A Smart Market for Industrial Procurement with Capacity Constraints: Appendix

Jérémie Gallien
Sloan School of Management, MIT
Cambridge, MA 02139
e-mail: jgallien@mit.edu

Lawrence M. Wein
Sloan School of Management, MIT
Cambridge, MA 02139
e-mail: lwein@mit.edu

June 10, 2002

This appendix has two parts: In the first one, we describe some extended formulations for the allocation engine \( AE[\cdot] \) in our mechanism that take into account the possible economies or diseconomies of scale resulting from nonlinear supplier costs. The second part contains the proof of Proposition 4.

1. Extended Allocation Engine Formulations

Whereas a bid in our manuscript represents a commitment to sell at a price per unit of \( b_i \) that is completely independent of the actual quantity eventually awarded, suppliers with nonlinear costs want to submit bids specifying quantity discounts/premiums, i.e., a schedule of bids on unit selling prices depending on the final quantity awarded. From a mechanism design standpoint it seems also in the interest of the buyer to allow suppliers to submit these kinds of bids in these situations. For the sake of exposition, we begin by describing how to formulate such allocation problems with only one component, and then extend the formulation to the case of multiple components.

Let the range \((0, q]\) of all possible positive allocations that a supplier can receive be partitioned into \( h \) sub-intervals \((y_\ell, y_{\ell+1}]\) for \( \ell \in \{1, ..., h\} \) such that \( 0 = y_1 \leq y_2 \leq ... \leq y_h \leq y_{h+1} = q \). Each supplier \( i \) will now be able to submit a schedule of bids \( b_i = (b^1_i, ..., b^h_i) \) dependent on the eventual allocation awarded. More precisely, we propose here three simple
types of quantity/price schedule bids:

1. With linear schedule bids, a supplier $i$ having submitted a schedule of bids $b_i = (b_i^1, ..., b_i^h)$ commits to selling a quantity of $x_i \in (y_\ell, y_{\ell+1}]$ for a total price of $G_i(x_i) = b_i^\ell x_i$. In words, supplier $i$ only commits to a unit selling price of $b_i^\ell$ if the final allocation $x_i$ he receives belongs to the interval $(y_\ell, y_{\ell+1}]$. Note that supplier $i$’s pricing function $G_i(.)$ is discontinuous at every point $y_\ell$ for $\ell \in \{2, ..., h\}$.

With continuous schedule bids, a supplier $i$ having submitted a schedule of bids $b_i = (b_i^1, ..., b_i^h)$ commits to selling a quantity of $x_i \in (y_{\ell'}, y_{\ell'+1}]$ for a total price of

$$F_i(x_i) = \begin{cases} \sum_{\ell'=1}^{\ell'-1} y_{\ell'} b_i^\ell + b_i^{\ell'} (x_i - y_{\ell'}) & \text{if } \ell' > 1; \\ b_i^1 x_i & \text{if } \ell' = 1. \end{cases}$$

That is, $F_i(.)$ is a continuous piecewise linear function such that $F(0) = 0$ and $F_i(x_i) = b_i^{\ell'}$ for $x_i \in (y_{\ell'}, y_{\ell'+1}]$. In words, the revenue of supplier $i$ for a given allocation $x_i$ is determined by decomposing this allocation into a sum of sub-allocations belonging to each sub-interval $(y_\ell, y_{\ell+1}]$ of the bid schedule definition, and then pricing each of those sub-allocations at a marginal price of $b_i^\ell$.

Finally, a hybrid schedule bid is essentially a continuous bid schedule with an initial setup component. In this model, each supplier $i$ submits a bid schedule $b_i^* = (b_i^0, b_i^1, ..., b_i^h)$ with one more component, $b_i^0$ representing an additional flat fee (machine purchase, setup) charged by supplier $i$ if he receives a positive allocation. This bid $b_i^*$ amounts to a commitment to sell a quantity of $x_i \in (y_{\ell'}, y_{\ell'+1}]$ for a total price of

$$F_i^*(x_i) = \begin{cases} b_i^0 + \sum_{\ell'=1}^{\ell'-1} y_{\ell'} b_i^\ell + b_i^{\ell'} (x_i - y_{\ell'}) & \text{if } \ell' > 1; \\ b_i^0 + b_i^1 x_i & \text{if } \ell' = 1. \end{cases}$$

Fortunately, all three bid schedule types can be formulated using the same general technique. For each supplier $i$, let $z_{i,1}, ..., z_{i,h}$ such that

$$\begin{align*}
&x_i = \sum_{\ell'=1}^h z_{i,\ell'}; \\
z_{i,\ell'} &\in [0, y_{\ell+1} - y_\ell] \quad \forall \ell; \\
z_{i,\ell+1} &> 0 \Rightarrow z_{i,\ell} = y_{\ell+1} - y_\ell \quad \forall \ell \in \{1, ..., h-1\}.
\end{align*}$$

With this definition, the buyer’s cost resulting from an allocation of $x_i$ to supplier $i$ (who submitted a schedule of bids of $b_i = (b_i^1, ..., b_i^h)$) in the continuous bid schedule model is
\[\sum_{\ell = 1}^{h} b_{i\ell} z_{i\ell}.\]  

Moreover, conditions (3) can be expressed as a feasible solution to a mixed integer program by introducing \( h \) binary variables \( v_{i\ell} \in \{0, 1\} \) for \( \ell \in \{1, \ldots, h\} \) such that

\[
\begin{cases}
  x_i = \sum_{\ell = 1}^{h} z_{i\ell}; \\
  (y_{\ell+1} - y_{\ell}) v_{i\ell+1} \leq z_{i\ell} \leq (y_{\ell+1} - y_{\ell}) v_{i\ell} \quad \forall \ell \in \{1, \ldots, h - 1\}; \\
  0 \leq z_{ih} \leq (y_{h+1} - y_{h}) v_{ih}.
\end{cases}
\]

The portion of the buyer’s cost resulting from an allocation \( x_i \) to supplier \( i \) in the hybrid bid schedule model is then

\[b_0^i v_{i1} + \sum_{\ell = 1}^{h} b_{i\ell} z_{i\ell}.\]  

The portion of the buyer’s cost resulting from an allocation \( x_i \) to supplier \( i \) in the linear bid schedule model is

\[
\sum_{\ell = 1}^{h} b_{i\ell} z_{i\ell} + \sum_{\ell = 1}^{h-1} (b_{i_{\ell+1}} - b_{i\ell}) y_{\ell+1} v_{i\ell+1}
\]

We are now ready to propose a formulation for \( AE[b] \). When all suppliers are submitting linear bid schedules, \( AE[b] \) can be expressed as

\[
\begin{align*}
\text{Min} & \quad \sum_{i=1}^{n} \sum_{\ell = 1}^{h} b_{i\ell} z_{i\ell} + \sum_{i=1}^{n} \sum_{\ell = 1}^{h-1} (b_{i_{\ell+1}} - b_{i\ell}) y_{\ell+1} v_{i\ell+1} \\
\text{s.t.} & \quad (y_{\ell+1} - y_{\ell}) v_{i\ell+1} \leq z_{i\ell} \leq (y_{\ell+1} - y_{\ell}) v_{i\ell} \quad \forall \ell \in \{1, \ldots, h - 1\}; \\
& \quad 0 \leq z_{ih} \leq (y_{h+1} - y_{h}) v_{ih} \quad \forall i; \\
& \quad \sum_{i=1}^{n} \sum_{\ell = 1}^{h} z_{i\ell} = q; \\
& \quad v_{i\ell} \in \{0, 1\} \quad \forall (i, \ell).
\end{align*}
\]

Note that the schedule constraints in the above formulation suffice to enforce the conditions \( z_{i\ell} \in [0, y_{\ell+1} - y_{\ell}] \) \( \forall (i, \ell) \). Also, once an optimal solution \( z_{i\ell} \) to the previous allocation problem \( AE[b] \) is known, then the potential allocation \( x_i \) of each supplier \( i \) can be calculated through
\[ x_i = \sum_{\ell=1}^{h} z_{i\ell} \quad \forall i. \]  

(9)

The formulation of \( AE[b] \) when all suppliers submit continuous schedule bids instead is obtained by substituting the objective function in (8) with the expression

\[
\text{Min}_{z_{i\ell}, v_{i\ell}} \sum_{i=1}^{n} \sum_{\ell=1}^{h} b_i^\ell z_{i\ell},
\]

(10) and the formulation of \( AE[b] \) when all suppliers submit hybrid bids is obtained when the objective function in (8) is given by

\[
\text{Min}_{z_{i\ell}, v_{i\ell}} \sum_{i=1}^{n} \sum_{\ell=1}^{h} \left( b_i^0 v_{i1} + \sum_{\ell=1}^{h} b_i^\ell z_{i\ell} \right). \]

(11)

Interestingly, all three types of schedule bids can be handled simultaneously, and if \( \mathcal{L}, \mathcal{C} \) and \( \mathcal{H} \) are the sets of suppliers submitting linear, continuous and hybrid schedule bids, respectively, the appropriate objective in \( AE[b] \) is given by

\[
\text{Min}_{z_{i\ell}, v_{i\ell}} \sum_{i=1}^{n} \sum_{\ell=1}^{h} b_i^\ell z_{i\ell} + \sum_{i\in\mathcal{L}} \sum_{\ell=1}^{h-1} \left( b_i^{\ell+1} - b_i^{\ell} \right) y_{\ell+1} v_{i\ell+1} + \sum_{i\in\mathcal{H}} b_i^0 v_{i1}. \]

(12)

Finally, using \( j \) as the index for components and relation (9), the extension of (8) to a procurement problem with multiple components and capacity constraints, for example with linear bid schedules, can be formulated as:

\[
\text{Min}_{z_{ij\ell}, v_{ij\ell}} \sum_{j=1}^{m} \sum_{\ell=1}^{h} b_i^\ell z_{ij\ell} + \sum_{j=1}^{m} \sum_{\ell=1}^{h-1} \left( b_i^{\ell+1} - b_i^{\ell} \right) y_{\ell+1} v_{ij\ell+1} \quad \text{buyer's cost}
\]

s.t. \( (y_{\ell+1} - y_{\ell}) v_{ij\ell+1} \leq z_{ij\ell} \leq (y_{\ell+1} - y_{\ell}) v_{ij\ell} \quad \forall(i,j), \forall \ell \in \{1, \ldots, h-1\}; \)
\( 0 \leq z_{ij\ell} \leq (y_{h+1} - y_h) v_{ijh} \quad \forall(i,j); \)
\( \sum_{i=1}^{n} \sum_{\ell=1}^{h} z_{ij\ell} = q_j \quad \forall j; \)
\( \sum_{j=1}^{m} \sum_{\ell=1}^{h} a_{ij} z_{ij\ell} \leq c_i \quad \forall i; \)
\( v_{ij\ell} \in \{0,1\} \quad \forall(i,j,\ell). \)

2. Proof of Proposition 4

Notation. In this market environment, \( AE[\cdot] \) has six extreme points, namely \((x_1, x_2) \in \)
\{(A, \tilde{C}), (\tilde{C}, A), (B, \tilde{B}), (\tilde{B}, B), (\tilde{A}, C), (C, \tilde{A})\}, \text{ where } A = (2q - c, 0)^T, \ B = (q, 0)^T, \ C = (q, c - q)^T \text{ and } \sim \text{ (tilde) denotes an exchange of the two elements in the vector (i.e., } \tilde{A} = (0, 2q - c)^T). \text{ In this proof, we refer to } h(t) \text{ and } h(t + 1) \text{ as } h \text{ and } \tilde{h}, \text{ respectively, where } h \text{ denotes any variable or vector of interest (allocation, bids). Also, to describe a player’s strategy for the next round, we use the notation } b_Y \text{ where } Y \in \{A, B, C, \tilde{A}, \tilde{B}, \tilde{C}\} \text{ with } Y = (AE[\tilde{b}_i = b_Y, \ b_{-i}])_i \text{ (i.e. under the MBR assumption that } \tilde{b}_{-i} = b_{-i}, \text{ player } i \text{ will obtain an allocation } x_i = Y \text{ if he plays } b_Y). \text{ As shown in Gallien (2000), for a given selection rule, } b_Y \text{ can be uniquely defined among all bids yielding allocation } Y \text{ under strategy profile stability as the one requiring the smallest decrease in margins (so that each player’s strategy space at each round is practically discrete). When comparing the impact of two different bid choices on a player’s payoff function, we refer to the case } \Pi_i(\tilde{b}_i = b_Y, \ b_{-i}) \geq \Pi_i(\tilde{b}_i = b_Z, \ b_{-i}) \text{ as } b_Y \geq b_Z. \text{ Finally, we use in the bidding space the distance metric defined by } d(b_1, b_2) = \max(|b_{21} - b_{11}|, |b_{22} - b_{12}|).

**Selection Rule.** We consider a multiple optima selection rule symmetric across bidders and component types, where the selected allocation is always an extreme point, and that is characterized by

\begin{itemize}
  \item \begin{align*}
  \begin{cases}
  \tilde{b}_{11} = \tilde{b}_{21} \\
  \tilde{b}_{12} > \tilde{b}_{22}
  \end{cases} \Rightarrow (x_1, x_2) = (A, \tilde{C});
\end{align*}
  \item \begin{align*}
  \begin{cases}
  \tilde{b}_{11} - \tilde{b}_{21} = \tilde{b}_{12} - \tilde{b}_{22} \\
  \tilde{b}_{11} > \tilde{b}_{21}
  \end{cases} \Rightarrow (x_1, x_2) = \begin{cases}
  (A, \tilde{C}) \text{ if } (x_1, x_2) \in \{(A, \tilde{C}), (\tilde{B}, B), (\tilde{A}, C)\} \\
  (A, C) \text{ otherwise}
  \end{cases};
\end{align*}
  \item \begin{align*}
  \tilde{b}_1 = \tilde{b}_2 \Rightarrow (x_1, x_2) = (x_1, x_2).
\end{align*}
\end{itemize}

Note that all missing cases in the definition above can be resolved by symmetry, and a random selection rule can be used at the first round if necessary.

**Strategy Space Restrictions.** While at every round the strategy space of each player is a priori \{\(b_A, b_B, b_C, b_{\tilde{A}}, b_{\tilde{B}}, b_{\tilde{C}}\}\), it can typically be reduced to a smaller set by considering both the non-reneging rule (e.g., if \(x_1 = B\) then \(\tilde{b}_1 \in \{b_B, b_C, b_{C'}\}\)) and an adaptation of Lemma 4 in Part I of Gallien (2000) to the selection rule described earlier, which shows that under the MBR rationale:

1. If \(b_{22} - v_{12} \geq b_{21} - v_{11}\) (player 2’s bid is above player 1’s margin switching line), then \(b_{\tilde{A}} \geq b_A\);
2. If \(b_{22} - v_{12} \geq b_{21} - v_{11}\) and \(x_1 \in \{A, \tilde{A}, \tilde{B}, \tilde{C}\}\), then \(b_{\tilde{C}} \geq b_C\);
3. \(b_{22} \geq v_{12} \iff b_C \geq b_B\) and \(b_{21} \geq v_{11} \iff b_{\tilde{C}} \geq b_{\tilde{B}}\).
Note here again that many more such statements follow by symmetry. In summary, the set of possible bid choices $\tilde{b}_1$ for player 1 can be obtained from the following table (in order to avoid a long and uninteresting discussion of the tie cases, we assume that the production costs $v_1$ and $v_2$ do not belong to the $\epsilon$-grid):

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$A$</th>
<th>$\tilde{A}$</th>
<th>$B$</th>
<th>$\tilde{B}$</th>
<th>$C$</th>
<th>$\tilde{C}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b_{22} - v_{12} &gt; b_{21} - v_{11}$ and $b_2 &gt; v_1$</td>
<td>${b_{\tilde{A}}, b_{\tilde{C}}}$</td>
<td>${b_{\tilde{A}}, b_{\tilde{C}}}$</td>
<td>${b_{\tilde{C}}}$</td>
<td>${b_{\tilde{C}}}$</td>
<td>${b_{\tilde{C}}}$</td>
<td>${b_{\tilde{C}}}$</td>
</tr>
<tr>
<td>$b_{22} - v_{12} &gt; b_{21} - v_{11}$ and ${b_{21} &lt; v_{11}$, $b_{22} &gt; v_{12}}$</td>
<td>${b_{\tilde{A}}, b_{\tilde{B}}}$</td>
<td>${b_{\tilde{A}}, b_{\tilde{B}}}$</td>
<td>$N/A$</td>
<td>$b_{\tilde{B}}$</td>
<td>$N/A$</td>
<td>$N/A$</td>
</tr>
</tbody>
</table>

**Neighborhood Stability and Convergence.** Let us now assume that $d(b_1, b_2) \leq \epsilon$.

We can use the above table to specify the bid chosen by player 1 under the MBR rationale (results for player 2 and for missing cases follow by symmetry):

**Case 1:** $b_{22} - v_{12} > b_{21} - v_{11}$ and $b_2 > v_1$

- When $x_1 \in \{A, \tilde{A}\}$, we have
  \[
  b_{\tilde{A}} \geq b_{\tilde{C}} \iff (2q - c)(b_{22} + \epsilon - v_{12}) \geq q(b_{22} - \epsilon - v_{12}) + (c - q)(b_{21} - v_{11}) ;
  \]
  \[
  \iff b_{22} - v_{12} + b_{21} - v_{11} \leq \frac{3q - c}{c - q} \epsilon
  \]
- When $x_1 = B$,
  \[
  b_{\tilde{B}} \geq b_{\tilde{C}} \iff (c - q)(b_{22} - v_{12}) + q(b_{11} - v_{11}) \geq q(b_{22} - \epsilon - v_{12}) + (c - q)(b_{11} - v_{11}) ;
  \]
  \[
  \iff b_{22} - v_{12} - b_{21} + v_{11} \leq \frac{c - q}{2q - c} \epsilon
  \]
- When $x_1 = C$,
  \[
  b_{\tilde{C}} \geq b_{\tilde{C}} \iff q(b_{11} - v_{11}) + (c - q)(b_{12} - v_{12}) \geq (c - q)(b_{11} - v_{11}) + q(b_{12} - \epsilon - v_{12}) ;
  \]
  \[
  \iff b_{22} - v_{12} - b_{21} + v_{11} \leq \frac{q}{2q - c} \epsilon
  \]

**Case 2:** $b_{22} - v_{12} > b_{21} - v_{11}$ and $\{b_{21} < v_{11}$, $b_{22} > v_{12}\}$

- When $x_1 \in \{A, \tilde{A}\}$, we have
  \[
  b_{\tilde{A}} \geq b_{\tilde{B}} \iff (2q - c)(b_{22} - v_{12}) \geq q(b_{22} - \epsilon - v_{12}) .
  \]
  \[
  \iff b_{22} - v_{12} \leq \frac{q}{c - q} \epsilon
  \]

In words, supplier 1 prefers $b_{\tilde{C}}$ to $b_{\tilde{A}}$ as long as supplier 2’s bid $b_2$ remains sufficiently greater than $v_1$ (i.e., when $b_{22} - v_{12} + b_{21} - v_{11} > \frac{3q - c}{c - q} \epsilon$), and prefers $b_{\tilde{C}}$ to $b_{\tilde{C}}$ when $b_2$ is far enough above his margin switching line (i.e., when $b_{12} - v_{12} - b_{11} + v_{11} > \frac{q}{2q - c} \epsilon$). When $b_{21}$ drops below $v_{11}$, supplier 1 plays $b_{\tilde{B}}$ if $b_{22}$ is sufficiently greater than $v_{12}$ (i.e., when $b_{22} - v_{12} > \frac{q}{c - q} \epsilon$).
In words, if at one given round the bids of the two suppliers are within a distance \( \epsilon \) of each other and sufficiently far away from the production costs, then under the MBR rationale the bids at the next round will still be within \( \epsilon \) of each other, and at least one of the bids is strictly above \( \epsilon \). Therefore, \( \epsilon \) decreases and \( b_1, b_2 \) will converge to \( b_1, b_2 \) in the next round. 

When there are \( \epsilon \) entries in the two tables corresponding to cases (1, 2) in this table provides \( (\epsilon, \epsilon) \) when \( (x, x) = (1, 2) \). Constructing the two-way table for other cases follow by symmetry, we can observe that the rows correspond to player 2’s allocation \( x_2 \), the columns to \( x_1 \), and each entry \( \left\{ d(b_i, b_j) \leq \epsilon; \begin{cases} b_{i2} - b_{j2} > b_{i1} - b_{j1}, \quad b_i > b_j; \\ b_{i2} - b_{j2} < b_{i1} - b_{j1}, \quad b_i < b_j; \\ b_{i2} = b_{j2}, \quad b_i = b_j; \end{cases} \right\} \) for \( i, j \in \{1, 2\} \) and \( \{ A, B \} \).

Applying the same reasoning to player 2, we can construct for all relevant cases the tables describing the joint bidding strategies and resulting allocation for both players. This characterizes the dynamics of the bidding sequence. For example, the table corresponding to the case \( b_{22} = b_{22} > b_{11} = b_{11} \) and \( b_2 > b_1 \) is unique allocation can be derived by symmetry.
bids will have been strictly decreased. Applying this reasoning iteratively, observing that
\( \frac{3q-c}{c-q} \epsilon > \frac{q}{c-q} \epsilon \) since \( 2q - c > 0 \), and that \( b_{i_{2}} - v_{i_{2}} + b_{i_{1}} - v_{i_{1}} \leq \frac{3q-c}{c-q} \epsilon \) for \( i = 1 \) or \( 2 \) implies
\( d(b_{i}, v_{1} \lor v_{2}) \leq \frac{3q-c}{c-q} \epsilon \), we can conclude that \( d(b_{i}(T), v_{1} \lor v_{2}) \leq (\frac{3q-c}{c-q} + 1) \epsilon \) for \( i \in \{1, 2\} \),
which completes the proof.

References