Decoherence: An Explanation of Quantum Measurement

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The description of the world given by quantum mechanics is at odds with our classical experience. Most of this conflict resides in the concept of "measurement". After explaining the origins of the controversy, and introducing and explaining some of the relevant mathematics behind density matrices and partial trace, I will introduce decoherence as a way of describing classical measurements from an entirely quantum perspective. I will discuss basis ambiguity and the problem of information flow in a system + observer model, and explain how introducing the environment removes this ambiguity via environmentally-induced superselection. I will use a controlled `not` gate as a toy model of measurement; including a one-bit environment will provide an example of how interaction can pick out a preferred basis, and included an N-bit environment will give a toy model of decoherence.

I. INTRODUCTION

For any physicist who learns classical mechanics before quantum mechanics, the theory of quantum mechanics at first appears to be a strange and counter-intuitive way to describe the world. The correspondence between classical experience and quantum theory is not obvious, and has been at the core of debates about the interpretation of quantum mechanics since its inception.

In this paper, I will begin by describing the measurement and interpretation problems of quantum mechanics. I will talk about interference, one of the essential characteristics of quantum systems. After describing a criterion for determining how close to classical a quantum system is, I will introduce decoherence, the thermodynamically irreversible transfer of information from a system to the environment; it provides a description of how to recover the illusion of classical mechanics from quantum mechanics. I will present a toy model of decoherence—a one-bit system with a one-bit measuring device, coupled to an N-bit environment—and calculate decoherence time. I will close with remarks about how close decoherence gets us to solving the interpretation and measurement problems, and what's left to be explained.

A. The Problem: Classicality

The so-called "measurement problem" is the problem of determining the correspondence between quantum mechanical descriptions of a system and our classical experience with measurements. One of the essential problems in finding such a correspondence is that there are far more permissible quantum mechanical states of a system than there are classical states. Classically, to specify the state of a system, you give the values of some small set of numbers, such as position, velocity, mass, charge, etc. Quantum mechanically, you must pick out an element of a vast Hilbert space, i.e., you must specify a complex function over each independent observable. Moreover, the law of superposition, which states that any linear combination of valid quantum states is also valid, gives rise to states that do not seem to correspond to any classical system anyone has experienced. The most famous example of such a state is that of Schrödinger's hypothetical cat, which is in a superposition of being alive and being dead.

As Zurek notes in [1], one might naïvely assume that the law of superposition should always be taken literally, that fundamentally, reality consists of quantum wavefunctions. From this point of view, there is no *a priori* reason to expect systems to have the well-defined localized behavior that we observe (such as definite position and momenta). Einstein noted in a 1954 letter to Born that such localization is incompatible with quantum mechanics, because time evolution tends to smear localized states. [1, 3] Additionally, such superpositions often involve interference effects, where probabilities do not add classically.

B. The Role of The Environment

Most systems in the world are open in the thermodynamic sense; they interact with the environment, which can be modeled as a "bath" with which a system can exchange information and energy. Decoherence is a description of the thermodynamically irreversible exchange of information with the environment. If we are largely incapable of measuring the environment, then decoherence quickly renders most quantum systems indistinguishable from classical ensembles of their possible (classical) states. We will see how quickly this happens by developing a toy model for decoherence and calculating the decoherence time in section VI C.

The environment is constantly interacting with most systems, especially large ones, and most measuring devices are rather large. The strength of that interaction usually dominates the strength of the interaction between the system being measured and the device being used to measure it. This results in what is called *environmentally induced super-selection*, or *einselection* for short. [1] The

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interaction between the observer and the environment picks out a preferred basis of states that are not (significantly) perturbed by further interactions; these are the *pointer states* that remain correlated with the system despite further interactions with the environment. Other states don't make for good measuring devices; it would be terribly inconvenient if your measurement results were scrambled every time a bit of light shined on them, or every time a bit of air passed by. We will see effect this more formally in section IV B and section VI B.

Though it is beyond the scope of this paper, Zurek develops a complementary description of this process in [1, 4], which he calls quantum Darwinism, in which the classically observable, einselected states are the ones that are best at propagating information about themselves through the environment. Very roughly, the idea is that we usually measure a system by interacting with a small subset of the environment (e.g., by measuring some small subset of all the photons that bounce off the system in question). So the relative frequencies of states that we observe are the relative "fitnesses" of those states in encoding their information in the environment. This description has the benefit that it does not necessarily require us to specify the measuring device as fundamental; the intrinsic characteristics of a system should not depend on how we measure that system.

II. ENVIRONMENT, SYSTEM, OBSERVERS

The formalism of decoherence relies on partitioning the universe into three disjoint, interacting systems: the environment, which will generally be denoted $\mathcal{E}$, the system we are studying, which will generally be denoted $\mathcal{S}$ or $\mathcal{A}$, and the observer, or the measuring device, which will often be denoted $\mathcal{O}$. The environment and the system are continuously interacting, and this forms the basis of decoherence. The measurement device interacts with the system for some short period of time, after which it is considered to have measured the system. We assume that the observers are largely incapable of measuring the environment. As we will see in section IV B and in our toy model in section VI C, after a short time, parts of the wave-function that project on to orthogonal elements of the Hilbert space $\mathcal{E}$ have little effect on each other; that is, there is very little interference between different states of the environment. Thus, from the point of view of the observer, the outcome of any measurement will be equivalent to the ensemble average of that measurement over the states of the environment. This will be made more rigorous in section IV B.

III. PRELIMINARY DEFINITIONS

This paper uses standard bra-ket notation ($\langle\psi|$and $|\psi\rangle$) for states; an arrow $|\psi_0\rangle \longrightarrow |\psi_t\rangle$ to denote the unitary time evolution from the state $|\psi_0\rangle$ at time 0 to the state $|\psi_t\rangle$ at time t; and tensor product $|\psi\rangle_1 \otimes |\phi\rangle_2$ to denote the entangled state of "object 1 in a state ψ and object 2 in a state ϕ". Occasionally, the tensor product symbol will be dropped and $|\psi\rangle_1|\phi\rangle_2$ will be used to save space. Quantum systems are denoted by script letters (e.g., $\mathcal{S}$), and sets of states by subscripted Latin letters in curly braces (e.g., $\{|s_i\rangle\}$). Unless otherwise specified, the states in these sets are assumed to be orthonormal.

IV. THE HALLMARK OF THE QUANTUM: INTERFERENCE AND RELATIVE PHASES

The primary difference between quantum mechanical states and classical states is the quantum mechanical law of superposition. It states that if $|\psi_1\rangle$ and $|\psi_2\rangle$ are valid states, then so is (up to normalization) $\alpha|\psi_1\rangle + \beta|\psi_2\rangle$ for arbitrary complex α and β. This is most easily seen in relative phases between terms and in interference experiments. [5]

A. Interference Terms

Classically, if there is a system $\mathcal{S}$ which has possible states ψ_i, we can suppose that we have an ensemble of such systems, in which the probability of one such system being in state ψ_i is $P(\psi_i)$. If we make a measurement of a physical observable ϕ which can take on values ϕ_j, then the probability of measuring it to be ϕ_j is the sum of the products of conditional probabilities:

$$P_{\mathrm{C}}(\phi_j) = \sum_i P(\phi_j|\psi_i)P(\psi_i). \tag{1}$$

Quantum mechanically, we may suppose that there is a system $\mathcal{S}$ which has possible states spanned by an orthonormal basis of (classically) observable states $\{|\psi_i\rangle\}$. Consider an ensemble of identically prepared systems, each in state

$$|\psi\rangle \equiv \sum_i \alpha_i|\psi_i\rangle. \tag{2}$$

If we measure the observable with eigenvectors $|\phi_j\rangle$, then the probability of measuring a system to be in a state $|\phi_j\rangle$ is

$$\begin{aligned} P_{\mathrm{QM}}(\phi_j) &\equiv |\langle\phi_j|\psi\rangle|^2 = \langle\phi_j|\psi\rangle\langle\psi|\phi_j\rangle \\ &= \sum_i |\alpha_i\langle\phi_j|\psi_i\rangle|^2 + \sum_{\substack{i,k \\ i\neq k}} \alpha_i\alpha_k^*\langle\phi_j|\psi_i\rangle\langle\psi_k|\phi_j\rangle. \end{aligned} \tag{3}$$

Define $P(\psi_i) \equiv |\langle\psi_i|\psi\rangle|^2 = |\alpha_i|^2$ to be the probability of measuring a system to be in state $|\psi_i\rangle$ (when measuring the observable with eigenvectors $|\psi_i\rangle$). Define $P(\phi_j|\psi_i) \equiv |\langle\phi_j|\psi_i\rangle|^2$ to be the probability of measuring a system initially in state $|\psi_i\rangle$ to be in state $|\phi_j\rangle$.

Then Equation 3 becomes

$$P_{\mathrm{QM}}(\phi_j) = \sum_i P(\phi_j|\psi_i)P(\psi_i) + \sum_{\substack{i,k \\ i\neq k}} \alpha_i\alpha_k^*\langle\phi_j|\psi_i\rangle\langle\psi_k|\phi_j\rangle$$
$$= P_{\mathrm{C}}(\phi_j) + \left\langle \phi_j \left| \sum_{\substack{i,k \\ i\neq k}} \alpha_i\alpha_k^*|\psi_i\rangle\langle\psi_k| \right| \phi_j \right\rangle. \qquad (4)$$

Note that for every term appearing in the sum over i and k on the first line, the conjugate of that term also appears, and so the sum is purely real.

Thus, the quantum mechanical probability of measuring a particular outcome is, in some sense, the classical probability of measuring that outcome, plus the expectation value of the sum of the cross-terms, the interference terms, *when we have picked a basis of classically observable states*; which terms are considered "interference terms" depends on which states we consider to be classically observable.

B. Ensemble Averages

Suppose now that the states of our system $\mathcal{S}$ live in the tensor product of the Hilbert spaces $\mathcal{A}$ and $\mathcal{E}$, which have orthonormal bases $\{|\psi_{\mathcal{A},i}\rangle\}$ and $\{|\psi_{\mathcal{E},k}\rangle\}$ of (classically) observable states of $\mathcal{A}$ and $\mathcal{E}$, respectively. If we have the ability to measure states of $\mathcal{A}$, but are incapable of distinguishing states of $\mathcal{E}$, so that the states $|\phi_j\rangle$ that we are measuring are elements of $\mathcal{A}$, then we can define the environment-agnostic overlap of a state $|\psi\rangle$ with a vector $|\psi_{\mathcal{A},i}\rangle \in \mathcal{A}$ to be

$$P_{\mathcal{E}}(\psi_{\mathcal{A},i}) \equiv \sum_k \langle\psi_{\mathcal{A}_i}| \otimes \langle\psi_{\mathcal{E},k}||\psi\rangle\langle\psi||\psi_{\mathcal{E},k}\rangle \otimes |\psi_{\mathcal{A}_i}\rangle$$
$$= \sum_k |\langle\psi_{\mathcal{A}_i}| \otimes \langle\psi_{\mathcal{E},k}||\psi\rangle|^2 \qquad (5)$$

which is a sum over an orthonormal basis of $\mathcal{E}$.[1] This is the norm squared of the overlap of the wave-function with $|\psi_{\mathcal{A},i}\rangle$. If we assume that all basis states $|\psi_{\mathcal{E},k}\rangle$ of $\mathcal{E}$ are equally likely, then we are simply adding up the joint probabilities $P(\psi_{\mathcal{A},i}$ and $\psi_{\mathcal{E},k})$.

If we express $|\psi\rangle$ in the tensor basis, then we can say

$$|\psi\rangle = \sum_m \alpha_m|\psi_{\mathcal{A},i_m}\rangle \otimes |\psi_{\mathcal{E},k_m}\rangle \qquad (6)$$

Then the probability of measuring the system $\mathcal{S}$ to be in a state $|\phi_j\rangle \in \mathcal{A}$ is

$$P_{\mathcal{E}}(\phi_j) \equiv \sum_k \langle\phi_j| \otimes \langle\psi_{\mathcal{E},k}||\psi\rangle\langle\psi||\psi_{\mathcal{E},k}\rangle \otimes |\phi_j\rangle$$
$$= \sum_m |\alpha_m\langle\phi_j|\psi_{\mathcal{A},i_m}\rangle|^2 \qquad (7)$$
$$+ \sum_{\substack{m,m' \\ m\neq m'}} \alpha_m\alpha_{m'}^*\langle\phi_j|\psi_{\mathcal{A},i_m}\rangle\langle\psi_{\mathcal{A},i_{m'}}|\phi_j\rangle\delta_{k_m,k_{m'}}$$

If we again define $P(\phi_j|\psi_{\mathcal{A},i}) \equiv |\alpha_m\langle\phi_j|\psi_{\mathcal{A},i}\rangle|^2$, then the first term is just the classical probability of measuring ϕ_j, and so Equation 7 becomes

$$P_{\mathcal{E}}(\phi_j) = P_{\mathrm{C}}(\phi_j) + \sum_{\substack{m,m' \\ m\neq m' \\ k_m=k_{m'}}} \alpha_m\alpha_{m'}^*\langle\phi_j|\psi_{\mathcal{A},i_m}\rangle\langle\psi_{\mathcal{A},i_{m'}}|\phi_j\rangle \qquad (8)$$

We again see that the quantum mechanical probability is the classical probability, plus an interference term. Note, again, that the interference term is dependent on our choice of basis for $\mathcal{E}$. (The overall value, however, is basis independent. This is due to a general property of the operation that we are doing, better known as a partial trace.) This time, the interference term dependent on how many terms of $|\phi_j\rangle$ project on to the same vector in $\mathcal{E}$ (i.e., for how many $m \neq m'$ we have $k_m = k_{m'}$). If, by and large, when considering only the terms in ψ with non-vanishing coefficients, there is a one-to-one correspondence between basis elements $|\psi_{\mathcal{A},i}\rangle$ of $\mathcal{A}$ and basis elements $|\psi_{\mathcal{E},k}\rangle$ of $\mathcal{E}$, then we see that the probability of any particular outcome is the same as it is classically, and we cannot distinguish this quantum system from the corresponding ensemble of classical ones.

This treatment, and the result that the relevant quantities to determining how far a system is from being classical are the terms that couple different basis elements of the environment, will be important for section VI C.

V. MEASUREMENT

A. Premeasurement

Consider a quantum system $\mathcal{S}$ initially in a state

$$|\psi\rangle = \sum_i a_i|s_i\rangle \qquad (9)$$

and a measuring apparatus $\mathcal{A}$ initially in a state $|A_0\rangle$. Let $\{|s_i\rangle\}$ and $\{|A_i\rangle\}$ be orthonormal bases for the system and the apparatus, respectively; the $|s_i\rangle$ are the possible states of the system that we can measure with our apparatus, and each $|A_i\rangle$ is the state that we observe the apparatus to have when it measures the system to be in the corresponding state $|s_i\rangle$. Letting $|A_0\rangle$ denote the

[1] I have tacitly assumed in this presentation that the environment is equally likely to be found in any classically observable pure state, and that this is a good representation of the environment. The more accurate way to model this is to construct a density matrix of combined $\mathcal{A}\otimes\mathcal{E}$ system, and then take the partial trace over the environment. See, for example, [6, §2.4] and [7, §2.1.8], for a good introduction to density matrices. Equation 5 is the appropriate matrix element of the partial trace over $\mathcal{E}$ of the density matrix $|\psi\rangle\langle\psi|$, denoted $Tr_{\mathcal{E}}(|\psi\rangle\langle\psi|)$. In this language, Equation 5 becomes $P_{\mathcal{E}}(\psi_{\mathcal{A},i}) = \langle\psi_{\mathcal{A},i}|\, Tr_{\mathcal{E}}(|\psi\rangle\langle\psi|)|\psi_{\mathcal{A},i}\rangle$.

initial state of the apparatus, a *premeasurement* on the system is a unitary time evolution from (10) to (11):

$$|\Psi_0\rangle \equiv |\psi\rangle \otimes |A_0\rangle = \left(\sum_i a_i |s_i\rangle\right) \otimes |A_0\rangle \tag{10}$$

$$\downarrow$$

$$|\Psi_t\rangle \equiv \sum_i a_i |s_i\rangle \otimes |A_i\rangle. \tag{11}$$

Informally, a premeasurement is an interaction between the system and the apparatus which does not change the state of the system when the apparatus is ignored.[2]

This step of premeasurement is not sufficient to measure $\mathcal{S}$, as we shall see in the next section, but it expresses the reason that we call $\mathcal{A}$ a measuring device: $\mathcal{A}$ attains a specific state corresponding to each possible state of the system, without disturbing the system.

B. Toy Model: c-`not`

Two-state systems are useful toy models for some quantum processes. The simplest toy model for measurement is when the system we are trying to measure is a two state system, often called the "target bit", and our measuring apparatus is also a two state system, often called the "control bit". When we refer to this objects as "bits", the two states are labeled 0 and 1.

Consider, as in [1], the logical operation "controlled `not`", which flips a target bit when a control bit is 1, and leaves the target bit alone when the control bit is 0:

$$0_c a_t \longrightarrow 0_c a_t; \qquad\qquad 1_c a_t \longrightarrow 1_c \neg a_t \tag{12}$$

where $\neg a_t$ is logical negation, sending 0 to 1 and 1 to 0. The quantum analog of this classical operation evolves by the rule

$$(\alpha|0\rangle_c + \beta|1\rangle_c)|a\rangle_t \to \alpha|0\rangle_c|a\rangle_t + \beta|1\rangle_c|\neg a\rangle_t \tag{13}$$

where the logical negation $|\neg a\rangle_t \equiv \neg|a\rangle_t$ is given by the rule

$$\neg(\gamma|0\rangle + \delta|1\rangle) \equiv \gamma|1\rangle + \delta|0\rangle \tag{14}$$

Letting $|A_0\rangle \equiv |0\rangle_t$ and $|A_1\rangle \equiv |1\rangle_t$ turns c-`not` in to a toy model for premeasurement.

In the classical controlled `not`, information unambiguously flows from the control to the target; knowing the state of the target bit before and after premeasurement is sufficient to determine the control bit. Knowing the state of the control bit before and after the premeasurement, on the other hand, tells you nothing; the control bit is unchanged by the target bit.

[2] More precisely, a premeasurement fixes the projection of the state onto $\mathcal{S}$.

It is tempting to think that the flow of information in the quantum controlled `not` system is also unambiguous. However, this is not the case. If we prepare the target bit in the state $|0\rangle_t$ or $|1\rangle_t$, then premeasurement transfers information from the control to the target. Consider, however, the $|\pm\rangle$ basis defined by

$$|\pm\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right). \tag{15}$$

Application of the transformation rule of Equation 13 to Equation 15, after straightforward calculation, gives us the new transformation rule

$$\begin{aligned} |\pm\rangle_c \otimes |+\rangle_t &\longrightarrow |\pm\rangle_c \otimes |+\rangle_t \\ |\pm\rangle_c \otimes |-\rangle_t &\longrightarrow |\mp\rangle_c \otimes |-\rangle_t. \end{aligned} \tag{16}$$

If we prepare the "target" (the "measuring device") in the state $|+\rangle_t$ or $|-\rangle_t$, then the target bit is unaffected by the premeasurement, and we have precisely the reverse of the situation we had before; knowing the state of the target bit before and after premeasurement gives no information about the control bit, but knowing the state of the control bit before and after gives complete information about the target bit. The same result occurs if we prepare the control bit in $|+\rangle_c$ or $|-\rangle_c$. Zurek claims in [1] that this basis ambiguity is present in all such target-control systems.

C. Three Descriptions of Measurement

Following Zurek in [1], I describe three different quantum mechanical descriptions of the same measurement.

Consider a quantum system $\mathcal{S}$ with orthonormal states $\{|s_i\rangle\}$, and three potential observers (measuring devices) $\mathcal{I}$, $\mathcal{D}$, and $\mathcal{O}$, all of which can measure $\mathcal{S}$. Say that the possible (classical) states of the measuring devices are $\{|I_i\rangle\}$, $\{|D_i\rangle\}$, and $\{|O_i\rangle\}$, respectively; say that "$\mathcal{I}$ is in state $|I_i\rangle$" corresponds to $\mathcal{I}$ measuring $\mathcal{S}$ to be in state $|s_i\rangle$ (for that i), and similarly for $\mathcal{D}$ and $\mathcal{O}$.

1. *The Insider*

The simplest case of quantum measurement corresponds to when the (classical) state of the system is already known with certainty; this is usually because it has just been measured. An *insider* is an observer already perfectly correlated with the state of the system; the insider can predict the outcome of his next measurement with certainty.

More formally, an insider, who already knows the state of $\mathcal{S}$ to be s_i, has this information recorded somewhere, say in measuring device $\mathcal{I}$ which is now in state I_i. A subsequent measurement of the system, using a measuring device $\mathcal{I}'$, looks like

$$|I'_0\rangle \otimes |I_i\rangle \otimes |s_i\rangle \longrightarrow |I'_i\rangle \otimes |I_i\rangle \otimes |s_i\rangle. \tag{17}$$

This is the familiar classical case of measurement; if you know the initial state of a classical system, and you know the (deterministic) laws according to which the system evolves, then you can predict with certainty what the outcome of a future measurement will be.

As Zurek notes in [1], every classical observer could aim, in principle, to be the ultimate insider; classically, if you ignore details about information storage limits, it is possible, in principle, to know the initial conditions of every particle in the universe, and thus be able to calculate, in principle, the outcome of every measurement ahead of time. In quantum mechanics, this is not possible, even in principle.

2. The Discoverer

Consider the standard description of quantum measurement: Observer $\mathcal{D}$ measures the system $\mathcal{S}$, and records a particular outcome s_i. If $\mathcal{D}$ did not ahead of time what the outcome of the measurement would be (i.e., if $\mathcal{D}$ describes the state of the system, before measurement, as a superposition of the states $|s_i\rangle$), then this observer is called a *discoverer*. If the discoverer also observes an insider $\mathcal{I}$, in addition to the system $\mathcal{S}$, then we can describe this observation mathematically by the (non-unitary) time evolution

$$|D_0\rangle \otimes \sum_i a_i |I_i\rangle \otimes |s_i\rangle \longrightarrow |D_i\rangle \otimes |I_i\rangle \otimes |s_i\rangle. \qquad (18)$$

This discoverer describes his observation with what is commonly called "collapse of the wave-function"; the discoverer might describe his measurement by saying: "When I measure $\mathcal{S}$, which is currently in a superposition state, I see it collapse into one of the classical states $|s_i\rangle$ with respective probabilities $|a_i|^2$. No matter which state I measure $\mathcal{S}$ to be in, I will percieve the insider $\mathcal{I}$ to be in agreement with my measurement, and to have known this outcome ahead of time."

3. The Outsider

An *outsider* is an observer who describes the process of $\mathcal{I}$ and $\mathcal{D}$ measuring $\mathcal{S}$, but taking no measurements of $\mathcal{S}$ (nor of $\mathcal{I}$ nor of $\mathcal{D}$) himself. Such an observer $\mathcal{O}$ would say that the insider starts off entangled with the state of the system, and, in measuring the system, the discoverer becomes entangled with the state of the system, but no collapse occurs. The mathematical description of this process (excluding the state of $\mathcal{O}$ for simplicity) is the time evolution

$$|D_0\rangle \sum_i a_i |I_i\rangle |s_i\rangle \longrightarrow \sum_i a_i |D_i\rangle |I_i\rangle |s_i\rangle. \qquad (19)$$

This is a unitary process, and well-described by what we called premeasurement above.

VI. DECOHERENCE

A. Environmentally Induced Superselection

B. Toy Model: c-not + Environment

Consider again the controlled `not` not gate of section V B, but instead of just a control/system $\mathcal{S}$ and a target/apparatus $\mathcal{A}$, consider also the environment $\mathcal{E}$.

As before, the first step is premeasurement by the apparatus of the system, in which the system, apparatus, and environment evolve as

$$\begin{aligned} |0\rangle_c \otimes |A_0\rangle \otimes |\varepsilon_*\rangle &\longrightarrow |0\rangle_c \otimes |A_0\rangle \otimes |\varepsilon_*\rangle \\ |1\rangle_c \otimes |A_0\rangle \otimes |\varepsilon_*\rangle &\longrightarrow |1\rangle_c \otimes |A_1\rangle \otimes |\varepsilon_*\rangle. \end{aligned} \qquad (20)$$

where $|\varepsilon_*\rangle$ is a generic state of the environment, and the other states are as they were defined above. In general, we have the transformation rule

$$(a|0\rangle_c + b|1\rangle_c)|A_0\rangle|\varepsilon_*\rangle \longrightarrow (a|0\rangle_c|A_0\rangle + b|1\rangle_c|A_1\rangle)|\varepsilon_*\rangle \qquad (21)$$

Let us suppose that the pointer states of the apparatus are einselected. That is, suppose that the states of the apparatus corresponding to our possible measurements, $|A_0\rangle$ and $|A_1\rangle$, are eigenstates of the interaction between the measurement device $\mathcal{A}$ and the environment $\mathcal{E}$. The environment then preforms a premeasurement on the apparatus and system, resulting in the transformation

$$\begin{gathered} (a|0\rangle_c|A_0\rangle + b|1\rangle_c|A_1\rangle)|\varepsilon_*\rangle \\ \downarrow \\ a|0\rangle_c|A_0\rangle|\varepsilon_0\rangle + b|1\rangle_c|A_1\rangle|\varepsilon_1\rangle \end{gathered} \qquad (22)$$

where $|\epsilon_0\rangle$ and $|\epsilon_1\rangle$ are the states of the environment that have premeasured the system. [3]

As Zurek notes in [1], the basis ambiguity of section V B is gone; in a tensor product of three or more spaces, the form of the interaction Hamiltonian and time evolution pick out an unambiguously preferred basis for us.

C. Toy Model: One Qubit + N Spins

1. The Model

Following Zurek, Cucchietti, and Paz in [1, 8], suppose that instead of a single bit, the environment consists of

[3] I am abusing notation somewhat by not specifying $|\varepsilon_*\rangle$; to make this a unitary transformation, either the coefficients have to change slightly, other terms have to be introduced, or the environment must be prepared in a special state. When we explicitly calculate decoherence time in the next section, we will see that this representation of the transformation is only approximate, and that in general other terms with exponentially suppressed coefficients are introduced.

N bits, or spins, initially in states $\alpha_k|0\rangle_e + \beta_k|1\rangle_e$, each coupled to the apparatus by a coupling coefficient g_k. In the simplest case, we can say that the self-Hamiltonian of both the system and the apparatus are zero. I consider the case where the system is first premeasured by the apparatus, and then undergoes decoherence. In analyzing decoherence, I drop the state of the system from the wave-function, for convenience, because it does not change. The interaction Hamiltonian is then

$$H = \sigma_z^{(\mathcal{S})} \otimes \sum_k g_k \sigma_z^{(e_k)} \tag{23}$$

where $\sigma_z^{(\mathcal{S})}$ and $\sigma_z^{(e_k)}$ the Pauli spin matrices for the system and environment respectively:

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|. \tag{24}$$

If we make the simplifying assumption that the environment begins in a pure state, then the Schrödinger equation gives the time evolution transformation rule

$$\begin{aligned} |\Psi(0)\rangle &\equiv (a|0\rangle_a + b|1\rangle_a) \bigotimes_{k=1}^{N} g_k \left(\alpha_k|0\rangle_{e_k} + \beta_k|1\rangle_{e_k}\right) \\ &\downarrow \\ |\Psi(t)\rangle &= a|0\rangle_a|\mathcal{E}_0(t)\rangle + b|1\rangle_a|\mathcal{E}_1(t)\rangle \end{aligned} \tag{25}$$

where

$$\begin{aligned} |\mathcal{E}_0(t)\rangle &\equiv \bigotimes_{k=1}^{N} \left(\alpha_k e^{ig_k t/\hbar}|0\rangle_{e_k} + \beta_k e^{-ig_k t/\hbar}|1\rangle_{e_k}\right) \\ &\equiv |\mathcal{E}_1(-t)\rangle \end{aligned} \tag{26}$$

and where $\bigotimes_k$ denotes a tensor product indexed by k.

2. Calculating Decoherence Time

We can ask the following question: as a function of time, how far from classical should a discoverer associated with the apparatus expect his measurements to be? Recall from section IV that what makes the quantum law of superposition different from the classical (probabilistic) law of superposition is the interference terms, or cross terms. Thus asking this question is thus approximately equivalent to asking for the expectation value of the sum of the interference terms, that is, the expectation over the environment of

$$ab^*|0\rangle_a|\mathcal{E}_0(t)\rangle\langle 1|_a\langle\mathcal{E}_1(t)| + a^*b|1\rangle_a|\mathcal{E}_1(t)\rangle\langle 0|_a\langle\mathcal{E}_0(t)| \tag{27}$$

which can be shortened to

$$2\Re\left(ab^*|0\rangle_a\langle 1|_a \otimes |\mathcal{E}_0(t)\rangle\langle\mathcal{E}_1(t)|\right). \tag{28}$$

Pick an orthonormal basis of the environment $\{|\varepsilon_i\rangle\}$. Then, as seen in section IV B, if we assume that all basis states of the environment are equally likely, the ensemble average over the environment is

$$\begin{aligned} &\sum_i \langle\varepsilon_i|2\Re\left(ab^*|0\rangle_a\langle 1|_a \otimes |\mathcal{E}_0(t)\rangle\langle\mathcal{E}_1(t)|\right)|\varepsilon_i\rangle \\ &= 2\Re\left(ab^*|0\rangle_a\langle 1|_a\langle\mathcal{E}_1(t)|\mathcal{E}_0(t)\rangle\right) \end{aligned} \tag{29}$$

Thus we see that the time dependence of the interference terms is given by

$$r(t) \equiv \langle\mathcal{E}_1(t)|\mathcal{E}_0(t)\rangle = \prod_{k=1}^{N}\left(|\alpha_k|^2 e^{2ig_k t/\hbar} + |\beta_k|^2 e^{-2ig_k t/\hbar}\right) \tag{30}$$

Since $|\alpha_k|^2 + |\beta_k|^2 = 1$ by normalization, this simplifies to

$$r(t) = \prod_{k=1}^{N}\left(\cos(2g_k t/\hbar) + i(1 - 2|\beta_k|^2)\sin(2g_k t/\hbar)\right). \tag{31}$$

We are interested primarily in how quickly this drops off to zero, if at all.

Suppose that the β_k and the g_k are randomly sampled from some distribution. If we assume that we are sampling from and ensemble of systems where $|\beta_k|$ are uniformly distributed between 0 and 1, then we can calculate the ensemble average of $r(t)$ by integrating over each $|\beta_k|$ from 0 to 1. If we suppose further, for simplicity, that the g_k are distributed normally with mean μ_g and variance σ_g^2, then we can calculate the ensemble average over g by integrating over a normal distribution for each g_k:

$$\langle r(t)\rangle = \left(e^{-2(\sigma_g t/\hbar)^2}\left(\cos(2t\mu_g/\hbar) + \frac{1}{3}i\sin(2t\mu_g/\hbar)\right)\right)^N. \tag{32}$$

Thus we see that we expect the interference terms to drop off as $e^{-2N(\sigma_g t/\hbar)^2}$, and the characteristic decoherence time scale of the system is $\hbar/(\sqrt{2N}\sigma_g)$.

Zurek, Cucchietti, and Paz treat this model more generally in [8], making fewer assumptions about the form of the interaction Hamiltonian.

VII. CONCLUDING REMARKS; WHAT'S LEFT?

This paper provides only a brief introduction to decoherence. For more complicated systems, density matrices and reduced density matrices are powerful tools that unify the calculations I presented. (See [6, §2.4] and [7, §2.1.8] for more details.) Zurek develops an alternative description of decoherence in [1, 4] based on information transfer and the ability of a state to propagate its information through the environment.

Decoherence provides a mathematical formalism for talking about the apparent "collapse" of the wave-function. With the aid of decoherence, we can recover the illusion of the classical; we can describe why observers of open quantum systems, both of which are constantly interacting with (and being measured by) the environment, observe systems to behave classically, on average.

Everett and Gleason independently proved in [9, 10] that the norm squared measure is the only possible probability measure on a Hilbert space of dimension ≥ 3. Zurek derived Born's Rule in [1, 4, 11] from the standpoint of decoherence, using the idea that if there are multiple possible states, and we are completely ignorant of which state the system is in, then there must be an equal probability of measuring the system to be in any of those given states.

As stated by Yudkowsky in [12], however, decoherence does not completely solve the measurement problem, at least in so far as I can tell. When we talk about probabilities classically, we talk about the probability of an event happening. We specify the set of possible outcomes of a particular process, and after the process completes, we will have observed one of those outcomes. One way of characterizing the probability of a particular outcome is by the frequency of that outcome in the limit of an infinite sequence of identically prepared processes. From the point of view of an outsider (see section V C 3), there seems to be no similar description for probability in describing a discoverer making a measurement. There is no ambiguity about the outcome of the measurement, as described by the outsider. So not only does probability seem to be subjective, in the sense that its definition requires you to be a discoverer, but the probability space itself, the set of possible outcomes that you label with likelihoods, seems to be subjective, in that it does not admit any remotely familiar classical description, so long as you are an outsider. While Zurek *et al.* give an operational definition of probability from decoherence, it does not seem to me be philosophically satisfying; decoherence is almost, but not quite, enough, to solve the measurement and interpretation problems.

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