

ϵ -Optimization Schemes and L -Bit Precision:
Alternative Perspectives in Combinatorial Optimization

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OUTLINE

- **Input** with L -Bit Precision.
 - A strongly polynomial-time algorithm for the KNAPSACK PROBLEM.
 - A strongly polynomial-time algorithm for the 3-PARTITION PROBLEM.
 - The GROUP KNAPSACK PROBLEM remains NP-complete.
- **Output** within ε of Optimum.
 - A different notion of approximation: ε -optimality.
 - A fully polynomial-time ε -optimization scheme for a problem without (F)PTAS.
- **Summary** — Dealing with Imprecise Data.
 - Inverse optimization.
 - ε -optimization.
 - L -bit precision.

THE L -BIT PRECISION MODEL

All numbers are of the form $a \cdot 2^t$, with $a < 2^L$.

Needs space $O(L + \log t)$.

We assume either that

- L is constant (fixed precision), or
- $L = O(\log n)$ (logarithmic precision).

Polynomial time: polynomial in n and $\log T$.

Strongly polynomial time: polynomial in n .

A FIRST EXAMPLE — THE KNAPSACK PROBLEM

$$\begin{aligned} &\text{maximize} && \sum_{i=1}^n c_i x_i \\ &\text{s.t.} && \sum_{i=1}^n a_i x_i \leq b \\ &&& x \in \{0, 1\}^n \end{aligned}$$

Theorem. The KNAPSACK PROBLEM can be solved in strongly polynomial time under the fixed and logarithmic precision models.

A PSEUDO-POLYNOMIAL TIME ALGORITHM FOR KNAPSACK

Nodes: $\langle 0, 0 \rangle$ and $\langle k, w \rangle$ for each $k \in \{1, 2, \dots, n\}$ and $w \in \{1, 2, \dots, nC\}$.

Arcs: From $\langle k-1, w \rangle$ to $\langle k, w \rangle$ with length 0, and from $\langle k-1, w \rangle$ to $\langle k, w + c_k \rangle$ with length a_k .

Let $s_k = \max\{w : \text{there is a path from } \langle 0, 0 \rangle \text{ to } \langle k, w \rangle \text{ with length } \leq b\}$.

Observation. The maximum value for the knapsack problem is s_n , and a solution can be determined in $O(n^2 C)$ time.

A POLYNOMIAL-TIME ALGORITHM UNDER L -BIT PRECISION

Assume $c_j = d_j \cdot 2^{t_j}$, where $d_j < 2^L$. Assume $t_1 \geq t_2 \geq \dots \geq t_n$.

Let $C_i = \sum_{j=i}^n c_j$.

Let G^* denote the subgraph of G with node set

$$\{\langle k, w \rangle : 1 \leq k \leq n \text{ and } s_k - C_k \leq w \leq s_k\} \cup \{\langle 0, 0 \rangle\}$$

.

Observation 1. The shortest path from node $\langle 0, 0 \rangle$ to node $\langle k, w \rangle$ in G^* is the shortest path from node $\langle 0, 0 \rangle$ to node $\langle k, w \rangle$ in G .

Observation 2. If there is a path from node $\langle 0, 0 \rangle$ to node $\langle k, w \rangle$, then w is an integral multiple of 2^{t_k} .

COUNTING REACHABLE NODES IN G^*

The number of nodes $\langle k, w \rangle$ for fixed value of k is at most C_k , which is at most $n 2^L 2^{t_k}$.

The number of such nodes for which w is an integral multiple of 2^{t_k} is at most $n 2^L$.

If we consider nodes $\langle k, w \rangle$ for all values of $k \in \{1, 2, \dots, n\}$, the number of nodes in G^* reachable from node $\langle 0, 0 \rangle$ is at most $n^2 2^L$.

The knapsack problem can be solved in $O(n^2 2^L)$ time when the input values c_j are expressed with L -bit precision. This is strongly polynomial time when L is fixed or logarithmic.

A SECOND EXAMPLE — THE 3-PARTITION PROBLEM

Given $3n$ non-negative integers a_1, a_2, \dots, a_{3n} such that $\sum_{i=1}^{3n} a_i = nb$, is there a partition of the integers into n groups S_1, S_2, \dots, S_n of three elements each, such that $\sum_{a_i \in S_j} a_i = b$ for all j ?

- The 3-PARTITION PROBLEM is NP-complete under logarithmic precision.
- It can be solved in strongly polynomial time in the fixed precision model.

Lemma. There are at most $(L + 2)^2 2^{2L}$ triples of L -bit precision integers which sum to b .

Proof. Let $z_i = x_i 2^{y_i}$ for $1 \leq i \leq 3$ with $y_1 \geq y_2 \geq y_3$ and $z_1 + z_2 + z_3 = b$. Let $2^T \leq b < 2^{T+1}$. Clearly, $y_1 \leq T$. We also have

$$\begin{aligned}x_1 2^{y_1} &\geq b/3 \\ \Rightarrow x_1 2^{y_1} &> 2^{T-2} \text{ since } b \geq 2^T \\ \Rightarrow 2^{y_1} &> 2^{T-2}/x_1 \\ \Rightarrow 2^{y_1} &> 2^{T-2}/2^L \text{ since } x_1 < 2^L \\ \Rightarrow y_1 &> T - 2 - L\end{aligned}$$

Thus, $T - 1 - L \leq y_1 \leq T$, and the total possible values for $z_1 = x_1 2^{y_1}$ is $(L + 2)2^L$. \square

AN INTEGER PROGRAM IN FIXED DIMENSION

$$\sum_{j \in S_i} \alpha_{ij} x_j = n_i \quad \text{for } 1 \leq i \leq m$$

$$x_j \geq 0 \quad \text{for all } j$$

$$x_j \text{ integer} \quad \text{for all } j$$

x_j = number of times triple j occurs.

S_i = set of triples containing integer a'_i .

n_i = number of occurrences of a'_i .

α_{ij} = number of times that a'_i appears in triple j .

When the input integers are expressed with L -bit precision, for fixed L , the 3-PARTITION PROBLEM can be solved in strongly polynomial time.

FURTHER FACTS AND FIRST SUMMARY

- The **KNAPSACK PROBLEM** can be solved in strongly polynomial time under the fixed and the logarithmic precision model. A simple variant is NP-hard with one bit precision.
- **3-PARTITION** is NP-complete under logarithmic precision, but it enjoys a strongly polynomial-time algorithm under the fixed precision model.
- **IDENTICAL PARALLEL MACHINE SCHEDULING** can be solved in strongly polynomial time when m is fixed and L is fixed or logarithmic. It is NP-hard under logarithmic precision when m is part of the input.

TRANSITION

We are given an integer objective function c . If we round c and express it using L bits of accuracy and solve the resulting problem, then the returned solution is $(1/2^{L-1})$ -optimal.

ϵ -OPTIMIZATION

c' is an ϵ -perturbation of c if

$$\begin{aligned} c_j(1 - \epsilon) &\leq c'_j \leq c_j(1 + \epsilon) && , \text{ if } c_j > 0, \\ -c_j(1 - \epsilon) &\leq -c'_j \leq -c_j(1 + \epsilon) && , \text{ if } c_j < 0. \end{aligned}$$

A solution x' to $\min\{cx : x \in X\}$ is ϵ -optimal if there exists an ϵ -perturbation c' of c such that x' is optimal for $\min\{c'x : x \in X\}$.

An ϵ -optimization algorithm is an algorithm that produces an ϵ -optimal solution for every instance.

We will particularly be interested in (fully) polynomial-time ϵ -optimization schemes.

APPROXIMATION ALGORITHMS

An ε -approximation algorithm for $\min\{cx : x \in X\}$ gives in polynomial time for any instance a solution \bar{x} with

$$\frac{c\bar{x} - cx}{cx} \leq \varepsilon \quad \text{for all } x \in X .$$

Some drawbacks:

- If the optimal objective function value cx^* is non-positive, the relative error is an inappropriate measure of performance.
- A translation of variables has a dramatic impact; replacing x by $x - a$ will lead to very different results.

TWO GOALS

1. Investigate the relationship between ε -optimization schemes and preexisting approximation schemes.
2. Design ε -optimization schemes for specific combinatorial problems.

FPTAS vs. FPTEOS

Observation. If all cost coefficients and all feasible solutions are non-negative, and if x is ε -optimal, then x has worst-case relative error at most $2\varepsilon/(1 - \varepsilon)$.

Theorem. Whenever a 0/1-optimization problem $\min\{cx : x \in X\}$ has an FPTAS which can handle the fixing of variables, then this problem also has an FPTEOS.

FPTAS \longrightarrow FPTEOS

Initialization:

Let $q := \varepsilon/2n$.

Let $I := J := \emptyset$; $K := \{1, 2, \dots, n\}$.

Let x^* be any feasible solution.

Main Loop:

Find a q -approximate completion x' of $\langle I, J \rangle$.

if $cx' < cx^*$, **then** $x^* := x'$.

let $U := \max\{c_k : k \in K\}$.

for all indices $k \in K$ with $c_k > U/2$, **do**

if $x_k^* = 1$, **then** $J := J \cup k$, and $K := K - k$;

if $x_k^* = 0$, **then** $I := I \cup k$, and $K := K - k$;

if $K = \emptyset$, **then** stop; **else return to** Main Loop.

PROOF (CONTD.)

Moreover, we have $(c_j^* - c_j)(x_j^* - x_j) \leq 0$ for each $j \in K$,

and $(c_r^* - c_r)(x_r^* - x_r) = \epsilon c_r$.

Hence,

$$c^*(x^* - \bar{x}) = \sum_{j \in K} c_j^*(x_j^* - \bar{x}_j) = \sum_{j \in K} c_j(x_j^* - x_j) + \sum_{j \in K} (c_j^* - c_j)(x_j^* - x_j) \leq 0 .$$

FURTHER FACTS AND SECOND SUMMARY

- Notion of ϵ -optimality can be extended to the perturbation of other parameters. BIN PACKING and IDENTICAL PARALLEL MACHINE SCHEDULING have a PTEOS.
- There is a close linkage between pseudopolynomial-time algorithms, FPTAS's, and FPTEOS's.

CONCLUDING REMARKS (ON IMPRECISE DATA OPTIMIZATION)

- **Inverse Optimization** is an alternative approach to measure distance from optimality.
- **ϵ -Optimization** is an alternative approach to produce near-optimal solutions.
- **L -Bit Precision** is an alternative way to measure complexity.