TABLE II
CHANGE IN kth DEPARTURE TIME

		PA % Error			
k	Actual δd_k	M=2	M=8	M=128	M=∞
100	-44.68	-14.7 %	-14.8 %	0.0 %	0.0 %
200	-85.36	- 4.9 %	6.2 %	0.0 %	0.0 %
400	-190.33	-16.1 %	2.3 %	1.5 %	0.0 %
800	-387.67	- 9.3 %	0.1 %	-2.9 %	0.0 %
1,600	-789.24	-11.5 %	- 2.0 %	-5.6 %	0.0 %
3,200	-1,611.96	-14.2 %	- 6.1 %	-4.2 %	0.0 %
6,400	-3,132.84	- 8.8 %	- 3.1 %	-0.5 %	0.0 %
12,800	-6,380.58	- 8.4 %	- 4.4 %	-0.9 %	0.0 %
25,600	-12,837.95	- 7.9 %	- 3.7 %	-0.8 %	0.0 %
51,200	-25,631.25	- 7.9 %	- 4.2 %	-0.4 %	0.0 %

TABLE III
CHANGE IN MEAN DELAY

		PA % Error			
k	Actual δD	M=2	M=8	M=128	M=∞
100	0.4032	7.5 %	-1.4 %	0.0 %	0.0 %
200	0.5136	2.6 %	-3.0 %	-0.2 %	0.0 %
400	0.4957	0.7 %	1.0 %	-3.5 %	0.0 %
800	0.4805	-1.0 %	-1.2 %	0.0 %	0.0 %
1,600	0.4918	-2.9 %	-2.8 %	0.8 %	0.0 %
3,200	0.4976	-5.6 %	-3.6 %	-0.7 %	0.0 %
6,400	0.4947	-4.8 %	-2.6 %	-1.0 %	0.0 %
12,800	0.5037	-5.1 %	-2.5 %	-1.4 %	0.0 %
25,600	0.5052	-5.5 %	-1.4 %	-0.9 %	0.0 %
51,200	0.4966	-5.9 %	-2.4 %	-0.6 %	0.0 %

V. CONCLUSIONS

For stochastic discrete event systems, PA provides a methodology for performing on-line optimization by estimating performance gradients. PA is particularly efficient when the parametric perturbations of interest affect event times only—not queue lengths. In this note, we have attempted to provide extensions for the latter case, by considering a system with a flow control strategy based on the queue length seen by arriving customers. The resulting algorithm is simple, but is limited by the amount of state memory required when the arrival process is not deterministic. Constraining the state memory, we have included in Section IV experimental results suggesting that the approach can still provide accurate estimates.

REFERENCES

[1] X. Cao, "Convergence of parameter sensitivity estimates in a stochastic experiment," *IEEE Trans. Automat. Contr.*, vol. AC-30, Sept. 1985.

[2] C. G. Cassandras and Y. C. Ho, "An event domain formalism for sample path perturbation analysis of discrete event dynamic systems," *IEEE Trans. Automat. Contr.*, vol. AC-30, Dec. 1985.

[3] C. G. Cassandras, "Sensitivity analysis of a simple routing strategy," in *Proc. 24th IEEE Conf. Decision Contr.*, 1985.

[4] ——, "Sample path analysis and optimization of a threshold-dependent routing strategy," Dept. Elec. Comput. Eng., Univ. Massachusetts, Amherst, Tech. Rep., 1986.

[5] Y. C. Ho and C. G. Cassandras, "A new approach to the analysis of discrete event dynamic systems," *Automatica*, vol. 19, 1983.

[6] Y. C. Ho and X. Cao, "Perturbation analysis and optimization of queueing networks," *J. Optimiz. Theory Appl.*, vol. 40, 1983.

[7] Y. C. Ho, X. Cao, and C. G. Cassandras, "Infinitesimal and finite perturbation analysis for queueing networks," *Automatica*, vol. 19, 1983.

[8] A. Segall, "The modeling of adaptive routing in data-communication networks," *IEEE Trans. Commun.*, vol. COM-25, 1977.

[9] S. Stidham, "Optimal control of admission to a queueing system," *IEEE Trans. Automat. Contr.*, vol. AC-30, Aug. 1985.

[10] R. Suri and M. A. Zazanis, "Comparison of perturbation analysis with conventional sensitivity estimates for regenerative stochastic systems," in *Proc. 24th IEEE Conf. Decision Contr.*, 1985.

Analysis of a Multiaccess Control Scheme

JOHN N. TSITSIKLIS

Abstract—**We consider a multiaccess channel under the infinite source model and ternary feedback. We consider a recently proposed scheme for the decentralized control of transmissions through the channel, and we prove that it is stable, as long as the rate of generation of new packets is smaller than e^{-1}.**

I. DESCRIPTION OF THE CONTROL SCHEME

Consider the usual slotted ALOHA model, under the infinite source assumption and ternary feedback. In more detail, there is an infinite number of stations and, at the beginning of any time slot, each station may have at most one packet to transmit. Any station with an available packet may decide to attempt transmission (possibly using a probabilistic rule) or to decide to defer this attempt for later. Let Y_t be the number of attempted transmissions during the tth slot. If $Y_t = 0$, we say that a "hole" has occurred. If $Y_t = 1$, the (single) attempted transmission is successful. Finally, if $Y_t \geq 2$, there is a collision and no packet is successfully transmitted. At the end of the tth slot, all stations learn whether a hole, a success, or a collision has occurred. Accordingly, we define the variable Z_t to be equal to Y_t, if $Y_t < 2$, and equal to 2, if $Y_t \geq 2$. The information available to any station at the beginning of the tth slot is the collection of variables $Z_1, \cdots, Z_{t-1}$. The decision of a station, whether it will attempt transmission during the tth slot, is constrained to be a function of $Z_1, \cdots, Z_{t-1}$ and possibly an internal random number generator.

We assume that during the tth slot, a random number A_t of new stations generate a packet which they would like to eventually transmit. We assume that the random variables A_t are independent and identically distributed according to a Poisson distribution with mean λ. Let N_t be the number of stations with a packet available for transmission at the beginning of the tth slot. Then, N_t evolves as follows: $N_{t+1} = N_t + A_t - 1$, if $Z_t = 1$; $N_{t+1} = N_t + A_t$, otherwise.

The objective is to find a probabilistic rule that lets each station decide at any given time, using only the information available to it, whether it will transmit or not. (Of course, this rule will be used only by those stations that have an available packet.) This rule should be stable, that is, the stochastic process N_t should not "explode" in a suitable mathematical sense.

Rivest [1] has suggested the following strategy. At the beginning of the tth slot, each station has available the same estimate $\hat{N}_t$ of N_t. Each station with an available packet attempts transmission with probability $1/\hat{N}_t$. Conditioned on $\hat{N}_t$, the decisions of different stations are statistically independent and independent of any other events that have occurred in the past. (It is not hard to show that if N_t is large and if $\hat{N}_t = N_t$, then the above choice of transmission probability is optimal, in the sense that it maximizes the probability of a successful transmission during the tth slot.) The novelty of the scheme lies in the procedure for updating the estimate $\hat{N}_t$, which is the following:

$$\text{i) If } Z_t < 2, \quad \text{then } \hat{N}_{t+1} = \max\{1, \hat{N}_t - 1 + \hat{\lambda}\}; \tag{1.1}$$

$$\text{ii) If } Z_t = 2, \quad \text{then } \hat{N}_{t+1} = \hat{N}_t + \frac{1}{e-2} + \hat{\lambda}. \tag{1.2}$$

In these equations, $\hat{\lambda}$ is an estimate of λ.

This updating procedure is motivated in [1] as an approximation of the exact Bayesian formula for updating the optimal estimate $E[N_t | Z_0, \cdots,$

Manuscript received April 21, 1986; revised December 8, 1986. This work was suported by the Army Research Office under Contract DAAAG-29-84-K-005 and by the Office of Naval Research under Grant ONR/N00014-77-C-0532.

The author is with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA 02139.

IEEE Log Number 8613538.

Z_{t-1}]. We define $X_t = (N_t, \hat{N}_t)$ and we notice that, for any fixed values of λ and $\hat{\lambda}$, X_t is a Markov process taking values in a countable state space, assuming that $\hat{N}$ is initialized with an integer value.

In this correspondence, we analyze the stability of this scheme. The technique we employ is somewhat simpler than the one used in [2]–[3] and is based on a suitable Lyapunov function. Our main results may be summarized as follows.

a) If λ is known exactly, and therefore, $\hat{\lambda} = \lambda$, then the Markov process $(N_t, \hat{N}_t)$ is geometrically ergodic if $\lambda < e^{-1}$. We should point out here that no control strategy, in which all stations use the same probability of transmission, could achieve throughput larger than or equal to e^{-1} and, in this sense, the above scheme is optimal.

b) If λ is not known exactly, but rather an inexact estimate $\hat{\lambda}$ is used in the updating equations (1.1), (1.2), then the scheme is stable (geometrically ergodic) if $\lambda < e^{-1}$, $\hat{\lambda} \le e^{-1}$, and $\lambda \le \hat{\lambda}$. We also provide a heuristic argument which suggests that if $\hat{\lambda} < \lambda$ and if the difference between λ and $\hat{\lambda}$ exceeds a certain threshold, then instability may result, even if $\lambda < e^{-1}$.

In [1], it is suggested that $\hat{\lambda}$ could be formed by estimating λ on line. In particular, one may let $\hat{\lambda}_t$, the estimate at time t, be equal to the number of successful transmissions so far, divided by the time elapsed. Alternatively, one may use a sliding window, or discount past successes, so that the estimators $\hat{\lambda}_t$ retain their adaptivity, as $t \to \infty$. It is not known whether stability is preserved when such an estimator for λ is used. Nevertheless, our results show that an estimator for λ is not needed. We may simply use $\hat{\lambda} = e^{-1}$ and this guarantees the same throughput as the throughput which would have been obtained for the case where λ is known.

Related Research: References [2] and [3] have presented and proved stability of a scheme which achieves a throughput of e^{-1}. Its difference from the scheme in (1.1)–(1.2) is that $\hat{N}_t$ is incremented in a multiplicative, as opposed to additive, fashion. The performance of that scheme has been studied in [7]. This stability result of [2]–[3] has been extended in [6], for the case where the information available to the stations is corrupted by a discrete memoryless channel. Reference [8] (see also [9]) presents a related scheme based on the exact implementation of the optimal (least squares) estimator for $\hat{N}_t$, and analyzes it for the limiting case of a large but finite number of stations.

Reference [4] mentions a related class of schemes which have been introduced and analyzed in [5]. The discussion in [4] indicates that the scheme introduced in [1], with $\hat{\lambda} = e^{-1}$, is a special case of those analyzed in [5] and that, in particular, it is known that the scheme is stable, although the proof has not been published. In this light, the main contribution of this correspondence is a simple technique for rederiving this stability result.

II. Mathematical Preliminaries

A random variable W is exponential-type if there exist $d > 0$, D such that $E[e^{d|W|}] \le D$. Let $\{X_k\}$ be an irreducible aperiodic Markov chain on a countable state space. We say that $\{X_k\}$ is geometrically ergodic if there exists a state x such that the stopping time $\tau = \min\{t > 0: X_t = x\}$ is exponential-type, for any initial state X_0.

Let $\{W_t\}$ be a sequence of random variables adapted to an increasing family $\{\mathfrak{F}_t\}$ of σ-fields. We say that $\{W_t, \mathfrak{F}_t\}$ is exponential-type if there exist $d > 0$, D, such that

$$E[e^{d\cdot|W_{t+1}-W_t|}|\mathfrak{F}_t] \le D, \quad \forall t \ge 0. \tag{2.1}$$

We will use the following result of [2].

Proposition 2.1: Suppose that W_0 is deterministic, that $\{W_t, \mathfrak{F}_t\}$ is exponential-type, and that for some $\epsilon > 0$, $a \in \mathbb{R}$, we have

$$E[W_{t+1} - W_t + \epsilon; \quad W_t > a|\mathfrak{F}_t] \le 0, \quad \forall t \ge 0.^1 \tag{2.2}$$

Then, for each value of W_0, the stopping time $\tau = \min\{t \ge 0: W_t \le a\}$ is exponential-type.

We will also need the following result which is proved in a way similar to the results of [2].

Proposition 2.2: Let $\{W_t, \mathfrak{F}_t\}$ be exponential-type, with $W_0 = 0$, and let J be a positive integer. Let τ be a stopping time (with respect to $\{\mathfrak{F}_t\}$) and assume that there exists some $\epsilon > 0$ such that $E[W_{t+1} - W_t + \epsilon; \tau > t|\mathfrak{F}_t] \le 0$, $\forall t$. Then there exists some B (depending only on d, D, ϵ, but not on J or the statistics of τ) such that $E[\max\{0, W_J\}; \tau > J|\mathfrak{F}_0] \le B$.

Proof: Since $\{W_t, \mathfrak{F}_t\}$ is exponential-type and using our assumption on $W_{t+1} - W_t$, there exists some $\eta > 0$ (depending only on d, D, ϵ) such that

$$E[e^{\eta(W_{t+1}-W_t)}; \quad \tau > t|\mathfrak{F}_t] \le 1, \quad \forall t. \tag{2.3}$$

(This is proved in Lemma 2.1 of [2].) We use the inequality $\max\{0, x\} \le (1/\eta)e^{\eta x}$, $\forall x$, to obtain

$$E[\max\{0, W_J\}; \quad \tau > J|\mathfrak{F}_0] = E[\max\{0, W_{J\wedge\tau}\}; \quad \tau > J|\mathfrak{F}_0]$$
$$\le \frac{1}{\eta} E[e^{\eta W_{J\wedge\tau}}; \quad \tau > J|\mathfrak{F}_0] \le \frac{1}{\eta} E[e^{\eta W_{J\wedge\tau}}|\mathfrak{F}_0].$$

We now notice that the stochastic process $e^{\eta W_{k\wedge\tau}}$ is a supermartingale, as a consequence of (2.3). Therefore, $E[e^{\eta W_{J\wedge\tau}}|\mathfrak{F}_0] \le 1$, which gives the desired result with $B = 1/\eta$. ∎

III. Main Result

Theorem 3.1: If $0 < \hat{\lambda} \le e^{-1}$, $0 < \lambda < e^{-1}$, and $\lambda \le \hat{\lambda}$, then the Markov process X_t, defined in Section I, is geometrically ergodic.

Proof: We will be using the notation $\tilde{N}_t = \hat{N}_t - N_t$ and $\tilde{\lambda} = \hat{\lambda} - \lambda$. We also define $\mathfrak{F}_t$ as the σ-field generated by $\{A_{s-1}, N_s, \hat{N}_s: s \le t\}$. We start by establishing approximate formulas for the drift of N_t and $\tilde{N}_t$. We define two functions on the state space: $c(N, \hat{N}) = E[N_{t+1} - N_t|X_t = (N, \hat{N})]$ and $d(N, \hat{N}) = E[\tilde{N}_{t+1} - \tilde{N}_t|X_t = (N, \hat{N})]$. Using the binomial probability formulas, we obtain

$$c(N, \hat{N}) = \lambda - \frac{N}{\hat{N}}\left(1 - \frac{1}{\hat{N}}\right)^{N-1}, \tag{3.1}$$

$$d(N, \hat{N}) = \frac{1}{e-2}\left(1 - \frac{N}{\hat{N}}\left(1 - \frac{1}{\hat{N}}\right)^{N-1} - \left(1 - \frac{1}{\hat{N}}\right)^{N}\right) - \left(1 - \frac{1}{\hat{N}}\right)^{N} + \tilde{\lambda}. \tag{3.2}$$

We also introduce a function $f: [0, \infty) \times (0, e^{-1}] \mapsto \mathbb{R}$, defined by

$$f(\alpha, \tilde{\lambda}) = \frac{1}{e-2}(1 - \alpha e^{-\alpha} - e^{-\alpha}) - e^{-\alpha} + \tilde{\lambda}. \tag{3.3}$$

Lemma 3.1: There exists a function $h: \mathbb{R} \mapsto \mathbb{R}$ such that $\lim_{M\to\infty} h(M) = 0$ and such that, if $N \ge M$ or $\hat{N} \ge M$, then

$$|c(N, \hat{N}) - (\lambda - \alpha e^{-\alpha})| \le h(M),$$
$$|d(N, \hat{N}) - f(\alpha, \tilde{\lambda})| \le h(M),$$

where $\alpha = N/\hat{N}$.

Proof (Outline): For any fixed value of α, the result is immediate from the formulas (3.1), (3.2), and the fact $\lim_{\hat{N}\to\infty}(1 - 1/\hat{N})^{\hat{N}} = e^{-1}$. The fact that the bounds are actually uniform over all α may be easily demonstrated by working out an exact expression for the approximation error, or by appealing to the similar bounds developed in [3]. ∎

We now study the properties of the function f.

Lemma 3.2:

i) For any $\tilde{\lambda}$, the function f is strictly increasing in α.

ii) For any $\tilde{\lambda} \in (0, e^{-1}]$, there exists a unique $\alpha = g(\tilde{\lambda}) \in (0, 1]$ such that $f(\alpha, \tilde{\lambda}) = 0$.

iii) If $\lambda \in (0, e^{-1}) \cap (0, \hat{\lambda}]$, then $g(\tilde{\lambda})e^{-g(\tilde{\lambda})} > \lambda$.

Proof:

i) This is implied by the inequality $(\partial f/\partial\alpha)(\alpha, \tilde{\lambda}) = (1/(e-2))\alpha e^{-\alpha} + e^{-\alpha} > 0$, $\forall \alpha > 0$.

ii) Existence of a solution in the desired range follows from $f(0, \tilde{\lambda}) = -1 + \tilde{\lambda} < 0$, $f(1, \tilde{\lambda}) = \tilde{\lambda} \ge 0$, and the continuity of f. Uniqueness follows from the strict monotonicity of f.

[1] If X is a random variable and A is an event, the notation $E[X; A|\mathfrak{F}]$ stands for $E[XI_A|\mathfrak{F}]$, where I_A is the indicator function of the event A.

iii) Suppose first that $\lambda = \hat{\lambda}$. Then, $g(\tilde{\lambda}) = g(0) = 1$ and $g(0)e^{-g(0)} = e^{-1} > \lambda$, as desired. So suppose that $\lambda \neq \hat{\lambda}$. We use the equation $(df/d\tilde{\lambda})(g(\tilde{\lambda}), \tilde{\lambda}) = 0$, to obtain

$$\frac{dg}{d\tilde{\lambda}}(\tilde{\lambda}) = \frac{-1}{\frac{1}{e-2}g(\tilde{\lambda})e^{-g(\tilde{\lambda})} + e^{-g(\tilde{\lambda})}}.$$

Thus,

$$\frac{d}{d\tilde{\lambda}}[g(\tilde{\lambda})e^{-g(\tilde{\lambda})}] = [1-g(\tilde{\lambda})]e^{-g(\tilde{\lambda})}\frac{dg}{d\tilde{\lambda}}(\tilde{\lambda}) \geq \frac{-1}{\frac{1}{e-2}g(\tilde{\lambda})+1} > -1.$$

Hence, for $\tilde{\lambda} > 0$, we have $g(\tilde{\lambda})e^{-g(\tilde{\lambda})} > g(0)e^{-g(0)} - \tilde{\lambda} = e^{-1} - \tilde{\lambda} \geq \hat{\lambda} - \tilde{\lambda} = \lambda$. ■

From now on we use β to denote the value of $g(\tilde{\lambda})$. Given any $\gamma \in (0, \beta)$ and $M > 0$, we partition the state space into four regions as follows. We let

$$S_{\gamma,M} = \left\{(N, \hat{N}) : N \geq M \text{ or } \hat{N} \geq M, \quad \beta - \gamma \leq \frac{N}{\hat{N}} \leq 1 + \gamma\right\},$$

$$R^-_{\gamma,M} = \left\{(N, \hat{N}) : \hat{N} \geq M, \frac{N}{\hat{N}} < \beta - \gamma\right\},$$

$$R^+_{\gamma,M} = \left\{(N, \hat{N}) : N \geq M, \frac{N}{\hat{N}} > 1 + \gamma\right\},$$

$$Q_M = \{(N, \hat{N}) : N < M, \quad \hat{N} < M\}.$$

We also let $R_{\gamma,M} = R^-_{\gamma,M} \cup R^+_{\gamma,M}$.

Lemma 3.3: There exist some $M > 0$, $\gamma > 0$, $\delta > 0$, such that $5\gamma < \beta$ and

$$c(N, \hat{N}) \leq -\delta, \quad \forall (N, \hat{N}) \in S_{5\gamma,M}, \tag{3.4}$$

$$d(N, \hat{N}) \leq -\delta, \quad \forall (N, \hat{N}) \in R^-_{\gamma,M}, \tag{3.5}$$

$$d(N, \hat{N}) \geq \delta, \quad \forall (N, \hat{N}) \in R^+_{\gamma,M}. \tag{3.6}$$

Proof: Notice that $\lambda - \alpha e^{-\alpha}$ is negative when $\alpha = 1$ (because $\lambda < e^{-1}$) as well as when $\alpha = \beta$ [because of Lemma 3.2 iii)] and is monotonic in between. Furthermore, it is a continuous function of α and therefore there exist $\gamma > 0$ and $\delta_1 > 0$ such that $\lambda - \alpha e^{-\alpha} \leq -\delta_1$, $\forall \alpha \in [\beta - 5\gamma, 1 + 5\gamma]$. Hence, using Lemma 3.1, $c(N, \hat{N}) \leq -\delta_1 + h(M) \leq -\delta_1/2$, $\forall (N, \hat{N}) \in S_{5\gamma,M}$, provided that we take M large enough so that $h(M) \leq \delta_1/2$. This proves (3.4) and fixes our choice of γ. For inequalities (3.5) and (3.6), we use the strict monotonicity of f to conclude that $f(\alpha, \tilde{\lambda}) \leq f(\beta - \gamma, \tilde{\lambda}) < 0$, $\forall \alpha \leq \beta - \gamma$ and $f(\alpha, \tilde{\lambda}) \geq f(1 + \gamma, \tilde{\lambda}) > 0$, $\forall \alpha \geq 1 + \gamma$. The desired result follows again by choosing M large enough and using Lemma 3.1. ■

From now on we assume that M and γ have been fixed and that inequalities (3.4)–(3.6) hold. We introduce a Lyapunov function which exploits the properties of the drift of $(N_t, \hat{N}_t)$ in the regions we introduced earlier. Namely, we let

$$V(N, \hat{N}) = \max\left\{N, \frac{1+3\gamma}{3\gamma}(N - \hat{N}), \quad \frac{\beta - 3\gamma}{1 - \beta + 3\gamma}(\hat{N} - N)\right\} \tag{3.7}$$

and notice that the first, second, and third expression inside the brackets becomes effective when N belongs to $S_{3\gamma,M}$, $R^+_{3\gamma,M}$, $R^-_{3\gamma,M}$, respectively. Unfortunately, for any $\Delta > 0$, the inequality $E[V(N_{t+1}, \hat{N}_{t+1})|(N_t, \hat{N}_t)] \leq V(N_t, \hat{N}_t) - \Delta$ fails to hold at the boundary between adjacent regions. However, we will show below that, if J is chosen large enough, then there exists some $\Delta > 0$ such that

$$E[V(N_{t+J}, \hat{N}_{t+J})|(N_t, \hat{N}_t) = (N, \hat{N})] \leq V(N, \hat{N}) - \Delta;$$
$$\forall (N, \hat{N}) \notin Q_{M+J^2}. \tag{3.8}$$

Our method consists of estimating the decrease in V by separately considering likely and unlikely events, starting with unlikely ones. Given some integer J and some $t \geq 0$, we define a random variable τ_J by $\tau_J = \min\{s \geq t : \sum_{k=t}^{t+s} A_k \geq J\}$, where A_k is the number of new packets generated at time k. We then have the following two auxiliary results whose proof is straightforward (using, for example, the same methods as in the proof of Proposition 2.2) and is omitted:

$$\lim_{J\to\infty} JP(\tau_J \leq J) = 0, \tag{3.9}$$

$$\lim_{J\to\infty} E\left[c_1 J + c_2 \sum_{k=t}^{t+J} A_k; \tau_J \leq J\right] = 0, \quad \forall c_1, c_2. \tag{3.10}$$

We notice that $|N_{t+1} - N_t| \leq 1 + A_t$ and $|\hat{N}_{t+1} - \hat{N}_t| \leq 1/(e-2) + \hat{\lambda} \leq 2 + \hat{\lambda} \leq 3$. It then follows from (3.7) that

$$|V(N_{t+1}, \hat{N}_{t+1}) - V(N_t, \hat{N}_t)| \leq \max\left\{1, \frac{1+3\gamma}{3\gamma}, \frac{\beta - 3\gamma}{1 - \beta + 3\gamma}\right\}(4 + A_t)$$
$$\leq C(1 + A_t),$$

for some constant C. Therefore, there exists some C, independent of J, such that

$$|V(N_{t+J}, \hat{N}_{t+J}) - V(N_t, \hat{N}_t)| \leq CJ + C\sum_{k=t}^{t+J} A_t.$$

Using (3.10), we see that if J is chosen large enough, then

$$E[V(N_{t+J}, \hat{N}_{t+J}) - V(N_t, \hat{N}_t); \quad \tau_J \leq J|(N_t, \hat{N}_t)]$$

can be made as close to zero as desired.

We now consider the event $\tau_J > J$.

Lemma 3.4: J can be chosen large enough so that, if $\tau_J > J$, then the following are true.

i) If $X_t \in S_{2\gamma,M-J^2}$, then $X_{t+k} \in S_{3\gamma,M}$, $\forall k \in [0, J]$.

ii) If $X_t \in S_{4\gamma,M+J^2} \cap R^+_{2\gamma,M-J^2}$, then $X_{t+k} \in S_{5\gamma,M} \cap R^+_{\gamma,M}$, $\forall k \in [0, J]$.

iii) If $X_t \in R^+_{4\gamma,M+J^2}$, then $X_{t+k} \in R^+_{3\gamma,M}$, $\forall k \in [0, J]$.

iv) Statements ii) and iii) remain true if we replace R^+ by R^-.

Proof: If $\tau_J > J$, then $|N_{t+k} - N_t| \leq 2J$ and $|\hat{N}_{t+k} - \hat{N}_t| \leq 3J$, $\forall k \in [0, J]$. On the other hand, notice that the distance between $S_{2\gamma,M+J^2}$ and the complement of $S_{3\gamma,M}$ is of the order of J^2 and part i) follows. The proof is similar for the remaining parts of the Lemma and is omitted. ■

From now on, we assume that J is large enough so that the statements of Lemma 3.4 hold. We start by considering the case $(N_t, \hat{N}_t) \in S_{2\gamma,M-J^2}$. Then, $V(N_t, \hat{N}_t) = N_t$. If, in addition, $\tau_J > J$, then $(N_{t+k}, \hat{N}_{t+k}) \in S_{3\gamma,M+J^2}$, $\forall k \in [0, J]$ (by Lemma 3.4) and $V(N_{t+J}, \hat{N}_{t+J}) = N_{t+J}$. Thus, using (3.4) and assuming that J is large enough so that $P(\tau_J > J) > 1/2$ [which is possible, due to (3.9)] we obtain

$$E[V(N_{t+J}, \hat{N}_{t+J}) - V(N_t, \hat{N}_t); \quad \tau_J > J|(N_t, \hat{N}_t)]$$
$$= E[N_{t+J} - N_t; \quad \tau_J > J|(N_t, \hat{N}_t)]$$
$$= \sum_{k=0}^{J-1} E[c(N_{t+k}, \hat{N}_{t+k});$$
$$\tau_J > J|(N_t, \hat{N}_t)] \leq -\delta JP(\tau_J > J) \leq -\frac{\delta J}{2},$$

where $\delta > 0$ is the constant of Lemma 3.3.

Next we consider the case $(N_t, \hat{N}_t) \in R^+_{5\gamma,M-J^2}$. The same argument as above yields, for J large enough,

$$E[V(N_{t+J}, \hat{N}_{t+J}) - V(N_t, \hat{N}_t); \tau_J > J|(N_t, \hat{N}_t)]$$
$$= \frac{1+3\gamma}{3\gamma}E[-\tilde{N}_{t+J} + \tilde{N}_t; \quad \tau_J > J|(N_t, \hat{N}_t)]$$
$$= -\frac{1+3\gamma}{3\gamma}\sum_{k=0}^{J-1} E[d(N_{t+k}, \hat{N}_{t+k}); \quad \tau_J > J|(N_t, \hat{N}_t)] \leq -\frac{1+3\gamma}{3\gamma}\frac{\delta J}{2}.$$

A similar argument applies to the case where $X_t \in R^-_{5\gamma,M+J^2}$.

We now consider the slightly more complicated case where $(N_t, \hat{N}_t) \in S_{4\gamma,M+J^2}$ and $(N_t, \hat{N}_t) \notin S_{2\gamma,M+J^2}$. There are two subcases to consider: a) $N_t/\hat{N}_t \in (1 + 2\gamma, 1 + 4\gamma]$, and b) $N_t/\hat{N}_t \in [\beta - 4\gamma, \beta - 2\gamma)$. We only consider the first subcase, since the argument for the second one is identical. We therefore have $V(N_t, \hat{N}_t) = \max\{N_t, -(1+3\gamma)/(3\gamma)\tilde{N}_t\}$. Furthermore, if $\tau_J > J$, then $(N_{t-k}, \hat{N}_{t-k})$ stays inside $S_{5\gamma,M} \cap R^+_{\gamma,M}$, $\forall k \in [0, J]$. Thus, $V(N_{t+J}, \hat{N}_{t+J}) = \max\{N_{t-J}, -(1+3\gamma)/(3\gamma)\tilde{N}_{t+J}\}$. Consequently,

$$E[V(N_{t+J}, \hat{N}_{t+J}) - V(N_t, \hat{N}_t);\ \tau_J > J \mid (N_t, \hat{N}_t)]$$
$$\leq E\left[\max\left\{N_{t+J}-N_t,\ -\frac{1+3\gamma}{3\gamma}(\tilde{N}_{t+J}-\tilde{N}_t)\right\};\ \tau_J>J \mid (N_t, \hat{N}_t)\right]$$
$$\leq E\left[\max\left\{0,\ N_{t+J}-N_t+\frac{J\delta}{2}\right\};\ \tau_J>J \mid (N_t, \hat{N}_t)\right]$$
$$+E\left[\max\left\{0,\ -\frac{1+3\gamma}{3\gamma}(\tilde{N}_{t+J}-\tilde{N}_t)+\frac{J\delta}{2}\right\};\ \tau_J>J \mid (N_t, \hat{N}_t)\right]$$
$$+E\left[-\frac{J\delta}{2};\ \tau_J>J \mid (N_t, \hat{N}_t)\right]. \tag{3.11}$$

Here, δ is the constant of Lemma 3.3, and we have used the inequalities $\max\{a, b\} - \max\{c, d\} \leq \max\{a-c, b-d\}$ and $\max\{a, b\} \leq \max\{0, a+f\} + \max\{0, b+f\} - f$, with $f = J\delta/2$. We consider the first summand in the right-hand side of (3.11). Let $W_k = N_{t+k} - N_t + k\delta/2$. Clearly, $\{W_k, \mathfrak{F}_{t+k}\}$ is exponential-type because $|W_{k+1} - W_k| \leq 1 + A_{t-k} + \delta$. Furthermore, τ_J is a stopping time, with respect to $\{\mathfrak{F}_{t+k}\}$. Finally, using Lemma 3.3,

$$E\left[W_{k+1} - W_k + \frac{\delta}{2};\ \tau_J > k \mid \mathfrak{F}_{t-k}\right] = E[N_{t+k+1} - N_{t+k} + \delta;\ \tau_J > k \mid \mathfrak{F}_{t+k}] \leq 0.$$

Thus, Proposition 2.2 applies and shows that $E[\max\{0, W_J\};\ \tau_J > J] \leq B$, for some B independent of J. Equivalently, the first summand in (3.11) is bounded above by the same B. The same conclusion is obtained, by an identical argument, for the second summand in (3.11). Finally, the last term in (3.11) is equal to $-(J\delta/2)P(\tau_J > J)$. Taking J large enough and using (3.9), this term can be made arbitrarily negative. It follows that the right-hand side of (3.11) can become negative and bounded away from zero by proper choice of J. This concludes the proof of (3.8).

The proof of the theorem may be now completed as follows. Let $G = \max\{1, (1+3\gamma)/3\gamma, (\beta - 3\gamma)/(1 - \beta + 3\gamma)\}(M + J^2)$. Whenever $V(N_t, \hat{N}_t) \geq G$, then either $N \geq M + J^2$ or $\hat{N} \geq M + J^2$ and (3.8) holds. Furthermore, $\{V(N_t, \hat{N}_t), \mathfrak{F}_t\}$ is exponential-type. Hence, Proposition 2.1 applies and shows that the stopping time $\tau = \min\{k: V(N_{kJ}, \hat{N}_{kJ}) < G\}$ is exponential-type, for any initial state. From this it follows easily that the time until $(N_t, \hat{N}_t)$ becomes equal to (0, 1) is also exponential-type and concludes the proof of the theorem. ∎

Remark: It should be clear from the above proof that it is not necessary to assume that the arrival process A_t is Poisson or even that the random variables A_t are independent identically distributed. One only needs to assume that $\{A_t, \mathfrak{F}_t\}$ is exponential-type, in the sense of Section II.

IV. The Case Where $\hat{\lambda} < \lambda$

With a minor modification of the proof in Section III, it can be shown that for any fixed $\lambda < e^{-1}$ there exists some $\epsilon > 0$ such that if $|\lambda - \hat{\lambda}| < \epsilon$, then $\{X_t\}$ is geometrically ergodic. In general, however, ϵ will depend on λ and will tend to zero as λ approaches e^{-1}.

Suppose now that λ is very close to e^{-1} and that $\lambda - \hat{\lambda}$ is positive and sufficiently large. Then, $\{X_t\}$ will no longer be ergodic, as indicated by the following argument. If $\lambda \approx e^{-1}$, the only way of having a stable (ergodic) process is to have some mechanism that ensures that the probability of transmission by each station is very close to $1/N_t$, at least whenever N_t is large. Equivalently, we want $N_t/\hat{N}_t \approx 1$. However, when $\tilde{\lambda} \neq 0$, then $\tilde{N}$ drifts away from one, because $f(1, \tilde{\lambda}) \neq 1$, where f, the function defined in (3.3) (which is the approximate drift of $\tilde{N}$, according to Lemma 3.1). Therefore, X_t will tend to spend most of its time in a region where α is bounded away from 1 and, consequently, the probability of a successful transmission is bounded away from e^{-1}. Instability then results.

One might try to make a similar argument for the case $\lambda < \hat{\lambda}$. In this case, the probability of a successful transmission is again bounded away from e^{-1}. However, since $\lambda < \hat{\lambda} \leq e^{-1}$, there is less input traffic to be accommodated and instability does not arise. [This is the essence of part iii) of Lemma 3.2.]

It is suggested in [1] that λ could be estimated on line, if it is unknown. One possible method [1] is to let $\hat{\lambda}_t$ be the number of successful transmissions up to time t, divided by t. Such an estimator loses its ability to adapt to changes in the input traffic statistics, as time goes to infinity. For this reason an exponential weight was used in [1] to discount old data. It is unclear whether such a method can achieve stability with a throughput up to e^{-1}. Given the result of Section III, overestimating λ by using the estimate $\hat{\lambda} = e^{-1}$ cannot result to instability and this seems to be a reasonable choice.

Finally, let us point out that the stability proof presented in Section III extends relatively easily to the case where the stations acquire information on the state of the channel (whether a hole, success, or collision occurred) with a fixed finite delay [10].

Acknowledgment

I would like to thank D. Bertsekas and R. Gallager for suggesting the problem studied here and for several discussions, as well as B. Hajek who brought the existence of [4] and [5] to my attention.

References

[1] R. L. Rivest, "Network control by Bayesian broadcast," Lab. Comput. Sci., Mass. Inst. Technol., Tech. Rep. TM-287, Sept. 1985.

[2] B. Hajek, "Hitting-time and occupation-time bounds implied by drift analysis with applications," *Advances Appl. Probabil.*, vol. 14, pp. 502-525, 1982.

[3] B. Hajek and T. Van Loon, "Decentralized dynamic control of a multiaccess broadcast channel," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 559-569, 1982.

[4] F. P. Kelly, "Stochastic models of computer communication systems," *J. Roy. Statist. Soc. B*, vol. 47, no. 1, 1985.

[5] V. A. Mikhailov, "Methods of random multiple access," candidate Eng. thesis, Moscow Inst. Phys. and Technol., 1979.

[6] L. Merakos and D. Kazakos, "On retransmission control policies in multiple-access communication networks," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 109-117, 1985.

[7] R. L. Cruz, "Protocols for multi-access channels with continuous entry and noisy feedback," Lab. Inform. and Decision Syst., Mass. Inst. Technol., Cambridge, Tech. Rep. LIDS-TH-1213, 1982.

[8] S. C. A. Thomopoulos and D. Bougoulias, "Decentralized control in random access communication networks: Stability and performance analysis," in *Proc. IEEE INFOCOM'85*, 1985, pp. 246-254.

[9] ——, "Decentralized control in random access broadcast channels with unknown arrival rates," in *Proc. IEEE Int. Commun. Conf.*, 1985, pp. 278-282.

[10] A. Greenberg, personal communication.

Comments on "Exact Control of Linear Systems with Multiple Controls"

FRANCISCO CASIELLO AND KENNETH A. LOPARO

***Abstract*—Given a linear system $\dot{x} = Ax + Bu$, where A and B are $n \times n$ and $n \times m$ matrices, with $m \leq n$ and B is of full rank, Farlow's**

Manuscript received July 28, 1986; revised January 12, 1987.

F. Casiello is with E.N.A.C.E., Argentina.

K. A. Loparo is with the Department of Systems Engineering and the Center for Automation and Intelligent Systems, Case Western Reserve University, Cleveland, OH 44106.

IEEE Log Number 8613841.