A Smart Market for Industrial Procurement with Capacity Constraints: Appendix

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This appendix has two parts: In the first one, we describe some extended formulations for the allocation engine $AE[\cdot]$ in our mechanism that take into account the possible economies or diseconomies of scale resulting from nonlinear supplier costs. The second part contains the proof of Proposition 4.

1. Extended Allocation Engine Formulations

Whereas a bid in our manuscript represents a commitment to sell at a price per unit of b_i that is completely independent of the actual quantity eventually awarded, suppliers with nonlinear costs want to submit bids specifying quantity discounts/premiums, i.e., a schedule of bids on unit selling prices depending on the final quantity awarded. From a mechanism design standpoint it seems also in the interest of the buyer to allow suppliers to submit these kinds of bids in these situations. For the sake of exposition, we begin by describing how to formulate such allocation problems with only one component, and then extend the formulation to the case of multiple components.

Let the range $(0, q]$ of all possible positive allocations that a supplier can receive be partitioned into h sub-intervals $(y_\ell, y_{\ell+1}]$ for $\ell \in \{1, ..., h\}$ such that $0 = y_1 \leq y_2 \leq ... \leq y_h \leq y_{h+1} = q$. Each supplier i will now be able to submit a schedule of bids $\mathbf{b}_i = (b_i^1, ..., b_i^h)$ dependent on the eventual allocation awarded. More precisely, we propose here three simple types of quantity/price schedule bids:

1. With *linear* schedule bids, a supplier i having submitted a schedule of bids $\mathbf{b}_i = (b_i^1, ..., b_i^h)$ commits to selling a quantity of $x_i \in (y_\ell, y_{\ell+1}]$ for a total price of $G_i(x_i) = b_i^\ell x_i$. In words, supplier i only commits to a unit selling price of b_i^ℓ if the final allocation x_i he receives belongs to the interval $(y_\ell, y_{\ell+1}]$. Note that supplier i's pricing function $G_i(.)$ is discontinuous at every point y_ℓ for $\ell \in \{2, ..., h\}$.

   With *continuous* schedule bids, a supplier i having submitted a schedule of bids $\mathbf{b}_i = (b_i^1, ..., b_i^h)$ commits to selling a quantity of $x_i \in (y_{\ell'}, y_{\ell'+1}]$ for a total price of

$$F_i(x_i) = \begin{cases} \sum_{\ell=1}^{\ell'-1} y_{i\ell+1} b_i^\ell + b_i^{\ell'}(x_i - y_{\ell'}) \text{ if } \ell' > 1; \\ b_i^1 x_i \text{ if } \ell' = 1. \end{cases} \tag{1}$$

   That is, $F_i(.)$ is a continuous piecewise linear function such that $F(0) = 0$ and $F_i'(x_i) = b_i^{\ell'}$ for $x_i \in (y_{\ell'}, y_{\ell'+1}]$. In words, the revenue of supplier i for a given allocation x_i is determined by decomposing this allocation into a sum of sub-allocations belonging to each sub-interval $(y_\ell, y_{\ell+1}]$ of the bid schedule definition, and then pricing each of those sub-allocations at a marginal price of b_i^ℓ.

   Finally, a *hybrid* schedule bid is essentially a continuous bid schedule with an initial setup component. In this model, each supplier i submits a bid schedule $\mathbf{b}_i^+ = (b_i^0, b_i^1, ..., b_i^h)$ with one more component, b_i^0 representing an additional flat fee (machine purchase, setup) charged by supplier i if he receives a positive allocation. This bid $\mathbf{b}_i^+$ amounts to a commitment to sell a quantity of $x_i \in (y_{\ell'}, y_{\ell'+1}]$ for a total price of

$$F_i^+(x_i) = \begin{cases} b_i^0 + \sum_{\ell=1}^{\ell'-1} y_{i\ell+1} b_i^\ell + b_i^{\ell'}(x_i - y_{\ell'}) \text{ if } \ell' > 1; \\ b_i^0 + b_i^1 x_i \text{ if } \ell' = 1. \end{cases} \tag{2}$$

Fortunately, all three bid schedule types can be formulated using the same general technique. For each supplier i, let $z_{i1}, ..., z_{ih}$ such that

$$\begin{cases} x_i = \sum_{\ell=1}^{h} z_{i\ell}; \\ z_{i\ell} \in [0, y_{\ell+1} - y_\ell] \ \forall \ell; \\ z_{i\ell+1} > 0 \Rightarrow z_{i\ell} = y_{\ell+1} - y_\ell \ \forall \ell \in \{1, ..., h-1\}. \end{cases} \tag{3}$$

With this definition, the buyer's cost resulting from an allocation of x_i to supplier i (who submitted a schedule of bids of $\mathbf{b}_i = (b_i^1, ..., b_i^h)$) in the continuous bid schedule model is

$$\sum_{\ell=1}^{h} b_i^\ell z_{i\ell}. \tag{4}$$

Moreover, conditions (3) can be expressed as a feasible solution to a mixed integer program by introducing h binary variables $v_{i\ell} \in \{0,1\}$ for $\ell \in \{1,...,h\}$ such that

$$\begin{cases} x_i = \sum_{\ell=1}^{h} z_{i\ell}; \\ (y_{\ell+1} - y_\ell)v_{i\ell+1} \leq z_{i\ell} \leq (y_{\ell+1} - y_\ell)v_{i\ell} \ \forall \ell \in \{1,...,h-1\}; \\ 0 \leq z_{ih} \leq (y_{h+1} - y_h)v_{ih}. \end{cases} \tag{5}$$

The portion of the buyer's cost resulting from an allocation x_i to supplier i in the hybrid bid schedule model is then

$$b_i^0 v_{i1} + \sum_{\ell=1}^{h} b_i^\ell z_{i\ell}. \tag{6}$$

The portion of the buyer's cost resulting from an allocation x_i to supplier i in the linear bid schedule model is

$$\sum_{\ell=1}^{h} b_i^\ell z_{i\ell} + \sum_{\ell=1}^{h-1} (b_i^{\ell+1} - b_i^\ell) y_{\ell+1} v_{i\ell+1} \tag{7}$$

We are now ready to propose a formulation for $AE[\mathbf{b}]$.When all suppliers are submitting linear bid schedules, $AE[\mathbf{b}]$ can be expressed as

$$\begin{array}{lll} \underset{z_{i\ell}, v_{i\ell}}{\text{Min}} & \sum_{i=1}^{n}\sum_{\ell=1}^{h} b_i^\ell z_{i\ell} + \sum_{i=1}^{n}\sum_{\ell=1}^{h-1} (b_i^{\ell+1} - b_i^\ell) y_{\ell+1} v_{i\ell+1} & (\text{buyer's cost}) \\ \text{s.t.} & (y_{\ell+1} - y_\ell)v_{i\ell+1} \leq z_{i\ell} \leq (y_{\ell+1} - y_\ell)v_{i\ell} \ \forall i, \forall \ell \in \{1,...,h-1\}; \\ & 0 \leq z_{ih} \leq (y_{h+1} - y_h)v_{ih} \ \forall i; & \left(\begin{array}{l}\text{schedule} \\ \text{constraints}\end{array}\right) \\ & \sum_{i=1}^{n}\sum_{\ell=1}^{h} z_{i\ell} = q; & \left(\begin{array}{l}\text{quantity} \\ \text{requirement} \\ \text{constraint}\end{array}\right) \\ & v_{i\ell} \in \{0,1\} \ \forall (i,\ell). & \left(\begin{array}{l}\text{binary } 0-1 \\ \text{constraints}\end{array}\right) \end{array} \tag{8}$$

Note that the schedule constraints in the above formulation suffice to enforce the conditions $z_{i\ell} \in [0, y_{\ell+1} - y_\ell] \ \forall (i,\ell)$. Also, once an optimal solution $z_{i\ell}$ to the previous allocation problem $AE[\mathbf{b}]$ is known, then the potential allocation x_i of each supplier i can be calculated through

$$x_i = \sum_{\ell=1}^{h} z_{i\ell} \ \forall i. \tag{9}$$

The formulation of $AE[\mathbf{b}]$ when all suppliers submit continuous schedule bids instead is obtained by substituting the objective function in (8) with the expression

$$\operatorname*{Min}_{z_{i\ell}, v_{i\ell}} \quad \sum_{i=1}^{n}\sum_{\ell=1}^{h} b_i^{\ell} z_{i\ell}, \tag{10}$$

and the formulation of $AE[\mathbf{b}]$ when all suppliers submit hybrid bids is obtained when the objective function in (8) is given by

$$\operatorname*{Min}_{z_{i\ell}, v_{i\ell}} \quad \sum_{i=1}^{n} \left(b_i^0 v_{i1} + \sum_{\ell=1}^{h} b_i^{\ell} z_{i\ell} \right). \tag{11}$$

Interestingly, all three types of schedule bids can be handled simultaneously, and if $\mathcal{L}, \mathcal{C}$ and $\mathcal{H}$ are the sets of suppliers submitting linear, continuous and hybrid schedule bids, respectively, the appropriate objective in $AE[\mathbf{b}]$ is given by

$$\operatorname*{Min}_{z_{i\ell}, v_{i\ell}} \quad \sum_{i=1}^{n}\sum_{\ell=1}^{h} b_i^{\ell} z_{i\ell} + \sum_{i\in\mathcal{L}}\sum_{\ell=1}^{h-1} (b_i^{\ell+1} - b_i^{\ell}) y_{\ell+1} v_{i\ell+1} + \sum_{i\in\mathcal{H}} b_i^0 v_{i1}. \tag{12}$$

Finally, using j as the index for components and relation (9), the extension of (8) to a procurement problem with multiple components and capacity constraints, for example with linear bid schedules, can be formulated as:

$$\begin{array}{lll}
\operatorname*{Min}_{z_{ij\ell}, v_{ij\ell}} & \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{\ell=1}^{h} b_{ij}^{\ell} z_{ij\ell} + \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{\ell=1}^{h-1} (b_{ij}^{\ell+1} - b_{ij}^{\ell}) y_{\ell+1} v_{ij\ell+1} & (\text{buyer's cost}) \\
\text{s.t.} & (y_{\ell+1} - y_{\ell}) v_{ij\ell+1} \le z_{ij\ell} \le (y_{\ell+1} - y_{\ell}) v_{ij\ell} \ \forall (i,j), \forall \ell \in \{1, ..., h-1\}; & \\
 & 0 \le z_{ijh} \le (y_{h+1} - y_h) v_{ijh} \ \forall (i,j); & (\text{schedule constraints}) \\
 & \sum_{i=1}^{n}\sum_{\ell=1}^{h} z_{ij\ell} = q_j \ \forall j; & (\text{quantity requirement constraints}) \\
 & \sum_{j=1}^{m}\sum_{\ell=1}^{h} a_{ij} z_{ij\ell} \le c_i \ \forall i; & (\text{capacity constraints}) \\
 & v_{ij\ell} \in \{0,1\} \ \forall (i,j,\ell). & (\text{binary } 0-1 \text{ constraints})
\end{array}$$

2. Proof of Proposition 4

Notation. In this market environment, $AE[\cdot]$ has six extreme points, namely $(\mathbf{x}_1, \mathbf{x}_2) \in$

$\{(A,\tilde{C}),\ (\tilde{C},A),\ (B,\tilde{B}),\ (\tilde{B},B),\ (\tilde{A},C),\ (C,\tilde{A})\}$, where $A=(2q-c,0)^T$, $B=(q,0)^T$, $C=(q,c-q)^T$ and $\sim$ (tilde) denotes an exchange of the two elements in the vector (i.e., $\tilde{A}=(0,2q-c)^T$). In this proof, we refer to $h(t)$ and $h(t+1)$ as h and $\bar{h}$, respectively, where h denotes any variable or vector of interest (allocation, bids). Also, to describe a player's strategy for the next round, we use the notation $\mathbf{b}_Y$ where $Y\in\{A,B,C,\tilde{A},\tilde{B},\tilde{C}\}$ with $Y=\left(AE[\bar{\mathbf{b}}_i=\mathbf{b}_Y,\mathbf{b}_{-i}]\right)_i$ (i.e. under the MBR assumption that $\bar{\mathbf{b}}_{-i}=\mathbf{b}_{-i}$, player i will obtain an allocation $\bar{\mathbf{x}}_i=Y$ if he plays $\mathbf{b}_Y$). As shown in Gallien (2000), for a given selection rule, $\mathbf{b}_Y$ can be uniquely defined among all bids yielding allocation Y under strategy profile stability as the one requiring the smallest decrease in margins (so that each player's strategy space at each round is practically discrete). When comparing the impact of two different bid choices on a player's payoff function, we refer to the case $\Pi_i(\bar{\mathbf{b}}_i=\mathbf{b}_Y,\mathbf{b}_{-i})\geq\Pi_i(\bar{\mathbf{b}}_i=\mathbf{b}_Z,\mathbf{b}_{-i})$ as $\mathbf{b}_Y\overset{i}{\geq}\mathbf{b}_Z$. Finally, we use in the bidding space the distance metric defined by $d(\mathbf{b}_1,\mathbf{b}_2)=\max(|b_{21}-b_{11}|,|b_{22}-b_{12}|)$.

Selection Rule. We consider a multiple optima selection rule symmetric across bidders and component types, where the selected allocation is always an extreme point, and that is characterized by

- $\begin{cases}\bar{b}_{11}=\bar{b}_{21}\\ \bar{b}_{12}>\bar{b}_{22}\end{cases}\Rightarrow(\bar{\mathbf{x}}_1,\bar{\mathbf{x}}_2)=(A,\tilde{C});$
- $\begin{cases}\bar{b}_{11}-\bar{b}_{21}=\bar{b}_{12}-\bar{b}_{22}\\ \bar{b}_{11}>\bar{b}_{21}\end{cases}\Rightarrow(\bar{\mathbf{x}}_1,\bar{\mathbf{x}}_2)=\begin{cases}(A,\tilde{C})\text{ if }(\mathbf{x}_1,\mathbf{x}_2)\in\{(\tilde{C},A),(\tilde{B},B),(\tilde{A},C)\}\\ (\tilde{A},C)\text{ otherwise}\end{cases};$
- $\bar{\mathbf{b}}_1=\bar{\mathbf{b}}_2\Rightarrow(\bar{\mathbf{x}}_1,\bar{\mathbf{x}}_2)=(\mathbf{x}_1,\mathbf{x}_2)$.

Note that all missing cases in the definition above can be resolved by symmetry, and a random selection rule can be used at the first round if necessary.

Strategy Space Restrictions. While at every round the strategy space of each player is a priori $\{\mathbf{b}_A,\mathbf{b}_B,\mathbf{b}_C,\mathbf{b}_{\tilde{A}},\mathbf{b}_{\tilde{B}},\mathbf{b}_{\tilde{C}}\}$, it can typically be reduced to a smaller set by considering both the non-reneging rule (e.g., if $\mathbf{x}_1=B$ then $\bar{\mathbf{b}}_1\in\{b_B,b_C,b_{C'}\}$) and an adaptation of Lemma 4 in Part I of Gallien (2000) to the selection rule described earlier, which shows that under the MBR rationale:

1. If $b_{22}-v_{12}\geq b_{21}-v_{11}$ (player 2's bid is above player 1's margin switching line), then $\mathbf{b}_{\tilde{A}}\overset{1}{\geq}\mathbf{b}_A$;
2. If $b_{22}-v_{12}\geq b_{21}-v_{11}$ and $\mathbf{x}_1\in\{A,\tilde{A},\tilde{B},\tilde{C}\}$, then $\mathbf{b}_{\tilde{C}}\overset{1}{\geq}\mathbf{b}_C$;
3. $b_{22}\geq v_{12}\Leftrightarrow\mathbf{b}_C\overset{1}{\geq}\mathbf{b}_B$ and $b_{21}\geq v_{11}\Leftrightarrow\mathbf{b}_{\tilde{C}}\overset{1}{\geq}\mathbf{b}_{\tilde{B}}$.

Note here again that many more such statements follow by symmetry. In summary, the set of possible bid choices $\bar{\mathbf{b}}_1$ for player 1 can be obtained from the following table (in order to avoid a long and uninteresting discussion of the tie cases, we assume that the production costs $\mathbf{v}_1$ and $\mathbf{v}_2$ do not belong to the ϵ -grid):

	$\mathbf{x}_1$					
	A	$\tilde{A}$	B	$\tilde{B}$	C	$\tilde{C}$
Case:						
$b_{22} - v_{12} > b_{21} - v_{11}$ and $\mathbf{b}_2 > \mathbf{v}_1$	$\{\mathbf{b}_{\tilde{A}}, \mathbf{b}_{\tilde{C}}\}$	$\{\mathbf{b}_{\tilde{A}}, \mathbf{b}_{\tilde{C}}\}$	$\{\mathbf{b}_C, \mathbf{b}_{\tilde{C}}\}$	$\{\mathbf{b}_{\tilde{C}}\}$	$\{\mathbf{b}_C, \mathbf{b}_{\tilde{C}}\}$	$\{\mathbf{b}_{\tilde{C}}\}$
$b_{22} - v_{12} > b_{21} - v_{11}$ and $\begin{cases} b_{21} < v_{11} \\ b_{22} > v_{12} \end{cases}$	$\{\mathbf{b}_{\tilde{A}}, \mathbf{b}_{\tilde{B}}\}$	$\{\mathbf{b}_{\tilde{A}}, \mathbf{b}_{\tilde{B}}\}$	N/A	$\{\mathbf{b}_{\tilde{B}}\}$	N/A	N/A

Neighborhood Stability and Convergence. Let us now assume that $d(\mathbf{b}_1, \mathbf{b}_2) \leq \epsilon$. We can use the above table to specify the bid chosen by player 1 under the MBR rationale (results for player 2 and for missing cases follow by symmetry):

Case 1: $b_{22} - v_{12} > b_{21} - v_{11}$ and $\mathbf{b}_2 > \mathbf{v}_1$

– When $\mathbf{x}_1 \in \{A, \tilde{A}\}$, we have

$$\mathbf{b}_{\tilde{A}} \overset{1}{\geq} \mathbf{b}_{\tilde{C}} \quad \begin{array}{l} \Leftrightarrow (2q - c)(b_{22} + \epsilon - v_{12}) \geq q(b_{22} - \epsilon - v_{12}) + (c - q)(b_{21} - v_{11}) \\ \Leftrightarrow b_{22} - v_{12} + b_{21} - v_{11} \leq \frac{3q-c}{c-q}\epsilon \end{array} ;$$

– When $\mathbf{x}_1 = B$,

$$\mathbf{b}_C \overset{1}{\geq} \mathbf{b}_{\tilde{C}} \quad \begin{array}{l} \Leftrightarrow (c - q)(b_{22} - v_{12}) + q(b_{11} - v_{11}) \geq q(b_{22} - \epsilon - v_{12}) + (c - q)(b_{11} - v_{11}) \\ \Leftrightarrow b_{22} - v_{12} - b_{21} + v_{11} \leq \frac{c-q}{2q-c}\epsilon \end{array} ;$$

– When $\mathbf{x}_1 = C$,

$$\mathbf{b}_C \overset{1}{\geq} \mathbf{b}_{\tilde{C}} \quad \begin{array}{l} \Leftrightarrow q(b_{11} - v_{11}) + (c - q)(b_{12} - v_{12}) \geq (c - q)(b_{11} - v_{11}) + q(b_{12} - \epsilon - v_{12}) \\ \Leftrightarrow b_{12} - v_{12} - b_{11} + v_{11} \leq \frac{q}{2q-c}\epsilon \end{array} .$$

Case 2: $b_{22} - v_{12} > b_{21} - v_{11}$ and $\begin{cases} b_{21} < v_{11} \\ b_{22} > v_{12} \end{cases}$

– When $\mathbf{x}_1 \in \{A, \tilde{A}\}$, we have

$$\begin{array}{l} \mathbf{b}_{\tilde{A}} \overset{1}{\geq} \mathbf{b}_{\tilde{B}} \text{ if } (2q - c)(b_{22} - v_{12}) \geq q(b_{22} - \epsilon - v_{12}) \\ \Leftrightarrow b_{22} - v_{12} \leq \frac{q}{c-q}\epsilon \end{array} .$$

In words, supplier 1 prefers $\mathbf{b}_{\tilde{C}}$ to $\mathbf{b}_{\tilde{A}}$ as long as supplier 2's bid $\mathbf{b}_2$ remains sufficiently greater than $\mathbf{v}_1$ (i.e., when $b_{22} - v_{12} + b_{21} - v_{11} > \frac{3q-c}{c-q}\epsilon$), and prefers $\mathbf{b}_{\tilde{C}}$ to $\mathbf{b}_C$ when $\mathbf{b}_2$ is far enough above his margin switching line (i.e., when $b_{12} - v_{12} - b_{11} + v_{11} > \frac{q}{2q-c}\epsilon$). When b_{21} drops below v_{11}, supplier 1 plays $\mathbf{b}_{\tilde{B}}$ if b_{22} is sufficiently greater than v_{12} (i.e., when $b_{22} - v_{12} >$

$\frac{q}{c-q}\epsilon$), and $\mathbf{b}_{\tilde{A}}$ otherwise. Applying the same reasoning to player 2, we can construct for all relevant cases the tables describing the joint bidding strategies and resulting allocation for both players, thus characterizing the dynamics of the bidding sequence. For example, the table corresponding to the case $\begin{cases} b_{22} - v_{12} > b_{21} - v_{11} \text{ and } \mathbf{b}_2 > \mathbf{v}_1 \\ b_{12} - v_{22} > b_{11} - v_{21} \text{ and } \mathbf{b}_1 > \mathbf{v}_2 \end{cases}$ is (missing allocation cases can be derived by symmetry)

	$A:$		$\tilde{A}:$		$B:$	
	$d(b_2, v_1) > \frac{3q-c}{c-q}\epsilon$	$d(b_2, v_1) \leq \frac{3q-c}{c-q}\epsilon$	$d(b_2, v_1) > \frac{3q-c}{c-q}\epsilon$	$d(b_2, v_1) \leq \frac{3q-c}{c-q}\epsilon$	$b_{22} - v_{12} - b_{21} + v_{11} > \frac{c-q}{2q-c}\epsilon$	$b_{22} - v_{12} - b_{21} + v_{11} \leq \frac{c-q}{2q-c}\epsilon$
$\tilde{C}$	$(b_{\tilde{C}}, b_{\tilde{C}}) \to (A, \tilde{C})$	$(b_{\tilde{C}}, b_{\tilde{A}}) \to (C, \tilde{A})$				
$C: b_{12} - v_{12} - b_{11} + v_{11} > \frac{q}{2q-c}\epsilon$			$(b_{\tilde{C}}, b_{\tilde{C}}) \to (C, \tilde{A})$	$(b_{\tilde{C}}, b_{\tilde{A}}) \to (\tilde{C}, A)$		
$C: b_{12} - v_{12} - b_{11} + v_{11} \leq \frac{q}{2q-c}\epsilon$			$(b_C, b_{\tilde{C}}) \to (A, \tilde{C})$	$(b_C, b_{\tilde{A}}) \to (C, \tilde{A})$		
$\tilde{B}$					$(b_{\tilde{C}}, b_{\tilde{C}}) \to (A, \tilde{C})$	$(b_{\tilde{C}}, b_C) \to (\tilde{B}, B)$

where the rows correspond to player 2's allocation $\mathbf{x}_2$, the columns to $\mathbf{x}_1$, and each entry (I, J) in this table provides $(\bar{\mathbf{b}}_2, \bar{\mathbf{b}}_1) \to (\bar{\mathbf{x}}_2, \bar{\mathbf{x}}_1)$ when $(\mathbf{x}_2, \mathbf{x}_1) = (I, J)$. Constructing the tables corresponding to cases

$$\begin{cases} b_{22} - v_{12} > b_{21} - v_{11} \text{ and } \mathbf{b}_2 > \mathbf{v}_1 \\ b_{12} - v_{22} < b_{11} - v_{21} \text{ and } \mathbf{b}_1 > \mathbf{v}_2 \end{cases} \text{ and } \begin{cases} b_{22} - v_{12} > b_{21} - v_{11},\ b_{21} < v_{11} \text{ and } b_{22} > v_{12} \\ b_{12} - v_{22} > b_{11} - v_{21} \text{ and } \mathbf{b}_1 > \mathbf{v}_2 \end{cases}$$

(tables for other cases follow by symmetry), we can observe that

$$\begin{cases} d(\mathbf{b}_1, \mathbf{b}_2) \leq \epsilon; \\ b_{i2} - v_{-i2} + b_{i1} - v_{-i1} > \frac{3q-c}{c-q}\epsilon \text{ for } i \in \{1, 2\}; \text{ and} \\ b_{ij} - v_{-ij} > \frac{q}{c-q}\epsilon \text{ for } i, j \in \{1, 2\} \end{cases}$$

implies

$$\begin{cases} d(\bar{\mathbf{b}}_1, \bar{\mathbf{b}}_2) \leq \boldsymbol{\epsilon}; \text{ and} \\ \bar{b}_{11} < b_{11}, \bar{b}_{12} < b_{12}, \bar{b}_{21} < b_{21} \text{ or } \bar{b}_{22} < b_{22}. \end{cases}$$

In words, if at one given round the bids of the two suppliers are within a distance ϵ of each other and sufficiently far away from the production costs, then under the MBR rationale the bids at the next round will still be within ϵ of each other, and at least one of the bids will have been strictly decreased. Applying this reasoning iteratively, observing that $\frac{3q-c}{c-q}\epsilon > \frac{q}{c-q}\epsilon$ since $2q - c > 0$, and that $b_{i2} - v_{-i2} + b_{i1} - v_{-i1} \leq \frac{3q-c}{c-q}\epsilon$ for $i = 1$ or 2 implies $d(\mathbf{b}_i, \mathbf{v}_1 \vee \mathbf{v}_2) \leq \frac{3q-c}{c-q}\epsilon$, we can conclude that $d(\mathbf{b}_i(T), \mathbf{v}_1 \vee \mathbf{v}_2) \leq (\frac{3q-c}{c-q} + 1)\epsilon$ for $i \in \{1, 2\}$, which completes the proof.

References

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