Furuta Pendulum

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Project Goal and Motivation

**Goal:** Design, build, and control a rotational inverted pendulum, also known as a Furuta Pendulum

- Classic physical example for application of control theory
- Can be constructed with simple hardware and electronics
- Control can be demonstrated in class
Furuta Pendulum Design

- Encoder
- Coupling
- Shaft
- Casing
- Motor hub
- Bearings
Physical System

- Sparkfun 1024 P/R encoder
- mbed NXP LPC1768 microcontroller
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- Pololu 19:1 gearmotor w/encoder
- VNHI019 motor driver
- VNH5019 motor driver
- base
- rod
- pendulum
Electronics

- Motor power supply and encoder
- Pendulum shaft encoder
- USB-serial and programmer
- Power
System Model

Modeling Assumptions:

- Motor and pendulum have only viscous friction
- No backlash
- Rigid attachments from motor shaft to first link and from encoder to pendulum shaft
- All materials are homogeneous
- Encoder treated as black box with known weight
- When $\theta_2$ is in a range between ±15 degrees the system dynamics can be considered to be linear
- No base dynamics

Dynamics

- Modeled as two rotational rigid bodies
- Lagrangian formulation to solve for equations of motion
- Voltage-controlled motor model
- $\Theta_1$ is positive in the Z direction and $\Theta_2$ is positive in the x direction
- Velocities were calculated in the intermediate xyz frame
- The pendulum inertia is time-varying in the xyz frame:
  \[
  I_2^r = R(\theta_2)^T I_1 R(\theta_2)
  \]
\[ T = \frac{1}{2} m_1 v_{c1}^T v_{c1} + \frac{1}{2} \omega_1 \mathbf{I}_1 \omega_1 + \frac{1}{2} m_2 v_{c2}^T v_{c2} + \frac{1}{2} \omega_2 \mathbf{I}_2 \omega_2 \]

\[ P = m_2 g l_2 \cos(\theta_2) \]
Using Lagrange’s principle we can write the equation of motion for each variable:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = u - b_1 \dot{\theta}_1 \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = -b_2 \dot{\theta}_2
\]

And obtain the nonlinear equations in matrix form

\[
H \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + D(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) + G(\theta_1, \theta_2) = \begin{bmatrix} u \\ 0 \end{bmatrix}
\]

\[
\ddot{q} = H^{-1} \begin{bmatrix} u \\ 0 \end{bmatrix} - D - G = f(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, u)
\]
Finally, using tangent linearization for the upright position:

\[
\dot{X} = AX + Bu \\
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\hat{q}}
\end{bmatrix} = 
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24}
\end{bmatrix} X + 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} u
\]

\[
A_{11} = \left. \frac{\partial f_1}{\partial \theta_1} \right|_{x=x_0}; \quad A_{12} = \left. \frac{\partial f_1}{\partial \theta_2} \right|_{x=x_0}; \quad A_{13} = \left. \frac{\partial f_1}{\partial \dot{\theta}_1} \right|_{x=x_0}; \quad A_{14} = \left. \frac{\partial f_1}{\partial \dot{\dot{\theta}}_2} \right|_{x=x_0}
\]

\[
A_{21} = \left. \frac{\partial f_2}{\partial \theta_1} \right|_{x=x_0}; \quad A_{22} = \left. \frac{\partial f_2}{\partial \theta_2} \right|_{x=x_0}; \quad A_{23} = \left. \frac{\partial f_2}{\partial \dot{\theta}_1} \right|_{x=x_0}; \quad A_{24} = \left. \frac{\partial f_2}{\partial \dot{\dot{\theta}}_2} \right|_{x=x_0}
\]

\[
B_1 = \left. \frac{\partial f_1}{\partial u} \right|_{x=x_0}; \quad B_2 = \left. \frac{\partial f_2}{\partial u} \right|_{x=x_0}
\]

\[
C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad D[0]
\]

\[
X = [\theta_1 \quad \theta_2 \quad \dot{\theta}_1 \quad \dot{\theta}_2 \quad i]^T
\]

\[
u = e
\]

\[
X_0 = [0 \quad 0 \quad 0 \quad 0]^T
\]
Parameter Identification

- Link 2
  - \( C_2 \)

- Link 1
  - \( C_1 \)

Link inertias – from CAD

\[
I_{2x} \ddot{\theta}_2 + b_2 \dot{\theta}_2 + m_2 g l_2 \theta_2 = 0
\]

\[
\ddot{\theta}_2 + 2 \zeta \omega_n \dot{\theta}_2 + \omega_n^2 \theta_2 = 0
\]

\[
\frac{b_2}{l_{2x}} = 2 \zeta \omega_n
\]

Motor coefficients

- Pendulum damping
- Free pendulum decay

Motor resistance and inductance

- Angular velocity over time for different voltages
- Motor coefficients

\[
k_t = \frac{e - R_m i_{ss}}{\dot{\theta}_{1ss}}
\]

\[
b_1 = \frac{k_t i_{ss}}{\dot{\theta}_{1ss}}
\]
## Parameter Identification

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>9.8 m.$s^{-2}$</td>
<td>$l_{1z}$</td>
<td>0.0013 kg $m^2$</td>
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<tr>
<td>$b_1$</td>
<td>$6.0 \times 10^{-4} Nm/s$</td>
<td>$l_{2x}$</td>
<td>5.34 $10^{-4} kg m^2$</td>
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<tr>
<td>$b_2$</td>
<td>$5.52 \times 10^{-4} Nm/s$</td>
<td>$l_{2y}$</td>
<td>8.41 $10^{-4} kg m^2$</td>
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<tr>
<td>$m_1$</td>
<td>0.1862 kg</td>
<td>$l_{2z}$</td>
<td>3.10 $10^{-4} kg m^2$</td>
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<tr>
<td>$m_2$</td>
<td>0.065 kg</td>
<td>$l_{2xz}$</td>
<td>$-2.40 \times 10^{-4} kg m^2$</td>
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<tr>
<td>$L_1$</td>
<td>0.0955 m</td>
<td>$k_t$</td>
<td>0.182 Nm $A^{-1}$</td>
</tr>
<tr>
<td>$l_1$</td>
<td>0.03478 m</td>
<td>$R_m$</td>
<td>3.6 $\Omega$</td>
</tr>
<tr>
<td>$L_2$</td>
<td>0.2983 m</td>
<td>$L_m$</td>
<td>1.845 $mH$</td>
</tr>
<tr>
<td>$l_2$</td>
<td>0.05847 m</td>
<td>$J_m$</td>
<td>0.001 kg $m^2$</td>
</tr>
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</table>
Eigenstructure

\[ A = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0.0152 & 0.0003 & 0.0002 & 0.0928 .10^3 \\
0 & 0.0613 & 0.0002 & 0.0009 & 0.0740 \\
0 & 0 & 0.0986 & 0 & 1.9512 \\
\end{bmatrix} \]

\[ B = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
542.0 \\
\end{bmatrix} \]

\[ \lambda_1 = 0; \ \lambda_2 = -1946.5; \ \lambda_3 = 7.1; \ \lambda_4 = -9.3; \ \lambda_5 = -3.7; \]

\[ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \ v_2 = \begin{bmatrix} 0.0000 \\ 0.0000 \\ -0.0476 \\ -0.0380 \\ 0.9981 \end{bmatrix}; \ v_3 = \begin{bmatrix} -0.0218 \\ -0.1377 \\ -0.1545 \\ -0.9781 \\ 0.0078 \end{bmatrix}; \ v_4 = \begin{bmatrix} -0.0429 \\ -0.0982 \\ 0.3981 \\ 0.9108 \\ -0.0202 \end{bmatrix}; \ v_5 = \begin{bmatrix} 0.2470 \\ -0.0729 \\ -0.9256 \\ 0.2733 \\ 0.0469 \end{bmatrix}; \]
Controller Design

- Full-state feedback, LQR design
  - Calculated initial gains using a conservative, “expensive control” weighting matrix ratio, to avoid saturating our small motor
  - Increased max. allowable velocities while still allowing only small angle deviations
  - Gradually adjusted ratio towards “cheap control” to improve performance, while staying within actuator limits

- Angular velocities estimated with differentiators
  - Filtered differentiation, first order low-pass
  - Tuned filter roll-off frequencies manually to achieve a balance between velocity phase offset and noise rejection
  - Motor current also filtered with manually-tuned low-pass

- Non-linear swing up and down
  - Swing up by energy shaping (Tedrake Class 6.832)
  - Used only potential and major inertial term for axis 2
Linear Controller (within $\pm 15^\circ$)

$$\begin{align*}
K_{\text{upright}} &= \begin{bmatrix} -0.1 & 12.03 & -0.54 & 1.55 & 0.1684 \end{bmatrix}
\end{align*}$$
Swing Up Controller (Energy Shaping)

\( K_{\text{swing,up}} = 2.3 \)
\( K_{\text{swing,down}} = -1.8 \)

\[
E_{\text{upright}} = m \cdot g \cdot l_2 = 0.372 [J]
\]
\[
E_{\text{current}} = \frac{I_{2x} \cdot \omega_2^2}{2} + m_2 \cdot g \cdot l_2 \cdot \cos(\hat{\theta}_2)
\]
Simulation Results

- Applied Voltage
  - Graph showing voltage input over time.

- Estimated velocities
  - Graph showing estimated velocities over time.

- Angular position for pendulum and base
  - Graph showing angular position over time.
Experimental Results

- **Position**: Angular Positions During Swing-Up and Stabilization
  - $\theta_1$
  - $\theta_2$

- **Velocity**: Velocities During Swing-Up and Stabilization
  - $\dot{\theta}_1$
  - $\dot{\theta}_2$

- **Motor Voltage**: Motor Voltage During Swing-Up and Stabilization

- **Motor Current**: Motor Current During Swing-Up and Stabilization
  - Unfiltered
  - Filtered
Conclusions and Future Work

- Successful build and control of the Furuta pendulum
- Careful modeling and parameter estimation are key
- Highly nonlinear, un-modeled phenomena e.g. static friction and backlash are unavoidable (and annoying)
- Control of the Furuta pendulum can be developed much further – addition of more sophisticated nonlinear and/or adaptive control
Demo!