

Bayesian Detection in Bounded Height Tree Networks

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Abstract

We study the detection performance of large scale sensor networks, configured as trees with bounded height, in which information is progressively compressed as it moves towards the root of the tree. We show that, under a Bayesian formulation, the error probability decays exponentially fast, and we provide bounds for the error exponent. We then focus on the case where the tree has certain symmetry properties. We derive the form of the optimal exponent within a restricted class of easily implementable strategies, as well as optimal strategies within that class. We also find conditions under which (suitably defined) majority rules are optimal. Finally, we provide evidence that in designing a network it is preferable to keep the branching factor small for nodes other than the neighbors of the leaves.

Index Terms

Decentralized detection, error exponent, tree network, sensor networks.

I. INTRODUCTION

We consider a sensor network, configured as a directed tree, with a fusion center at its root. The objective of the network is to make a decision between two given hypotheses H_0 and H_1 . Observations are obtained at the nodes of the tree, and information is propagated from the leaves towards the root. However, because of resource constraints, e.g., a restriction to single-bit

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messages, every node is required to compress or quantize its information (its observation and the messages it has received) before forming its own message. Based on the received information, the root or fusion center makes a decision about the true hypothesis. Our objective is to understand the scaling of the error probability at the fusion center, as the number of nodes increases, and its dependence on qualitative properties of the tree.

In the well studied parallel configuration (see e.g. [1]–[10]), each node sends its compressed information directly to the fusion center. A tree, on the other hand, allows for shorter-range communications, thus making better use of communication resources. Tree networks have been studied in several references, such as [11]–[18]. It is known that under the assumptions to be made in this paper (conditioned on either hypothesis, the observations at the different nodes are i.i.d.), optimal quantization strategies take the form of likelihood-ratio quantizers, and one can obtain “person-by-person optimality” conditions that need to be satisfied by the optimal quantizers. Nevertheless, finding the optimal quantizers, and hence characterizing the detection performance, is a rather intractable problem even for a moderate number of nodes. For this reason, in the spirit of [19], we focus on the exponential rate of decay of error probabilities.

In [20], we studied the Neyman-Pearson variant of the problem considered in this paper. We showed that the error probability decays exponentially fast with the number of nodes (this is apparently not the case when the height is unbounded, e.g., in a tandem configuration [21]–[24]); furthermore, in some cases the error exponent associated with a tree configuration turned out to be the same as for the parallel configuration. In this paper, we continue this investigation by focusing on the Bayesian formulation. Similar to the Neyman-Pearson case, we will see that for bounded height trees error probabilities decay exponentially fast. However, the optimal error exponent is generically worse than the one associated with a parallel configuration (cf. Proposition 2), and is also harder to characterize exactly. In order to make further progress, we place some additional restrictions on the trees to be considered, as well as on the allowed quantization strategies. The following example serves to motivate some of our assumptions.

Example 1 (Random Nodes in the Unit Square). Suppose that we distribute n nodes randomly in the unit square and place a fusion center at the center of the square. We are interested in configuring the nodes so that every node is at most two hops away from the fusion center.

One possibility (to be referred to as Design I) is to fix some m , and divide the square into m

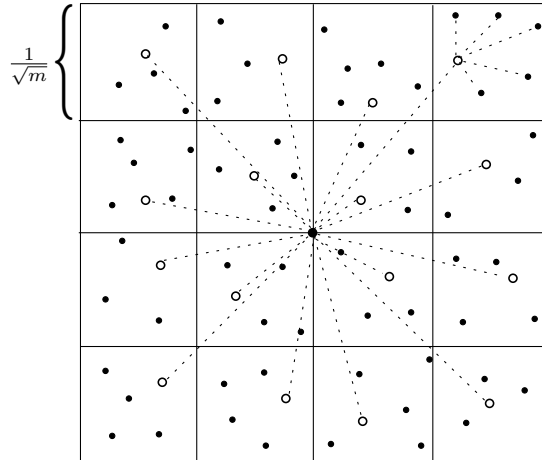


Fig. 1. Random nodes in a unit square. The hollow circles represent the local aggregators. The dotted lines represent communication links. Only one sub-square is shown with its communication links.

sub-squares, each with side of length $1/\sqrt{m}$ (see Figure 1). For large n , there are approximately n/m nodes in each of these sub-squares. We let all nodes within a sub-square transmit their messages to an “aggregator” node in that sub-square. In this way, we get a “symmetric” tree network, in which every aggregator is connected to roughly the same number of nodes, with high probability. Suppose now that the communication cost is proportional to the Euclidean distance between two communicating nodes. Since the number m is fixed, the communication cost in this strategy is $\Theta(n)$.

An alternative possibility (to be referred to as Design II) is to reduce the overall communication cost by using a 2-hop Minimum Spanning Tree (MST). Because finding a hop-constrained MST is NP-hard (see [25]), we consider a simple heuristic [26]. As before, we place an aggregator in each of the m sub-squares, and let the rest of the nodes in the sub-square send their messages to this aggregator. However, we allow m to be chosen optimally. The overall expected communication cost is

$$O(n/m) + O(m)$$

which we minimize by setting $m = m(n) = \Theta(n^{2/3})$, and thus reducing the $\Theta(n)$ cost of Design I to $\Theta(n^{2/3})$. On the other hand, one suspects that the detection performance of Design II will be inferior to that of Design I. The results in Lemma 3 and Proposition 3 provide evidence that this is indeed the case. \square

Motivated by the two designs introduced in Example 1, we will consider the detection performance of two different classes of tree networks. The first one consists of symmetric trees with a fixed number of aggregators or intermediate nodes, while the second consists of trees in which the number of intermediate nodes increases at a certain rate (we call these the rapidly branching tree sequences; cf. Section V). We characterize and compare the detection performance of these two classes, optimized over a restricted set of strategies that are easy to implement. In particular, we show in Proposition 3 that the second class performs worse than any of the tree networks in the first class.

The rest of this paper is organized as follows. In Section II, we introduce the problem formulation and some related concepts. In Section III, we show that for general tree networks, the error probability decays exponentially fast with the number of nodes in the network, and provide bounds for the rate of decay. In Sections IV and V, we consider specific classes of tree networks, characterize their performance, and provide simple (but suboptimal) strategies. Finally in Section VI, we summarize and conclude.

II. PROBLEM FORMULATION

In this section, we introduce the Bayesian version of the model in [20], describe the basic assumptions and notation, and recall a useful result from [20]. We are given two hypotheses H_0 and H_1 , each with prior probability $\pi_j > 0$, corresponding probability measures \mathbb{P}_j , and associated expectation operators \mathbb{E}_j , $j = 0, 1$. We model the sensor network as a directed rooted tree T_n , in which a node sends messages to another if there is a directed arc from the first to the second node. The root of the tree T_n is the *fusion center*, and will be denoted by f . The nodes that send messages directly to v are called its immediate *predecessors*, while v is called an immediate *successor* of each of these nodes. Let the set of immediate predecessors of a node v be $C_n(v)$.

A sequence of trees $(T_n)_{n \geq 1}$ represents the evolution of the network. We focus on tree sequences with bounded height, defined as the length of a longest directed path. For a tree with height h , a node is said to be at *level* k if it is connected to the fusion center via a path with $h - k$ hops. Hence the fusion center f is a level h node.

We assume that under each hypothesis H_j , where $j = 0, 1$, every node v makes an i.i.d. observation X_v , with marginal distribution \mathbb{P}_j^X . If v is a leaf node, it sends a summary $Y_v =$

$\gamma_v(X_v)$ of its observation to its immediate successor, where γ_v is constrained to belong to a given set Γ of allowed *quantization functions*. (For example, Γ can be the set of all binary functions of X_v .) If v is a non-leaf node, it summarizes its own observation and the messages it has received using a *transmission function* γ_v , to produce a message Y_v . This message is then sent to its immediate successor. Finally, the fusion center f uses a *fusion rule* to decide between the two hypotheses. Let Y_f be a random variable that represents the decision of the fusion center. A collection of quantization and transmission functions, one for each node, and a fusion rule will be called a *strategy*. A tree in which every non-leaf node ignores its own observation, and simply forwards a summary of its received messages, will be called a *relay tree*; in that case, non-leaf nodes will also be referred to as *relay nodes*.

Given a tree network T_n , our objective is to minimize the probability of error $P_e(T_n) = \pi_0 \mathbb{P}_0(Y_f = 1) + \pi_1 \mathbb{P}_1(Y_f = 0)$, over all strategies. Let $P_e^*(T_n)$ be the minimum probability of error (over all strategies) at the fusion center. From an asymptotic perspective, we are given a sequence of trees $(T_n)_{n \geq 1}$, and seek to characterize the optimal error exponent [27],

$$\begin{aligned} \mathcal{E}^* &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_e^*(T_n) \\ &= \max \left\{ \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_0(Y_f = 1), \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_1(Y_f = 0) \right\}. \end{aligned}$$

Let $l_n(v)$ be the number of leaves in the sub-tree rooted at node v . In particular, $l_n(f)$ is the total number of leaves of the tree T_n . For a relay tree, we consider instead the optimal error exponent,

$$\mathcal{E}_R^* = \limsup_{n \rightarrow \infty} \frac{1}{l_n(f)} \log P_e^*(T_n),$$

where we have normalized the log-error probability by $l_n(f)$, so that \mathcal{E}_R^* is the error exponent per observation.

Recall that \mathbb{P}_j^X is the distribution of an observation made by a node under hypothesis H_j . For any $\gamma \in \Gamma$, let \mathbb{P}_j^γ be the distribution of $\gamma(X)$, when X has distribution \mathbb{P}_j^X . We make the following assumptions, which are standard in the literature (see e.g. [5], [8], [19]). The Kullback-Leibler (KL) divergence between two probability measures \mathbb{P} and \mathbb{Q} is denoted by

$$D(\mathbb{P} \parallel \mathbb{Q}) = \mathbb{E} \left[\log \frac{d\mathbb{P}}{d\mathbb{Q}} \right],$$

where the expectation is taken with respect to (w.r.t.) the measure \mathbb{P} .

Assumption 1. *The measures \mathbb{P}_0^X and \mathbb{P}_1^X are equivalent, i.e., they are absolutely continuous w.r.t. each other. Furthermore, there exists some $\gamma \in \Gamma$ such that $-\mathrm{D}(\mathbb{P}_0^\gamma \parallel \mathbb{P}_1^\gamma) < 0 < \mathrm{D}(\mathbb{P}_1^\gamma \parallel \mathbb{P}_0^\gamma)$.*

To develop insights into how the error probabilities scale with the number of nodes, we will use an upper bound for the error probabilities at each node in the network. The next proposition allows us to recursively propagate error probabilities along a tree in which *all* leaves have paths with h hops to the fusion center. Such a tree is called a *h-uniform* tree.

Let $t^{(k)} = (t_1, t_2, \dots, t_k)$, for $k \geq 1$, and $t^{(0)} = \emptyset$. For $j = 0, 1$, $k \geq 1$, and $\lambda \in \mathbb{R}$, we define recursively [20]

$$\Lambda_{j,0}(\gamma; \lambda) = \Lambda_{j,0}(\gamma, t^{(0)}; \lambda) = \log \mathbb{E}_j \left[\left(\frac{d\mathbb{P}_1^\gamma}{d\mathbb{P}_0^\gamma} \right)^\lambda \right],$$

$$\Lambda_{j,k}^*(\gamma, t^{(k)}) = \sup_{\lambda \in \mathbb{R}} \{ \lambda t_k - \Lambda_{j,k-1}(\gamma, t^{(k-1)}; \lambda) \}, \quad (1)$$

$$\Lambda_{j,k}(\gamma, t^{(k)}; \lambda) = \max \{ -\Lambda_{1,k}^*(\gamma, t^{(k)})(j + \lambda), \Lambda_{0,k}^*(\gamma, t^{(k)})(j - 1 + \lambda) \}. \quad (2)$$

We make the following assumption. A prime denotes differentiation w.r.t. λ , and a double prime indicates the second derivative w.r.t. λ .

Assumption 2. *Both $\mathrm{D}(\mathbb{P}_0^X \parallel \mathbb{P}_1^X)$ and $\mathrm{D}(\mathbb{P}_1^X \parallel \mathbb{P}_0^X)$ are finite, and there exists some $b \in (0, \infty)$, such that for all $\gamma \in \Gamma$, we have $\Lambda_{0,0}''(\gamma; \lambda) \leq b$ for all $\lambda \in (0, 1)$, and $\Lambda_{1,0}''(\gamma; \lambda) \leq b$ for all $\lambda \in (-1, 0)$.*

The following result is proved as Proposition 1 in [20]. Let $S_n(v)$ be the log-likelihood ratio (or more formally, the logarithm of the associated Radon-Nikodym derivative) of the received messages at node v . A (one-bit) Log Likelihood Ratio Quantizer (LLRQ) with threshold t for a non-leaf node v is a quantizer that takes the form

$$Y_v = \begin{cases} 0, & S_n(v)/l_n(v) \leq t, \\ 1, & \text{otherwise.} \end{cases}$$

Proposition 1. *Consider a sequence of h -uniform relay trees. Suppose that Assumptions 1-2 hold. Suppose that the following strategy is used: every leaf employs the same quantization*

function $\gamma \in \Gamma$, and every level k node ($k \geq 1$) uses a LLRQ with threshold t_k , satisfying

$$-D(\mathbb{P}_0^\gamma \parallel \mathbb{P}_1^\gamma) < 0 < D(\mathbb{P}_1^\gamma \parallel \mathbb{P}_0^\gamma), \quad (3)$$

$$-D(\mathbb{P}_0^\gamma \parallel \mathbb{P}_1^\gamma) < t_1 < D(\mathbb{P}_1^\gamma \parallel \mathbb{P}_0^\gamma), \quad (4)$$

$$-\Lambda_{1,k-1}^*(\gamma, t^{(k-1)}) < t_k < \Lambda_{0,k-1}^*(\gamma, t^{(k-1)}), \quad \text{for } 1 < k \leq h. \quad (5)$$

Then,

$$\begin{aligned} \frac{1}{l_n(f)} \log \mathbb{P}_0(Y_f = 1) &\leq -\Lambda_{0,h}^*(\gamma, t^{(h)}) + \frac{n}{l_n(f)} - 1, \\ \frac{1}{l_n(f)} \log \mathbb{P}_1(Y_f = 0) &\leq -\Lambda_{1,h}^*(\gamma, t^{(h)}) + \frac{n}{l_n(f)} - 1, \end{aligned}$$

Proposition 1 shows that the Type I and II error exponents of h -uniform trees using the strategy described in the proposition are essentially upper bounded by $-\Lambda_{0,h}^*(\gamma, t^{(h)})$ and $-\Lambda_{1,h}^*(\gamma, t^{(h)})$ respectively. In Section V, we present a class of tree networks whose error exponents are precisely $-\Lambda_{j,h}^*(\gamma, t^{(h)})$, for $j = 0, 1$.

III. EXPONENTIAL DECAY

In this section, we show that the optimal error probability in a sequence of trees with bounded height h decays exponentially fast with the number of nodes n . (This is in contrast to general trees, where the decay can be sub-exponential [24].) When $h = 1$, we have the classical parallel configuration considered in [19], and the optimal error exponent is given by

$$\mathcal{E}_P^* = -\sup_{\gamma \in \Gamma} \Lambda_{0,1}^*(\gamma, 0) = \inf_{\gamma \in \Gamma} \min_{\lambda \in [0,1]} \Lambda_{0,0}(\gamma; \lambda) < 0. \quad (6)$$

Theorem 1. *Suppose that Assumptions 1 and 2 hold. Consider any sequence of trees of height h . Let $z = \liminf_{n \rightarrow \infty} l_n(f)/n$ be the asymptotic proportion of nodes that are leaves. Then,*

$$\mathcal{E}_P^* \leq \mathcal{E}_R^* < 0, \quad (7)$$

and

$$\min_{\lambda \in [0,1]} \log \mathbb{E}_0 \left[\left(\frac{d\mathbb{P}_1^X}{d\mathbb{P}_0^X} \right)^\lambda \right] \leq \mathcal{E}^* \leq z\mathcal{E}_R^* < 0. \quad (8)$$

Furthermore, if $z = 1$, we have

$$\mathcal{E}_P^* \leq \mathcal{E}^* \leq \mathcal{E}_R^* \leq \frac{1}{2^{h-1}} \mathcal{E}_P^*. \quad (9)$$

Proof: We first note that the inequality $\mathcal{E}_P^* \leq \mathcal{E}_R^*$ holds trivially since a parallel configuration can simulate any relay tree network. Also, since a relay strategy is a possible strategy for any tree network, $\mathcal{E}^* \leq z\mathcal{E}_R^*$; the factor z appears because we have normalized \mathcal{E}_R^* with $l_n(f)$, instead of n . Furthermore, the first inequality in (8) is easily arrived at by comparing a tree network to one in which the raw observations at every node are available at the fusion center. Finally, since $z \geq 1/h$, it remains to show $\mathcal{E}_R^* < 0$ and the last inequality in (9).

We first show $\mathcal{E}_P^* \leq \mathcal{E}^*$ when $z = 1$. Compare the tree network T_n to another network T'_n in which all the non-leaf nodes send their raw observations to the fusion center, and all the leaves send their messages directly to the fusion center. Clearly, T'_n can simulate T_n , and has at least as good performance as T_n . Moreover, T'_n is now a parallel configuration in which almost all nodes have i.i.d. observations (since $z = 1$). The nodes in T'_n that correspond to non-leaf nodes are too few to affect the error exponent. Then, using an argument similar to that in [19], it is not hard to see that the error exponent for T'_n is the same as for a parallel configuration with i.i.d. observations, namely \mathcal{E}_P^* . We conclude that $\mathcal{E}_P^* \leq \mathcal{E}^*$.

Next, we show that $\mathcal{E}_R^* \leq \mathcal{E}_P^*/2^{h-1} < 0$ when $z = 1$, and $\mathcal{E}_R^* < 0$ when $z < 1$. Any tree of height h can be transformed to an h -uniform tree using a height uniformization procedure [20]: For every node v of level greater than 1 that has at least one leaf as an immediate predecessor, we create a new node u . Remove all the arcs connecting leaves to v , and add arcs so that these leaves now send messages to u . Finally, add enough new nodes and arcs so that there is a path connecting u to v , and the leaves attached to u are of level 0.

Let $(T'_n)_{n \geq 1}$ be the new sequence of h -uniform trees obtained from $(T_n)_{n \geq 1}$, after applying the height uniformization procedure. Note that the number of leaves in T'_n remains the same as $l_n(f)$, the number of leaves in T_n . It is seen that, when only the leaves make observations, the performance of T'_n is no better than that of T_n , so we obtain

$$\mathcal{E}_R^* \leq \limsup_{n \rightarrow \infty} \frac{1}{l_n(f)} \log P_e^*(T'_n). \quad (10)$$

Therefore, it is sufficient to show the inequalities for the height uniform counterpart of $(T_n)_{n \geq 1}$.

Hence, without loss of generality, we can assume that the sequence of trees is h -uniform. We consider two cases below.

When $z = 1$, let all leaves use a common transmission function $\gamma \in \Gamma$ that satisfies (3), and let all non-leaf nodes use a LLRQ with threshold 0 (it is easy to verify that this satisfies (4)-(5)). Then, from Proposition 1, we have

$$\mathcal{E}_R^* \leq -\min\{\Lambda_{0,h}^*(\gamma, \mathbf{0}), \Lambda_{1,h}^*(\gamma, \mathbf{0})\},$$

where $\mathbf{0}$ is a zero vector of dimension h . We can then use the recursions (1) - (2) and some calculation (or alternatively, the formulas in Lemma 2 of [20]), to obtain

$$\Lambda_{0,h}^*(\gamma, \mathbf{0}) = \Lambda_{1,h}^*(\gamma, \mathbf{0}) = \frac{1}{2^{h-1}} \Lambda_{0,1}^*(\gamma, 0).$$

Taking the infimum over all $\gamma \in \Gamma$, and using (6), we obtain the desired bound.

We now consider a sequence of trees $(T_n)_{n \geq 1}$ with $z < 1$, and show that $\mathcal{E}_R^* < 0$. We only give an outline of the proof, because it is similar to that of Proposition 2 in [20]. Let $F_{N,n}$ be the set of level 1 nodes in T_n , with less than N leaves attached. Note that $|F_{N,n}| \leq |B_n|$, where B_n is the set of level 1 nodes. Since $z < 1$, there exists a positive N such that $\liminf_{n \rightarrow \infty} |F_{N,n}|/l_n(f) > 0$. We form a new tree T'_n by removing all leaves, and allowing each level 1 node to make its own observation. The new tree, of height $h - 1$, has performance no better than the original:

$$\limsup_{n \rightarrow \infty} \frac{1}{l_n(f)} \log P_e^*(T_n) \leq \limsup_{n \rightarrow \infty} \frac{1}{l_n(f)} \log P_e^*(T'_n) \leq \liminf_{n \rightarrow \infty} \frac{|F_{N,n}|}{l_n(f)} \limsup_{n \rightarrow \infty} \frac{1}{|B_n|} \log P_e^*(T'_n).$$

The desired conclusion then follows by induction on h . This concludes the proof. ■

We have shown that for trees of bounded height, the error probability decays exponentially with n , and in the case when almost all nodes are leaves, the error exponent is within a constant factor of the error exponent of a parallel configuration. However, the exact rate of decay depends on several factors, such as the probability distributions and the architecture of the network. For example, in architectures that are essentially the same as the parallel configuration or can be reduced to the parallel configuration, the error exponent is \mathcal{E}_P^* . However, in most other cases, the error exponent is in general strictly inferior to \mathcal{E}_P^* (cf. Proposition 2). To obtain some insights into the optimal error exponent, we consider specific classes of h -uniform tree networks in the next two sections. It turns out that finding optimal strategies is in general difficult, so we will

instead analyze simple, but suboptimal strategies.

IV. SYMMETRIC TREE SEQUENCES

In this section, we consider the asymptotic performance of a special class of h -uniform tree networks, which we call r -symmetric. These are relay trees, with a bounded number of relay nodes, as in Design I in Example 1. Throughout this section, we assume that nodes can only send binary messages, taking values $\{0, 1\}$. An r -symmetric tree network is defined as follows.

Definition 1 (r -symmetric tree). *For $h, r \geq 1$, a h -uniform tree sequence $(T_n)_{n \geq 1}$ is said to be r -symmetric if:*

- (i) *for all level k nodes v , where $k > 1$, $|C_n(v)| = r$, and*
- (ii) *for all level 1 nodes v , $l_n(v)/l_n(f) \rightarrow 1/r^{h-1}$ as $n \rightarrow \infty$.*

The second condition in this definition requires that when n is large, all the r^{h-1} level 1 nodes have approximately the same number of immediate predecessors.

We define a *counting quantizer (CQ)* with threshold s for a level k node v , where $k \geq 1$, as a transmission function of the form

$$Y_v = \begin{cases} 0, & \sum_{u \in C_n(v)} Y_u \leq s, \\ 1, & \text{otherwise,} \end{cases}$$

where $\sum_{u \in C_n(v)} Y_u$ is the total number of 1s that v receives from its immediate predecessors. A counting quantizer has arguably the simplest possible structure. Furthermore, it is equivalent to a LLRQ with an appropriate threshold if all the messages of v 's immediate predecessors are identically distributed. For tractability and to ensure that our strategies are easily implementable, we will now restrict all non-leaf nodes to using counting quantizers. We call such a strategy a *counting strategy*. Let $\mathcal{E}_S^*(r)$ denote the optimal (over all counting strategies) error exponent (in the worst-case over all r -symmetric tree sequences). We will show that with the restriction to a counting strategy, using the same transmission function at the leaves results in no loss of optimality.

Consider minimizing the following objective function,¹

$$\max\{\lambda_1 \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_0(Y_f = 1), \lambda_2 \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_1(Y_f = 0)\}, \quad (11)$$

where λ_1 and λ_2 are fixed positive constants. In the case of minimizing the error exponent, $\lambda_1 = \lambda_2 = 1$. We use this more general formulation because it proves to be useful later. We start with two preliminary lemmas, the first of which is proved in [19].

Lemma 1. *Suppose that Assumptions 1-2 hold. Consider minimizing the objective function (11) at the fusion center of a parallel configuration. Then, there is no loss in optimality if we restrict all nodes to use the same transmission function, and the fusion rule to use a counting quantizer.*

Consider a symmetric tree, and let the set of immediate predecessors of the fusion center f be $C_n(f) = \{v_1, \dots, v_r\}$. For any given strategy, let

$$\begin{aligned} \psi_i &= - \lim_{n \rightarrow \infty} \frac{1}{l_n(v_i)} \log \mathbb{P}_0(Y_{v_i} = 1), \\ \varphi_i &= - \lim_{n \rightarrow \infty} \frac{1}{l_n(v_i)} \log \mathbb{P}_1(Y_{v_i} = 0). \end{aligned}$$

From Definition 1, the subtrees rooted at the different predecessors of f are asymptotically the same. We also note that under an optimal strategy there is a tradeoff between the two different types of error probabilities, ψ_i and ϕ_i . It follows that without loss of generality, we can assume that

$$0 \leq \psi_1 \leq \psi_2 \leq \dots \leq \psi_r < \infty, \quad (12)$$

$$\infty > \varphi_1 \geq \varphi_2 \geq \dots \geq \varphi_r \geq 0. \quad (13)$$

Furthermore, if $\psi_i > \psi_j$, then $\varphi_i < \varphi_j$, and vice versa, for all i, j .

Lemma 2. *To minimize the objective function (11) at the fusion center using a counting quantizer as the fusion rule, there is no loss of optimality if we restrict all immediate predecessors of f to satisfy $\psi_i = \psi_j$, and $\varphi_i = \varphi_j$ for all i, j .*

¹We use the notation \lim here, without first showing that the limit exists. The subsequent arguments can be made completely rigorous by considering a subsequence of the tree sequence, in which limits of the Type I and II error exponents exist at each non-leaf node.

Proof: Suppose the fusion center uses a counting quantizer with threshold s . Then, we have

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1}{l_n(f)} \log \mathbb{P}_0(Y_f = 1) \\
&= \lim_{n \rightarrow \infty} \frac{1}{l_n(f)} \log \mathbb{P}_0\left(\sum_{i=1}^r Y_{v_i} > s\right) \\
&= \lim_{n \rightarrow \infty} \frac{1}{l_n(f)} \log \mathbb{P}_0(Y_{v_i} = 1, i = 1, 2, \dots, s+1) \\
&= \sum_{i=1}^{s+1} \lim_{n \rightarrow \infty} \frac{1}{l_n(f)} \log \mathbb{P}_0(Y_{v_i} = 1) \\
&= -\frac{1}{r} \sum_{i=1}^{s+1} \psi_i,
\end{aligned} \tag{14}$$

where the second equality follows because $\{Y_{v_i} = 1, i = 1, 2, \dots, s+1\}$ is the dominating error event, and the third equality follows from independence. Similarly, we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{l_n(f)} \log \mathbb{P}_1(Y_f = 0) = -\frac{1}{r} \sum_{i=s+1}^r \varphi_i. \tag{15}$$

Then, the objective function (11) is equal to

$$\begin{aligned}
& \frac{1}{r} \max \left\{ -\lambda_1 \sum_{j=1}^{s+1} \psi_j, -\lambda_2 \sum_{j=s+1}^r \varphi_j \right\} \\
& \geq \frac{1}{r} \max \{ -\lambda_1 (s+1) \psi_{s+1}, -\lambda_2 (r-s) \varphi_{s+1} \},
\end{aligned}$$

where equality holds if we set $\psi_i = \psi_{s+1}$ and $\varphi_i = \varphi_{s+1}$ for all i . Hence, it is optimal to use the same strategy for each of the sub-trees rooted at the nodes v_1, \dots, v_r . \blacksquare

Theorem 2. Consider an r -symmetric tree sequence $(T_n)_{n \geq 1}$, and suppose that Assumptions 1-2 hold. Within the set of counting strategies, there is no loss in optimality if we impose the following restrictions:

- (i) all leaves use the same transmission function;
- (ii) for each $k \geq 1$, all level k nodes use counting quantizers with the same threshold.

Furthermore, the optimal error exponent at the fusion center is given by²

$$\begin{aligned}
\mathcal{E}_S^*(r) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log P_e^*(T_n) \\
&= - \sup_{\{s_k\}, t} \left\{ \left[\left(\prod_{k=2}^h \frac{s_k + 1}{r} \right) \Lambda_{0,1}^*(\gamma, t) \right] \wedge \left[\left(\prod_{k=2}^h \frac{r - s_k}{r} \right) \Lambda_{1,1}^*(\gamma, t) \right] : \right. \\
&\quad \left. s_k \in \{0, \dots, r - 1\}, \text{ for } k = 2, \dots, h; \gamma \in \Gamma; \right. \\
&\quad \left. - D(\mathbb{P}_0^\gamma \| \mathbb{P}_1^\gamma) < t < D(\mathbb{P}_1^\gamma \| \mathbb{P}_0^\gamma) \right\}. \tag{16}
\end{aligned}$$

Proof: (Outline³) From Lemma 2, we can restrict attention to counting strategies that use the same strategy at every sub-tree rooted at each $v \in C_n(f)$. Suppose that the fusion center uses, as its fusion rule, a counting quantizer with threshold s_h . Then, the objective at each $v \in C_n(f)$ is to minimize

$$\frac{1}{r} \max \left\{ (s_h + 1) \lim_{n \rightarrow \infty} \frac{1}{l_n(v)} \log \mathbb{P}_0(Y_v = 1), (r - s_h) \lim_{n \rightarrow \infty} \frac{1}{l_n(v)} \log \mathbb{P}_1(Y_v = 0) \right\}.$$

We apply Lemma 2 on v , and repeat the same argument for $h - 2$ steps. Therefore, we conclude that for each $k \geq 2$, there is no loss in optimality if all nodes at the same level k , use counting quantizers with the same threshold s_k . Moreover, by the same argument, there is no loss in optimality if each level 1 node has the same Type I and II error exponents. Lemma 1, applied to each level 1 node, implies that it is asymptotically optimal for all leaves to use the same transmission function γ , and all level 1 nodes to use LLRQs with the same threshold t . (Note that these LLRQs must be equivalent to counting quantizers, since the leaves use the same transmission function.) Finally, the form of the optimal error exponent is obtained by optimizing over the thresholds s_k (for $k = 2, \dots, h$), the threshold t , and the transmission function γ . The theorem is now proved. ■

Suppose that the transmission function γ in (16) has been fixed, and suppose that $h > 1$ and

²The products are taken to be 1 when $h = 1$. We also use the notation $x \wedge y = \min\{x, y\}$.

³For any given counting strategy, a more rigorous proof will involve taking a subsequence of $(T_n)_{n \geq 1}$ along which the vector of thresholds that defines the counting strategy converges to a limit; see the proof of Theorem 3, for a similar argument.

$r > 1$. Then, we have

$$\frac{1}{r^{h-1}} \prod_{i=2}^h (s_i + 1) \leq 1,$$

$$\frac{1}{r^{h-1}} \prod_{i=2}^h (r - s_i) \leq 1,$$

and equality cannot hold simultaneously in both expressions above. Since for each $\gamma \in \Gamma$, $\Lambda_{0,1}^*(\gamma, t)$ and $\Lambda_{1,1}^*(\gamma, t)$ are continuous in t , the error exponent in (16) is achieved by setting

$$\left(\prod_{i=2}^h \frac{s_i + 1}{r} \right) \Lambda_{0,1}^*(\gamma, t) = \left(\prod_{i=2}^h \frac{r - s_i}{r} \right) \Lambda_{1,1}^*(\gamma, t). \quad (17)$$

Hence, the error exponent is *strictly* smaller than that for the parallel configuration. This shows that using a r -symmetric tree results in a loss of efficiency as compared to the parallel configuration, if we restrict to counting strategies. In fact, a stronger result is possible. The detection performance of a 2-symmetric tree is strictly worse than that of a parallel configuration, even without the restriction to counting strategies.

Proposition 2. *A 2-symmetric tree has strictly worse detection performance than a parallel configuration. Moreover, there is no loss in optimality restricting to counting strategies.*

Proof: Consider a 2-symmetric tree of the form shown in Figure 2. It is not hard to see that the only choices for the fusion rule are: (i) declare H_0 iff both v_1 and v_2 send 0; (ii) declare H_0 iff either v_1 or v_2 send a 0; (iii) declare H_0 iff v_1 sends a 0; and (iv) declare H_0 iff v_2 sends a 0. The latter two rules can achieve an error exponent at most half that of the parallel configuration since half of the leaves are ignored. Rules 1 and 2 are counting rules. It follows by the same argument as in the proof of Theorem 2, that there is no loss in optimality restricting the 2-symmetric tree to counting strategies. The lemma then follows immediately from our discussion after (17). ■

A. On the Worst Case Error Exponent

When $r = 1$, the network is essentially the same, and therefore achieves the same performance, as a parallel configuration, which is the best possible. Our next result provides evidence that performance degrades as r increases. In other words, for a fixed number of nodes, it is preferable

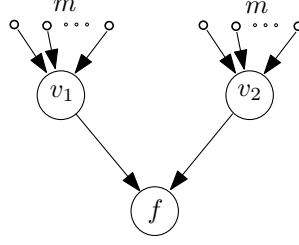


Fig. 2. A 2-symmetric tree.

to have a high branching factor at level 1, and a low branching factor, say $r = 2$, at the other levels. Let $(T_n(r))_{n \geq 1}$ be a r -symmetric tree sequence, for $r = 1, 2, \dots$

Lemma 3. *Suppose that Assumptions 1-2 hold, and that the network is restricted to counting strategies. Then, for any $r \geq 1$ and any positive integer $m > 1$, $\mathcal{E}_S^*(r) < \mathcal{E}_S^*(mr)$.*

Proof: Consider any sequence of integers k_i , where $i = 2, \dots, h$, such that $0 \leq k_i < mr$ for all i . For each i , we can find an integer $s_i \in [0, r)$, such that $ms_i \leq k_i < m(s_i + 1)$. We have

$$\frac{k_i + 1}{mr} \leq \frac{m(s_i + 1)}{mr} = \frac{s_i + 1}{r}, \quad (18)$$

$$1 - \frac{k_i}{mr} \leq 1 - \frac{ms_i}{mr} = 1 - \frac{s_i}{r}. \quad (19)$$

Then, it is an easy exercise to show that

$$\begin{aligned} & \left[\left(\prod_{i=2}^h \frac{k_i + 1}{mr} \right) \Lambda_{0,1}^*(\gamma, t) \right] \wedge \left[\left(\prod_{i=2}^h \left(1 - \frac{k_i}{mr} \right) \right) \Lambda_{1,1}^*(\gamma, t) \right] \\ & \leq \left(1 - \frac{1}{mr} \right) \left[\left(\prod_{i=2}^h \frac{s_i + 1}{r} \right) \Lambda_{0,1}^*(\gamma, t) \right] \wedge \left[\left(\prod_{i=2}^h \left(1 - \frac{s_i}{r} \right) \right) \Lambda_{1,1}^*(\gamma, t) \right] \\ & \leq -\left(1 - \frac{1}{mr} \right) \mathcal{E}_S^*(r). \end{aligned}$$

Taking the supremum over k_i , γ and t , yields $\mathcal{E}_S^*(mr) > \mathcal{E}_S^*(r)$. The proof is now complete. ■

The above lemma shows that for any $m > 1$ and $r \geq 1$, $(\mathcal{E}_S^*(m^l r))_{l \geq 0}$ is an increasing sequence, which is bounded above by zero, hence it converges. We provide an upper bound for this limit (cf. Proposition 6) below.

Proposition 3. *Suppose that Assumptions 1-2 hold. For any collection of symmetric tree se-*

quences, $\{(T_n(r))_{n \geq 1} : r = 1, 2, \dots\}$, where $(T_n(r))_{n \geq 1}$ is a r -symmetric tree sequence, we have

$$\limsup_{r \rightarrow \infty} \mathcal{E}_S^*(r) \leq - \sup_{\substack{\gamma \in \Gamma \\ t \in \mathbb{R}}} \left(\frac{\Lambda_{0,1}^*(\gamma, t)^{\frac{1}{h-1}} \Lambda_{1,1}^*(\gamma, t)^{\frac{1}{h-1}}}{\Lambda_{0,1}^*(\gamma, t)^{\frac{1}{h-1}} + \Lambda_{1,1}^*(\gamma, t)^{\frac{1}{h-1}}} \right)^{h-1}.$$

Proof: Given $\gamma \in \Gamma$, and t that satisfies $-\text{D}(\mathbb{P}_0^\gamma \parallel \mathbb{P}_1^\gamma) < t < \text{D}(\mathbb{P}_1^\gamma \parallel \mathbb{P}_0^\gamma)$, let

$$\delta = \frac{\Lambda_{1,1}^*(\gamma, t)^{\frac{1}{h-1}}}{\Lambda_{0,1}^*(\gamma, t)^{\frac{1}{h-1}} + \Lambda_{1,1}^*(\gamma, t)^{\frac{1}{h-1}}}, \quad (20)$$

and $s = \lfloor \delta r \rfloor$. We have

$$\mathcal{E}_S^*(r) \leq - \left[\left(\frac{s+1}{r} \right)^{h-1} \Lambda_{0,1}^*(\gamma, t) \right] \wedge \left[\left(\frac{r-s}{r} \right)^{h-1} \Lambda_{1,1}^*(\gamma, t) \right].$$

Since $s/r \rightarrow \delta$ as $r \rightarrow \infty$, we obtain

$$\begin{aligned} \limsup_{r \rightarrow \infty} \mathcal{E}_S^*(r) &\leq - [\delta^{h-1} \Lambda_{0,1}^*(\gamma, t)] \wedge [(1-\delta)^{h-1} \Lambda_{1,1}^*(\gamma, t)] \\ &= - \left(\frac{\Lambda_{0,1}^*(\gamma, t)^{\frac{1}{h-1}} \Lambda_{1,1}^*(\gamma, t)^{\frac{1}{h-1}}}{\Lambda_{0,1}^*(\gamma, t)^{\frac{1}{h-1}} + \Lambda_{1,1}^*(\gamma, t)^{\frac{1}{h-1}}} \right)^{h-1}, \end{aligned}$$

and taking the infimum over $\gamma \in \Gamma$ and $t \in \mathbb{R}$, the proposition is proved. \blacksquare

Under some additional symmetry assumptions, the inequality in the above proposition becomes an equality. This is shown in Proposition 6 in Section V.

B. Optimality of the Majority Decision Rule

Suppose that all leaves use the transmission function $\gamma \in \Gamma$. Finding an optimal counting strategy by carrying solving the optimization problem (16) requires us to search over a space with r^{h-1} elements, and also optimizing over t . The search can be daunting even for moderate values of r and h . For this reason, we now consider the case where r is odd, and the majority decision rule is used at every non-leaf node, i.e., a node transmits a 1 iff the majority of its immediate predecessors send a 1. For level 1 nodes, the majority decision rule corresponds to a LLRQ with threshold 0, while for nodes of level greater than 1, it corresponds to a counting quantizer with threshold $(r-1)/2$. In the proposition below, we develop a sufficient condition under which this strategy is optimal.

Proposition 4. Consider a r -symmetric tree network with $h > 1$, where r is an odd integer. Suppose that all leaves use the same transmission function γ . Let t_0 and t_1 be such that $\Lambda_{0,1}^*(\gamma, t_0) = r^{h-1}\Lambda_{1,1}^*(\gamma, t_0)$ and $\Lambda_{1,1}^*(\gamma, t_1) = r^{h-1}\Lambda_{0,1}^*(\gamma, t_1)$. Under Assumptions 1-2, and the restriction to counting strategies, if

$$\max \{ \Lambda_{0,1}^*(\gamma, t_0), \Lambda_{1,1}^*(\gamma, t_1) \} \leq \frac{2r^{h-1}(r+1)}{r^{h-1}(r-1) + r + 3} \Lambda_{0,1}^*(\gamma, 0), \quad (21)$$

the optimal error exponent is

$$\mathcal{E}_S^*(r) = -\left(\frac{r+1}{2r}\right)^{h-1} \Lambda_{0,1}^*(\gamma, 0),$$

and is achieved by using the majority decision rule at all relay nodes.

Proof: If $r = 1$, the network is equivalent to the parallel configuration, and there are no relay nodes to consider. In this case, $t_0 = t_1 = 0$ and the condition (21) holds with equality. Also, the formula for $\mathcal{E}_S^*(1)$ is the well known error exponent for the parallel configuration. Henceforth, we assume that $r > 1$.

For simplicity, let $\psi(t) = \Lambda_{0,1}^*(\gamma, t)$ and $\varphi(t) = \Lambda_{1,1}^*(\gamma, t)$. The sufficient condition (21) is obtained by approximating the convex functions ψ and φ with appropriate straight line segments as shown in Figure 3, and as we proceed to show.

Suppose that

$$b := \prod_{k=2}^h (s_k + 1) < a := \left(\frac{r+1}{2}\right)^{h-1} < c := \prod_{k=2}^h (r - s_k).$$

(The argument in the case when the above inequalities hold in the reverse direction will be similar.) We consider the solution to the equations

$$\begin{aligned} y &= \frac{b(\psi(t_0) - \psi(0))}{t_0} t + b\psi(0), \\ y &= -\frac{c(\varphi(0) - \varphi(t_0))}{t_0} t + c\varphi(0), \end{aligned}$$

which gives the intersection of the straight line approximations shown in Figure 3. Solving the linear equations, and observing that $\psi(0) = \varphi(0)$, we obtain

$$y = \frac{bc(1+d)}{c+bd} \psi(0),$$

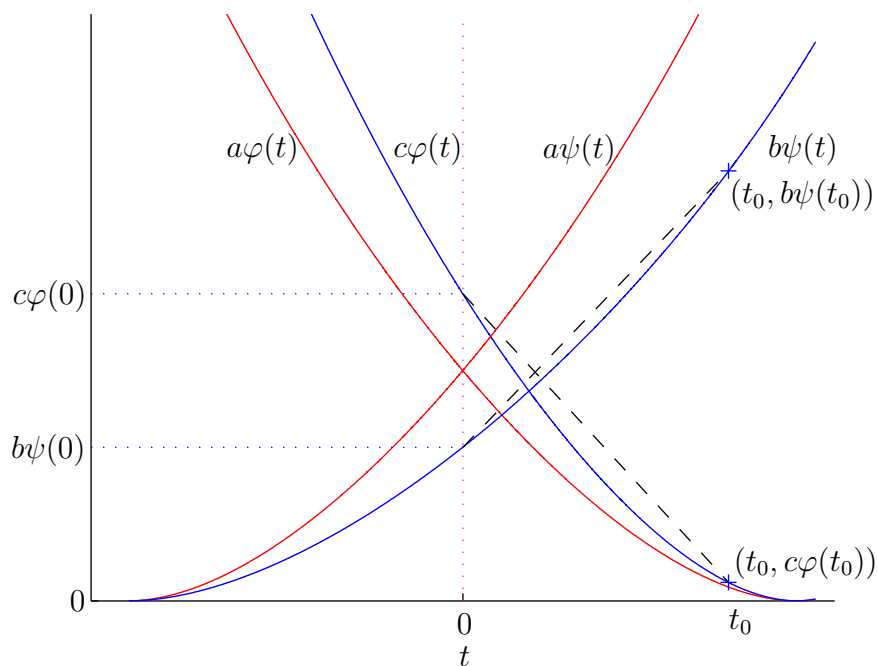


Fig. 3. A typical plot of the rate functions.

where $d = \frac{\psi(t_0) - \psi(0)}{\varphi(0) - \varphi(t_0)}$. Since ψ and φ are convex functions,

$$\sup_t \min\{b\psi(t), c\varphi(t)\} \leq y. \quad (22)$$

We first show that $y \leq a\psi(0)$, for all pairs (b, c) such that $b < a < c$. This is equivalent to checking that

$$d \leq \frac{c(a-b)}{b(c-a)} = \frac{a}{b} \left(1 - \frac{b-a}{c-a}\right) - 1, \quad (23)$$

for all (b, c) such that $b < a < c$. Using the condition

$$\Lambda_{0,1}^*(\gamma, t_0) \leq \frac{2r^{h-1}(r+1)}{r^{h-1}(r-1) + r + 3} \Lambda_{0,1}^*(\gamma, 0),$$

(which is a consequence of (21)), it can be shown (after some algebra) that

$$d \leq \frac{c^*(a-b^*)}{b^*(c^*-a)},$$

where $b^* = (r + 1)^{h-2}(r - 1)/2^{h-1}$ and $c^* = (r + 1)^{h-2}(r + 3)/2^{h-1}$. The right-hand side of (23) increases when b decreases (and c increases), hence the minimum value is achieved by $b = b^*$, and $c = c^*$. This shows that (23) holds for all (b, c) such that $b < a < c$, and therefore $y \leq a\psi(0)$. From (22), we then have

$$\sup_{b,c: b \stackrel{t}{\leq} a \leq c} \min\{b\psi(t), c\varphi(t)\} \leq a\psi(0).$$

A similar argument shows that

$$\sup_{b,c: c \stackrel{t}{\leq} a \leq b} \min\{b\psi(t), c\varphi(t)\} \leq a\psi(0).$$

Therefore, from Theorem 2, we obtain

$$\mathcal{E}_S^*(r) \geq -\frac{a}{r^{h-1}}\psi(0) = -\left(\frac{r+1}{2r}\right)^{h-1} \Lambda_{0,1}^*(\gamma, 0).$$

Finally, the proposition is proved by noting that the above inequality becomes an equality when we set each of the counting quantizer thresholds to $s_k = (r - 1)/2$. \blacksquare

To show that our sufficient condition in Proposition 4 is not vacuous, we provide an example in which the use of the majority decision rule does not give an optimal counting strategy.

Example 2. Consider a r -symmetric network, with $r = 45$ and $h = 3$. Suppose that each leaf sends the message 1 with probability $p_0 = 0.3$ under hypothesis H_0 , and with probability $p_1 = 0.9$ under hypothesis H_1 . If all non-leaf nodes use the majority decision rule (the counting quantizer thresholds are $s_2 = s_3 = 22$), we get an error exponent of $-129.2460/45^2$. If counting quantizers with thresholds $s_2 = s_3 = 23$ are used, our error exponent is $-129.5009/45^2$, which dominates (is more negative than) the one for the majority decision rule. In fact, it can be checked numerically that $s_2 = s_3 = 23$ is the optimal choice of counting quantizers. \square

The sufficient condition in (21) can be difficult to check if one does not have access to the functions $\Lambda_{j,1}^*(\gamma, t)$, $j = 0, 1$. A simpler but cruder sufficient condition is presented below; the proof is the same as in Proposition 4, except that we let $D(\mathbb{P}_1^\gamma \parallel \mathbb{P}_0^\gamma)$ play the role of t_0 , and $-D(\mathbb{P}_0^\gamma \parallel \mathbb{P}_1^\gamma)$ the role of t_1 .

Corollary 1. *Suppose that r is an odd integer greater than 1, and that all leaves use the same*

transmission function γ . Under Assumptions 1-2, and the restriction to counting strategies, if

$$\max\{D(\mathbb{P}_0^\gamma \parallel \mathbb{P}_1^\gamma), D(\mathbb{P}_1^\gamma \parallel \mathbb{P}_0^\gamma)\} \leq -\frac{2(r+1)}{r-1} \inf_{\lambda \in [0,1]} \Lambda_{0,0}(\gamma; \lambda),$$

then using the majority decision rule at all non-leaf nodes achieves the optimal error exponent.

V. RAPIDLY BRANCHING TREE SEQUENCES

In the previous section, we considered a symmetric tree sequence in which the number of non-leaf nodes is bounded. In this section, we consider tree sequences in which the number of non-leaf nodes becomes large, in a certain sense, as n increases. Motivated by Example 1, we define the following.

Definition 2. A rapidly branching tree sequence is a sequence of h -uniform trees $(T_n)_{n \geq 1}$, such that:

- (i) the number of immediate predecessors of each non-leaf node grows to infinity as n increases;
- (ii) there exists a sequence of positive reals $(\kappa_n)_{n \geq 1}$ such that κ_n decreases to 0 as n increases and such that for each level k node v , with $k \geq 2$, we have

$$\frac{\max_{u \in C_n(v)} l_n^2(u)}{\min_{u \in C_n(v)} l_n^2(u)} \leq \kappa_n |C_n(v)|.$$

A rapidly branching tree sequence is a sequence of trees in which the number of immediate predecessors of each node grows faster than the rate at which the tree becomes “unbalanced” The definition of a rapidly branching tree sequence implies that the number of immediate predecessors of every level 1 node grows uniformly fast, in a certain sense.

In Design II of Example 1, when n is large, with high probability, we have $l_n(u) \simeq l_n(v)$ for all level 1 nodes u and v . Therefore, this tree network fits our definition of a rapidly branching network with height $h = 2$. For a general h , a similar design can be used to approximate a h -hop MST [26]. In all of these designs, with high probability we get a rapidly branching tree network.

Since using LLRQs for every node is known to be optimal (see e.g. [5]), we assume that every node (including leaves) is allowed to use LLRQs. The number of nodes at each level k in a rapidly branching tree network grows with n . Similar to Section IV, the problem of finding

optimal LLRQs for each node in a rapidly branching tree network is, in general, intractable. Therefore, we make the following simplifying assumption.

Assumption 3. *Every node is allowed to use LLRQs, and every node at the same level k uses a LLRQ with the same threshold t_k .*

For notational simplicity, if each leaf uses a transmission function γ which is a LLRQ, we identify γ with the threshold of the LLRQ, i.e., $\gamma = t_0 \in \mathbb{R}$. We will first make a slight detour and show a limit theorem for a rapidly branching tree network. This result essentially shows that the bounds in Proposition 1 are tight, and is similar in spirit to tightness results for Chernoff bounds. The proof is provided in the Appendix.

Proposition 5. *Suppose that Assumptions 1-2 hold. Given a rapidly branching tree sequence $(T_n)_{n \geq 1}$, suppose each leaf sends its observation to its immediate successor using a transmission function $\gamma \in \Gamma$, and each level k node, where $k \geq 1$, uses a LLRQ with a common threshold t_k . Suppose that $\{\gamma, t_1, \dots, t_h\}$ satisfy (3)-(5). Then,*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{l_n(f)} \log \mathbb{P}_1(Y_f = 0) &= -\Lambda_{1,h}^*(\gamma, t^{(h)}), \\ \lim_{n \rightarrow \infty} \frac{1}{l_n(f)} \log \mathbb{P}_0(Y_f = 1) &= -\Lambda_{0,h}^*(\gamma, t^{(h)}). \end{aligned}$$

We now consider the Bayesian detection problem in a rapidly branching tree sequence, in which all nodes are constrained to sending binary messages.

Theorem 3. *Consider a rapidly branching tree sequence $(T_n)_{n \geq 1}$. Suppose that Assumptions 1-3 hold. Then, the optimal error exponent is*

$$\mathcal{E}_{RB}^* = - \sup_{\substack{\gamma \in \Gamma \\ t_1 \in \mathbb{R}}} \left(\frac{\Lambda_{0,1}^*(\gamma, t_1)^{\frac{1}{h-1}} \Lambda_{1,1}^*(\gamma, t_1)^{\frac{1}{h-1}}}{\Lambda_{0,1}^*(\gamma, t_1)^{\frac{1}{h-1}} + \Lambda_{1,1}^*(\gamma, t_1)^{\frac{1}{h-1}}} \right)^{h-1}. \quad (24)$$

Furthermore, if the supremum is achieved by $\gamma \in \Gamma$, and $t_1 \in (-D(\mathbb{P}_0^\gamma \parallel \mathbb{P}_1^\gamma), D(\mathbb{P}_1^\gamma \parallel \mathbb{P}_0^\gamma))$, then the optimal threshold for the fusion center is $t_h = 0$, and the optimal threshold t_k for level k nodes, where $k = 2, \dots, h-1$, is

$$t_k = \frac{\Lambda_{0,k-1}^*(\gamma, t^{(k-1)}) \Lambda_{1,k-1}^*(\gamma, t^{(k-1)})^{\frac{1}{h-k+1}} - \Lambda_{0,k-1}^*(\gamma, t^{(k-1)})^{\frac{1}{h-k+1}} \Lambda_{1,k-1}^*(\gamma, t^{(k-1)})}{\Lambda_{0,k-1}^*(\gamma, t^{(k-1)})^{\frac{1}{h-k+1}} + \Lambda_{1,k-1}^*(\gamma, t^{(k-1)})^{\frac{1}{h-k+1}}}.$$

We first state two lemmas that we will not prove. The proof of these two lemmas are easily obtained using simple algebra.

Lemma 4. *Given $k \geq 1$ and $a, b > 0$, we have*

$$\begin{aligned} & \min_{-b < x < a} \left(\left(\frac{a+b}{a(b+x)} \right)^{\frac{1}{k}} + \left(\frac{a+b}{b(a-x)} \right)^{\frac{1}{k}} \right)^k \\ &= \left(\left(\frac{1}{a} \right)^{\frac{1}{k+1}} + \left(\frac{1}{b} \right)^{\frac{1}{k+1}} \right)^{k+1}, \end{aligned}$$

and the minimizer is given by

$$x^* = \frac{ab^{\frac{1}{k+1}} - a^{\frac{1}{k+1}}b}{a^{\frac{1}{k+1}} + b^{\frac{1}{k+1}}}.$$

Lemma 5. *For $k \geq 2$, and $t^{(k)}$ satisfying (4)-(5), we have*

$$\inf_{\lambda \in [0,1]} \Lambda_{0,k}(\gamma, t^{(k)}; \lambda) = -\frac{\Lambda_{0,k}^*(\gamma, t^{(k)})\Lambda_{1,k}^*(\gamma, t^{(k)})}{\Lambda_{0,k}^*(\gamma, t^{(k)}) + \Lambda_{1,k}^*(\gamma, t^{(k)})},$$

and

$$\begin{aligned} \Lambda_{1,k}^*(\gamma, t^{(k)}) &= \frac{\Lambda_{1,k-1}^*(\gamma, t^{(k-1)})(\Lambda_{0,k-1}^*(\gamma, t^{(k-1)}) - t_k)}{\Lambda_{0,k-1}^*(\gamma, t^{(k-1)}) + \Lambda_{1,k-1}^*(\gamma, t^{(k-1)})}, \\ \Lambda_{0,k}^*(\gamma, t^{(k)}) &= \frac{\Lambda_{0,k-1}^*(\gamma, t^{(k-1)})(\Lambda_{1,k-1}^*(\gamma, t^{(k-1)}) + t_k)}{\Lambda_{0,k-1}^*(\gamma, t^{(k-1)}) + \Lambda_{1,k-1}^*(\gamma, t^{(k-1)})}. \end{aligned}$$

Proof of Theorem 3: Suppose that under Assumptions 1-3, an optimal strategy is for each leaf to use a LLRQ with threshold γ_n , and for each level k node, where $k \geq 1$, to use a LLRQ with threshold $t_{n,k}$. Let $(n_l)_{l \geq 1}$ be a subsequence such that

$$\lim_{l \rightarrow \infty} \frac{1}{l_{n_l}(f)} \log P_e(T_{n_l}) = \mathcal{E}_{RB}^*.$$

Since γ_n is bounded ($|\gamma_n|$ cannot diverge to infinity, otherwise every leaf reports either 1 or 0 with probability one asymptotically, under either hypothesis), there exists a subsequence $(u_l)_{l \geq 1}$ of $(n_l)_{l \geq 1}$ such that $\gamma_{u_l} \rightarrow \gamma \in \mathbb{R}$ as $l \rightarrow \infty$. Then, from Assumption 2, since $D(\mathbb{P}_0^\gamma \parallel \mathbb{P}_1^\gamma)$ and $D(\mathbb{P}_1^\gamma \parallel \mathbb{P}_0^\gamma)$ are bounded, the thresholds $t_{u_l,k}$ must satisfy $-D(\mathbb{P}_0^\gamma \parallel \mathbb{P}_1^\gamma) - 1 < t_{u_l,k} < D(\mathbb{P}_1^\gamma \parallel \mathbb{P}_0^\gamma) + 1$, for l sufficiently large; otherwise, it can be shown that either the Type I or Type II error exponent at the fusion center is zero.

Therefore, there exists a further subsequence $(m_l)_{l \geq 1}$ of $(u_l)_{l \geq 1}$ such that for all k , $\lim_{l \rightarrow \infty} t_{m_l, k} = t_k$, for some bounded t_k . Then, for all $\epsilon > 0$, from Proposition 5, we obtain

$$\mathcal{E}_{RB}^* \geq -\min\{\Lambda_{0,h}^*(\gamma + \epsilon, t_1 + \epsilon, \dots, t_k + \epsilon), \Lambda_{1,h}^*(\gamma - \epsilon, t_1 - \epsilon, \dots, t_k - \epsilon)\}.$$

Taking $\epsilon \rightarrow 0$, and noting that $\Lambda_{0,h}^*$ and $\Lambda_{1,h}^*$ are continuous in all their arguments, we get

$$\mathcal{E}_{RB}^* \geq -\min\{\Lambda_{0,h}^*(\gamma, t^{(h)}), \Lambda_{1,h}^*(\gamma, t^{(h)})\}.$$

This shows that there is no loss in optimality if we restrict the transmission functions to be the same for all n . Therefore, it remains to optimize over $\gamma \in \Gamma$ and over $t^{(h)}$. In this case, it is well known (using the same argument as in Corollary 3.4.6 of [27]) that the optimal fusion rule at the fusion center consists of a LLRQ with threshold $t_h = 0$. This yields

$$\begin{aligned} \mathcal{E}_{RB}^* &= \inf_{\substack{\lambda \in [0,1] \\ \gamma, t^{(h-1)}}} \Lambda_{0,h-1}(\gamma, t^{(h-1)}; \lambda) \\ &= -\sup_{\gamma, t^{(h-1)}} \frac{\Lambda_{0,h-1}^*(\gamma, t^{(h-1)}) \Lambda_{1,h-1}^*(\gamma, t^{(h-1)})}{\Lambda_{0,h-1}^*(\gamma, t^{(h-1)}) + \Lambda_{1,h-1}^*(\gamma, t^{(h-1)})} \\ &= -\left[\inf_{\gamma, t^{(h-2)}} \inf_{t_{h-1}} \left\{ \frac{1}{\Lambda_{0,h-1}^*(\gamma, t^{(h-1)})} + \frac{1}{\Lambda_{1,h-1}^*(\gamma, t^{(h-1)})} \right\} \right]^{-1} \\ &= -\left[\inf_{\gamma, t^{(h-2)}} \inf_{t_{h-1}} \left\{ \frac{\Lambda_{0,h-2}^*(\gamma, t^{(h-2)}) + \Lambda_{1,h-2}^*(\gamma, t^{(h-2)})}{\Lambda_{0,h-2}^*(\gamma, t^{(h-2)}) (\Lambda_{1,h-2}^*(\gamma, t^{(h-2)}) + t_{h-1})} \right. \right. \\ &\quad \left. \left. + \frac{\Lambda_{0,h-2}^*(\gamma, t^{(h-2)}) + \Lambda_{1,h-2}^*(\gamma, t^{(h-2)})}{\Lambda_{1,h-2}^*(\gamma, t^{(h-2)}) (\Lambda_{0,h-2}^*(\gamma, t^{(h-2)}) - t_{h-1})} \right\} \right]^{-1}, \end{aligned} \quad (25)$$

where (25) and (26) follow from Lemma 5. We take $a = \Lambda_{0,h-2}^*(\gamma, t^{(h-2)})$ and $b = \Lambda_{1,h-2}^*(\gamma, t^{(h-2)})$ in Lemma 4 to obtain

$$\mathcal{E}_{RB}^* = -\left[\inf_{\gamma, t^{(h-2)}} \left\{ \left(\frac{1}{\Lambda_{0,h-2}^*(\gamma, t^{(h-2)})} \right)^{1/2} + \left(\frac{1}{\Lambda_{1,h-2}^*(\gamma, t^{(h-2)})} \right)^{1/2} \right\}^2 \right]^{-1}.$$

The optimal error exponent and the optimal thresholds for the LLRQs then follow by repeating the above same argument for another $h - 2$ steps. The proof is now complete. \square

By taking $t_1 = 0$ in (24), we obtain a lower bound that matches the upper bound in (9). Hence one does no worse than by a factor of $1/2^{h-1}$ from the optimal error exponent of a parallel configuration.

For completeness, our next result shows that the bound in Proposition 3 is an equality if leaves can use LLRQs as transmission functions. In some sense, it is also a consistency result: trees with a fixed branching factor r , in the limit of large r , perform the same as rapidly branching trees.

Proposition 6. *Suppose that the set Γ of allowable transmission functions for the leaves includes LLRQs. Then, under Assumptions 1 and 2, we have*

$$\lim_{r \rightarrow \infty} \mathcal{E}_S^*(r) = - \sup_{\substack{\gamma \in \Gamma \\ t \in \mathbb{R}}} \left(\frac{\Lambda_{0,1}^*(\gamma, t)^{\frac{1}{h-1}} \Lambda_{1,1}^*(\gamma, t)^{\frac{1}{h-1}}}{\Lambda_{0,1}^*(\gamma, t)^{\frac{1}{h-1}} + \Lambda_{1,1}^*(\gamma, t)^{\frac{1}{h-1}}} \right)^{h-1} = \mathcal{E}_{RB}^*.$$

Proof: Consider a collection of tree sequences $\{(T(n, r))_{n \geq 1} : r \geq 1\}$ such that (a) each $(T(n, r))_{n \geq 1}$ is a r -symmetric tree sequence; and (b) for each r and for each n , every level 1 node in $T(n, r)$ has the same number of leaves attached to it. Then, from Theorem 2, the optimal error exponent for each tree sequence $(T(n, r))_{n \geq 1}$ is $\mathcal{E}_S^*(r)$.

Suppose that there exists a subsequence $(r_m)_{m \geq 1}$ such that $g = \lim_{m \rightarrow \infty} \mathcal{E}_S^*(r_m) < \mathcal{E}_{RB}^*$. Suppose that each tree sequence $(T(n, r_m))_{n \geq 1}$ uses the asymptotically optimal counting strategy proposed in Theorem 2. Note that this strategy also satisfies Assumption 3. We shall construct a rapidly branching tree sequence from $\{(T(n, r_m))_{n \geq 1} : m \geq 1\}$. Fix a positive $\epsilon < \mathcal{E}_{RB}^* - g$, and let $(n_m)_{m \geq 1}$ be an increasing sequence of positive integers such that

$$\frac{1}{l_{n_m}(f)} \log P_e(T(n_m, r_m)) \leq \mathcal{E}_S^*(r_m) + \epsilon.$$

Let $\tilde{T}_m = T(n_m, r_m)$. Then, it is an easy exercise to verify that $(\tilde{T}_m)_{m \geq 1}$ satisfies Definition 2 with $\kappa_m = 1/r_m$ (which goes to 0, as $m \rightarrow \infty$). We then have

$$\frac{1}{l_{n_m}(f)} \log P_e(\tilde{T}_m) = \frac{1}{l_{n_m}(f)} \log P_e(T(n_m, r_m)) \leq \mathcal{E}_S^*(r_m) + \epsilon.$$

Taking $m \rightarrow \infty$, we obtain $\limsup_{m \rightarrow \infty} (1/l_{n_m}(f)) \log P_e(\tilde{T}_m) \leq g + \epsilon < \mathcal{E}_{RB}^*$, a contradiction to Theorem 3. Therefore, we must have $\liminf_{r \rightarrow \infty} \mathcal{E}_S^*(r) \geq \mathcal{E}_{RB}^*$. Finally, from Proposition 3, we obtain the desired conclusion. ■

VI. CONCLUSION

In this paper, we studied the detection performance of large scale tree networks with bounded height, under a Bayesian formulation. We showed that the error probability decays exponentially fast with the number of nodes in the network, and provided bounds for the rate of decay. We also considered specific classes of tree networks to quantify the detection performance. In particular, we considered simple counting strategies in symmetric tree networks, and characterized the optimal detection performance over this class of strategies. We also compared the detection performance of symmetric tree networks (with a fixed number of relay nodes) to that of rapidly branching tree networks. We showed that for these classes of tree networks and transmission strategies, the Bayesian detection performance deteriorates with the height of the tree architecture, in contrast to the results for the Neyman-Pearson formulation [20].

Throughout this paper, we have assumed that every node makes a (conditionally) i.i.d. observation. A topic for further research is the case of correlated observations, which remains a relatively unexplored area, with work mainly limited to the parallel configuration [10], [28]–[32].

APPENDIX

A. Some Large Deviations Tools

In this section, we record some mathematical results from Large Deviations Theory that will be needed in the proofs. The first result comes from Cramèr’s Theorem (see [27] or [33]). We present here a modified version of the lower bound in Theorem 1.3.13 of [33].

Theorem 4. *Suppose that X has distribution \mathbb{P} , with log-moment generating function $\varphi(\lambda) = \log \mathbb{E}[\exp(\lambda X)]$. Let the Fenchel-Legendre transform of $\varphi(\lambda)$ be $\Phi(x) = \sup\{\lambda x - \varphi(\lambda) : \lambda \in \mathbb{R}\}$. Suppose that there exists an open interval I such that $\varphi(\lambda) < \infty$ for all $\lambda \in I$.⁴ Suppose that x lies inside the support of the distribution \mathbb{P} , and that there exists a corresponding $\lambda \in I$, such that $\varphi'(\lambda) = x$. Then, for any $\epsilon > 0$,*

$$\mathbb{P}(|X - x| < \epsilon) \geq \left(1 - \frac{\varphi''(\lambda)}{\epsilon^2}\right) \exp(-\Phi(x) - |\lambda|\epsilon).$$

⁴Note that the function $\varphi(\cdot)$ is convex and twice differentiable over I .

B. Proof of Proposition 5

We note that we cannot apply the Gärtner-Ellis Theorem directly here, because the asymptotic log-moment generating function is not differentiable (see the regularity conditions assumed in that theorem [27]). For this reason, our proof proceeds from first principles.

It can be shown that Definition 2 ensures that $n/l_n(f)$ goes to 1, as n grows large (cf. Lemma 4 of [20]). Therefore, Proposition 1 yields

$$\limsup_{n \rightarrow \infty} \frac{1}{l_n(f)} \log \mathbb{P}_1(Y_f = 0) \leq -\Lambda_{1,h}^*(\gamma, t^{(h)}), \quad (27)$$

$$\limsup_{n \rightarrow \infty} \frac{1}{l_n(f)} \log \mathbb{P}_0(Y_f = 1) \leq -\Lambda_{0,h}^*(\gamma, t^{(h)}). \quad (28)$$

Next, we show the following lower bound by induction on k : for all nodes v at level k , where $k \geq 1$, and for any $\epsilon > 0$, there exists a positive N such that for all $n \geq N$, we have

$$\frac{1}{l_n(v)} \log \mathbb{P}_1(Y_v = 0) \geq -\Lambda_{1,k}^*(\gamma, t^{(k)}) - \epsilon, \quad (29)$$

$$\frac{1}{l_n(v)} \log \mathbb{P}_0(Y_v = 1) \geq -\Lambda_{0,k}^*(\gamma, t^{(k)}) - \epsilon. \quad (30)$$

When $k = 1$, it is straightforward to apply Theorem 4 and Assumption 2 to obtain (29) and (30); the details are omitted. Now assume that the claim holds for all level k nodes. In what follows, we apply Theorem 4 to show that (29)-(30) hold for all level $k + 1$ nodes v . For each $u \in C_n(v)$, let $L_{u,n}$ be the log-likelihood ratio of the message sent by u . Let

$$\varphi(\lambda) = \log \mathbb{E}_1[e^{\lambda S_n(v)/l_n(v)}] = \sum_{u \in C_n(v)} \log \mathbb{E}_1[e^{\lambda L_{u,n}/l_n(v)}],$$

and consider the corresponding Fenchel-Legendre transform

$$\begin{aligned} \Phi(x) &= \sup_{\lambda \in \mathbb{R}} \{\lambda x - \varphi(\lambda)\} \\ &= \sup_{\lambda \in \mathbb{R}} \left\{ \lambda x - \sum_{u \in C_n(v)} \log \mathbb{E}_1[e^{\lambda L_{u,n}/l_n(v)}] \right\} \\ &\leq \sum_{u \in C_n(v)} l_n(u) \sup_{\lambda \in \mathbb{R}} \left\{ \frac{\lambda}{l_n(v)} x - \frac{1}{l_n(u)} \log \mathbb{E}_1[e^{\lambda L_{u,n}/l_n(v)}] \right\} \\ &= \sum_{u \in C_n(v)} l_n(u) \sup_{\lambda \in \mathbb{R}} \left\{ \lambda x - \frac{1}{l_n(u)} \log \mathbb{E}_1[e^{\lambda L_{u,n}}] \right\}. \end{aligned} \quad (31)$$

Note that the supremum inside the sum on the R.H.S. of (31) is achieved by some $\lambda \in [-1, 0]$.

For such a λ , we have

$$\begin{aligned} & \frac{1}{l_n(u)} \log \mathbb{E}_1 [e^{\lambda L_{u,n}}] \\ &= \frac{1}{l_n(u)} \log \left[\mathbb{P}_1(Y_u = 0)^{1+\lambda} \mathbb{P}_0(Y_u = 0)^{-\lambda} + \mathbb{P}_1(Y_u = 1)^{1+\lambda} \mathbb{P}_0(Y_u = 1)^{-\lambda} \right] \\ &\geq \max \left\{ \frac{1+\lambda}{l_n(u)} \log \mathbb{P}_1(Y_u = 0), -\frac{\lambda}{l_n(u)} \log \mathbb{P}_0(Y_u = 1) \right\} \\ &\quad + \frac{1}{l_n(u)} \log \min \{ \mathbb{P}_0(Y_u = 0), \mathbb{P}_1(Y_u = 1) \}. \end{aligned}$$

From (27)-(28), if n is large, then $\min \{ \mathbb{P}_0(Y_u = 0), \mathbb{P}_1(Y_u = 1) \}$ is close to 1. Together with the induction hypothesis on the node u , we then have for all n sufficiently large,

$$\begin{aligned} \frac{1}{l_n(u)} \log \mathbb{E}_1 [e^{\lambda L_{u,n}}] &\geq \max \left\{ -(1+\lambda) \Lambda_{1,k}^*(\gamma, t^{(k)}), \lambda \Lambda_{0,k}^*(\gamma, t^{(k)}) \right\} - \epsilon/3 \\ &= \Lambda_{1,k}(\gamma, t^{(k)}; \lambda) - \epsilon/3. \end{aligned}$$

Continuing from (31), we obtain

$$\begin{aligned} \Phi(x) &\leq \sum_{u \in C_n(v)} l_n(u) \sup_{\lambda \in \mathbb{R}} \left\{ \lambda x - \Lambda_{1,k}(\gamma, t^{(k)}; \lambda) \right\} + l_n(v) \frac{\epsilon}{3} \\ &= l_n(v) (\Lambda_{1,k+1}^*(\gamma, (t^{(k)}, x)) + \epsilon/3). \end{aligned} \tag{32}$$

Next, we show that there exists a finite positive constant c such that for all $\lambda \in [-l_n(v), 0]$, $\varphi''(\lambda) \leq c^2 \kappa_n$. For any level k node u , we have

$$\frac{L_{u,n}}{l_n(u)} = \frac{1}{l_n(u)} \log \frac{\mathbb{P}_1(Y_u = y)}{\mathbb{P}_0(Y_u = y)},$$

where $y \in \{0, 1\}$. The induction hypothesis yields that for n large, we have $|L_{u,n}| \leq l_n(u)c$, for

some constant c . Therefore, we obtain

$$\begin{aligned}
\varphi''(\lambda) &\leq \sum_{u \in C_n(v)} \frac{1}{l_n^2(v)} \frac{\mathbb{E}_1[L_{u,n}^2 \exp(\lambda L_{u,n}/l_n(v))]}{\mathbb{E}_1[\exp(\lambda L_{u,n}/l_n(v))]} \\
&\leq c^2 \sum_{u \in C_n(v)} \frac{l_n^2(u)}{l_n^2(v)} \\
&\leq \frac{c^2 \max_{u \in C_n(v)} l_n^2(u)}{|C_n(v)| \min_{u \in C_n(v)} l_n^2(u)} \\
&\leq c^2 \kappa_n,
\end{aligned} \tag{33}$$

where the last inequality follows from Definition 2. Fix a $x < t_{k+1}$, and a sufficiently small positive δ that is less than $\epsilon/3$, so that $x + \delta \leq t_{k+1}$. Then, it is an easy exercise to check that if t_{k+1} satisfies (5), and if n is large, the conditions in Theorem 4 are satisfied for $X = S_n(v)/l_n(v)$, and for some $\lambda \in [-l_n(v), 0]$. From Theorem 4, we have

$$\begin{aligned}
\frac{1}{l_n(v)} \log \mathbb{P}_1(Y_v = 0) &= \frac{1}{l_n(v)} \log \mathbb{P}_1(S_n(v)/l_n(v) \leq t_{k+1}) \\
&\geq \frac{1}{l_n(v)} \log \mathbb{P}_1(S_n(v)/l_n(v) \leq x + \delta) \\
&\geq -\frac{1}{l_n(v)} \Phi(x) - \frac{|\lambda|}{l_n(v)} \delta + \frac{1}{l_n(v)} \log(1 - \varphi''(\lambda)/\delta^2) \\
&\geq -\Lambda_{1,k+1}^*(\gamma, (t^{(k)}, x)) - \epsilon/3 - \epsilon/3 + \frac{1}{l(n)} \log(1 - c^2 \kappa_n/\delta^2) \\
&\geq -\Lambda_{1,k+1}^*(\gamma, t^{(k+1)}) - \epsilon,
\end{aligned}$$

where the penultimate inequality follows from (32) and (33), and $l(n) = \min_{v \in B_n} l_n(v)$. (Here, B_n is the set of level 1 nodes, so we have $l_n(v) \geq l(n)$ for all non-leaf nodes v .) The last inequality follows by taking n sufficiently large, and from the continuity of the function $\Lambda_{1,k+1}^*(\gamma, (t^{(k)}, \cdot))$. We have now shown (29) for all level $k+1$ nodes; the proof of (30) is similar. The induction is now complete, and the proposition is proved.

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