

Now assume, for contradiction, that $u_{\text{tmp}}(\nu^*(t), t)$ is not decreasing in t and indeed there exists an interval $[t_1, t_2] \subset [0, T]$ such that

$$u_{\text{tmp}}(\nu^*(t), t) \leq u_{\text{tmp}}(\nu^*(t_2), t_2) \quad \forall\, t \in [t_1, t_2]$$

and thus

$$\int_{t_1}^{t_2} u_{\text{tmp}}(\nu^*(t), t)\,dt \le \int_{t_1}^{t_2} u_{\text{tmp}}(\nu^*(t_2), t_2)\,dt = u_{\text{tmp}}(\nu^*(t_2), t_2)(t_2 - t_1). \tag{EC122}$$

Consider the following valuation trajectory

$$\hat{\nu}(t) = \begin{cases} \nu^*(t_2) & \forall\, t \in [t_1, t_2] \\ \nu^*(t) & \text{otherwise.} \end{cases}$$

Because $\nu^*(t_1) \ge \nu^*(t_2)$, $\hat{\nu} \in C_0^+$ and is thus feasible for the problem (EC121). Hence because ν^* is the optimal solution of the problem (EC121), the utility obtained from using threshold valuation $\hat{\nu}$ must be less than or equal to the optimal utility. Because $\hat{\nu}(t) = \nu^*(t)$ for all $t \notin [t_1, t_2]$, the optimality of ν^* implies

$$\begin{aligned} \int_{t_1}^{t_2} u_{\text{tmp}}(\nu^*(t), t)\,dt &\ge \int_{t_1}^{t_2} u_{\text{tmp}}(\hat{\nu}(t), t)\,dt \\ &= \int_{t_1}^{t_2} u_{\text{tmp}}(\nu^*(t_2), t)\,dt \\ &> \int_{t_1}^{t_2} u_{\text{tmp}}(\nu^*(t_2), t_2)\,dt = u_{\text{tmp}}(\nu^*(t_2), t_2)(t_2 - t_1), \end{aligned}$$

where the second inequality follows because $u_{\text{tmp}}(v, t)$ is decreasing in t. This contradicts (EC122). □

For any partition $\boldsymbol{\tau} \triangleq (\tau_j)_{j\in\{0,\ldots,m\}}$ of $[0, T]$ into m subintervals such that $\tau_0 = 0 < \tau_1 < \cdots < \tau_m = T$, define $\triangle\tau_j \triangleq \tau_{j+1} - \tau_j$ for $j \in \{0, \ldots, m-1\}$ and let $\triangle\boldsymbol{\tau} \triangleq \max_j \triangle\tau_j$ be the mesh size of $\boldsymbol{\tau}$:

$$\begin{aligned} Z^*_{\text{tmp}} \le \hat{Z}_{\text{tmp}} &= \int_0^T u_{\text{tmp}}(\nu^*(\tau), \tau)\lambda e^{-\lambda\tau}\,d\tau = \sum_{i=0}^{m-1} \int_{\tau_i}^{\tau_{i+1}} u_{\text{tmp}}(\nu^*(\tau), \tau)\lambda e^{-\lambda\tau}\,d\tau \\ &\le \sum_{i=0}^{m-1} u_{\text{tmp}}(\nu^*(\tau_i), \tau_i)\lambda e^{-\lambda\tau_i}\Delta\tau_i \\ &\le \bar{Z}_{\text{tmp}}(\tau), \end{aligned}$$

where the first equality follows since ν^* is the optimal threshold valuation. The next inequality follows from Lemma (16), whereas the second inequality follows because $\{\nu_i = \nu^*(\tau_i)\}_{i=0,1,\ldots,m-1}$ is a feasible solution to the discretized problem (16).

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