17 Vectors; dot product

**Vectors**
Two views: First the geometric and then the analytic.

**Geometric view**
Vector = length and direction: (Discuss scaling, scalars)
\[ \rightarrow \quad \rightarrow \quad (\text{same vector}) \]
Length: denoted \( |\mathbf{A}| \), also called magnitude or norm
\[ \mathbf{A} \quad \mathbf{B} \quad \mathbf{A} + \mathbf{B} \]
Addition: (head to tail)

\[ \mathbf{A} \quad \mathbf{B} \quad \mathbf{A} - \mathbf{B} \]
Subtraction: either tail to tail or \( \mathbf{A} + (-\mathbf{B}) \)

**Analytic or algebraic view**
Place the tail of \( \mathbf{A} \) at the origin ⇒ the coordinates of the head determine \( \mathbf{A} \):
\[ \mathbf{A} = (a_1, a_2) = a_1 \mathbf{i} + a_2 \mathbf{j}. \]

You’ve seen the vectors \( \mathbf{i} \) and \( \mathbf{j} \) in physics. They have coordinates \( \mathbf{i} = (1, 0), \quad \mathbf{j} = (0, 1) \)

**Notation and terminology**
1. \( (a_1, a_2) \) indicate as point in the plane.
2. \( (a_1, a_2) \) indicates the vector from the origin to the point \( (a_1, a_2) \). Of course, this vector can be translated anywhere and \( (a_1, a_2) = a_1 \mathbf{i} + a_2 \mathbf{j}. \)
3. For \( \mathbf{A} = a_1 \mathbf{i} + a_2 \mathbf{j}, \quad a_1 \) and \( a_2 \) are called the \( \mathbf{i} \) and \( \mathbf{j} \) components of \( \mathbf{A} \). Note: they are scalars.
4. \( \overrightarrow{OP} = \mathbf{P} \) is the vector from the origin to \( P \).
5. In print we will often drop the arrow and just use the bold face to indicate a vector, i.e. \( \mathbf{P} \equiv \overrightarrow{OP} \).
7. Scalars: a real number is a scalar, you can use it to scale a vector.

Length: $|\mathbf{A}| = \sqrt{a_1^2 + a_2^2}$

Addition: $(a_1 \mathbf{i} + a_2 \mathbf{j}) + (b_1 \mathbf{i} + b_2 \mathbf{j}) = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}$

$\leftrightarrow \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle$

$\overrightarrow{PQ} = \overrightarrow{Q} - \overrightarrow{P}$ (i.e. $\overrightarrow{PQ}$ is the displacement from $P$ to $Q$) –understand this geometrically and analytically

**Dot product (scalar product)**

Geometric definition: $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos(\theta)$

$\begin{align*}
\mathbf{A} & \quad \mathbf{B} \\
\theta &
\end{align*}$

Algebraic view

$\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2$ (Hard to get geometrically)

Proof: Law of cosines: (won’t do in class)

$|\mathbf{A} - \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}| \cos(\theta)$

$\Rightarrow (a_1^2 + a_2^2) + (b_1^2 + b_2^2) - ((a_1 - b_1)^2 + (a_2 - b_2)^2) = 2|\mathbf{A}||\mathbf{B}| \cos(\theta)$

$\Rightarrow a_1b_1 + a_2b_2 = |\mathbf{A}||\mathbf{B}| \cos(\theta)$. QED

Algebraic law: $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$. (This follows from the algebraic view of dot product.)

**Example 17.1.** Find the dot product of $\mathbf{A}$ and $\mathbf{B}$.

(i) $|\mathbf{A}| = 2$, $|\mathbf{B}| = 5$, $\theta = \pi/4$.

**answer:** (draw picture) $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta = 10\sqrt{2}/2 = 5\sqrt{2}.$

(ii) $\mathbf{A} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{B} = 3\mathbf{i} + 4\mathbf{j}$.

**answer:** $\mathbf{A} \cdot \mathbf{B} = 1 \cdot 3 + 2 \cdot 4 = 11$.

**Unit vectors**

Special vectors: $\mathbf{i}$ and $\mathbf{j}$. Note: $\mathbf{i} \cdot \mathbf{i} = 1 = \mathbf{j} \cdot \mathbf{j}$ and $\mathbf{i} \cdot \mathbf{j} = 0$.

Unit vector: $\mathbf{u}$: $|\mathbf{u}| = 1$. Often indicate by $\hat{\mathbf{u}}$.

**Example 17.2.** Are the following unit vectors?

(i) $\mathbf{i} + \mathbf{j}$, (ii) $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$.

**answer:** (i) No. (ii) Yes.

**Example 17.3.** Find two unit vectors parallel to $2\mathbf{i} + 3\mathbf{j}$.

**answer:** $(2\mathbf{i} + 3\mathbf{j})/\sqrt{13}$, $-(2\mathbf{i} + 3\mathbf{j})/\sqrt{13}$

**Components or projection:**

$\mathbf{A} \cdot \mathbf{u} = |\mathbf{A}| \cos(\theta)$

$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2$
\[ A \perp B \iff A \cdot B = 0 \]

The component of \( A \) in the direction of \( \hat{u} \) is \( A \cdot \hat{u} \). (Note: it is a scalar.)

For a non-unit vector: the component of \( A \) in the direction of \( B \) is the component of \( A \) in the direction of \( \frac{B}{|B|} \).

**Example 17.4.** Find the component of \( A \) in the direction of \( B \).

(i) \(|A| = 2, |B| = 5, \theta = \pi/4\).

**answer:** (draw picture)

(ii) \( A = i + 2j, \ B = 3i + 4j \).

**answer:** Unit vector in direction of \( B \) is \( \frac{B}{|B|} = \frac{3}{5}i + \frac{4}{5}j \Rightarrow \) component is \( \frac{A \cdot B}{|B|} = \frac{3}{5} + \frac{8}{5} = 11/5 \).

**Trig identity** \( \cos(\beta - \alpha) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \)

Unit vectors: \( u = \cos \alpha i + \sin \alpha j, \ v = \cos \beta i + \sin \beta j \).

Angle between them is \( \theta = \beta - \alpha \)

Geometric: \( u \cdot v = |u||v| \cos \theta = \cos \theta = \cos(\beta - \alpha) \)

Analytic: \( u \cdot v = u_1v_1 + u_2v_2 = \cos \alpha \cos \beta + \sin \alpha \sin \beta \).

**Example 17.5.** \( P = (-5, 0), Q = (1, 3) \Rightarrow \vec{PQ} = 6i + 3j = \langle 6, 3 \rangle \).

**Example 17.6.** Show \( \vec{PQ} + \vec{QR} + \vec{QP} = 0 \)

**Example 17.7.** Find 2 unit vectors parallel to \( v = 3i - 4j \).

\(|v| = 5: \ u_1 = \frac{3}{5}i - \frac{4}{5}j, \ u_2 = -u_1 \).

**Example 17.8.** Let \( A = (1, 2), \ B = (2, 3) \) \ and \( C = (2, -1) \). Find the cosine of \( \angle BAC \).

**answer:** Let \( \theta \) be the angle \( \Rightarrow \cos \theta = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}||\vec{AC}|} \).

\( \vec{AB} = \langle 1, 1 \rangle, \ \vec{AC} = \langle 1, -3 \rangle \)

\( \Rightarrow \cos \theta = \frac{1 - 3}{\sqrt{2} \sqrt{10}} = -\frac{2}{\sqrt{20}} = \frac{1}{\sqrt{5}} \).
Example 17.9. Velocities are vectors
A river flows at 3mph and a rower rows at 6mph. What heading should he use to get straight across a river?

answer: Need \( \sin(\theta) = \frac{3}{6} \Rightarrow \theta = \pi/6 \)
Answer: Head at angle of \( \pi/6 \) radians upstream from straight across.

Example 17.10. Same question with river=2 mph, row=2\( \sqrt{2} \) mph:

answer: \( \sin(\theta) = \frac{2}{2\sqrt{2}} \Rightarrow \theta = \pi/4 \).

Example 17.11. Same question with river=6 mph, row=3 mph:

answer: \( \sin(\theta) = \frac{6}{3} \Rightarrow \) No such \( \theta \)

Example 17.12. Show the sum of the medians of a triangle = 0.
Median of \( \overrightarrow{AB} = P = \frac{1}{2}(A + B) \Rightarrow \overrightarrow{CP} = \frac{1}{2}(B + A) - C. \)
Likewise: \( \overrightarrow{BQ} = \frac{1}{2}(A + C) - B, \overrightarrow{AR} = \frac{1}{2}(B + C) - A. \)
Thus, the sum of medians = \( \overrightarrow{CP} + \overrightarrow{BQ} + \overrightarrow{AR} = 0. \)

Three dimensions
Exactly the same except we have a third coordinate:
\[
a_1i + a_2j + a_3k = \langle a_1, a_2, a_3 \rangle
\]

Example 17.13. Show that \( A = (4, 3, 6), B = (-2, 0, 8), C = (1, 5, 0) \) are the vertices of a right triangle.

answer: Two legs of the triangle are \( \overrightarrow{AC} = \langle -3, 2, -6 \rangle \) and \( \overrightarrow{AB} = \langle -6, -3, 2 \rangle \)
Taking their dot product we have
\[
\overrightarrow{AC} \cdot \overrightarrow{AB} = 18 - 6 - 12 = 0.
\]
The dot product is 0, implies the vectors are orthogonal, i.e. they are the legs of a right triangle.
18 Determinants; cross-product

\[
\begin{vmatrix}
 a & b \\
 c & d \\
\end{vmatrix} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc = \text{"diag" - "antidiag"}
\]

Example 18.1. \[\begin{vmatrix} 6 & 5 \\ 1 & 2 \end{vmatrix} = 7.\]

Row operations: (starting with \(\begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}\) = 7)

Scale row \(\rightarrow\) scale det, e.g. \[\begin{vmatrix} 12 & 10 \\ 1 & 2 \end{vmatrix} = 14\]

Swap rows \(\rightarrow\) change sign: e.g. \[\begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix} = -7.\]

Algebraic facts for \(\det(A)\)
1. 1 row or column all 0’s \(\Rightarrow\) \(\det(A) = 0\).
2. Multiply row (or column) by \(c\) \(\Rightarrow\) det multiplied by \(c\).
3. Interchange two rows (or columns) \(\Rightarrow\) \(\det = -\det(A)\)
4. Add multiple of one row (or column) to another \(\Rightarrow\) det unchanged.

More examples

Scale row 2 by 2: \[\begin{vmatrix} 6 & 5 \\ 2 & 4 \end{vmatrix} = 14\]

Add row 1 to row 2: \[\begin{vmatrix} 6 & 5 \\ 7 & 7 \end{vmatrix} = 7\]

Example 18.2. Compute \[\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}\]
We first simplify using row operations:

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{vmatrix} = (\text{add } R_1 \text{ to } R_3) \\
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
8 & 10 & 12
\end{vmatrix}
\]

\[-2R_2 \text{ to } R_3\]

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
0 & 0 & 0
\end{vmatrix} = 0
\]

Therefore,

\[
\begin{vmatrix}
1 & 1 & 1 \\
3 & 3 & 3 \\
5 & 6 & 7
\end{vmatrix} = 0
\]

**Laplace formula for determinants** of $3 \times 3$ matrices and bigger.

\[a_{i,j} \text{ notation: } i=\text{row}, \; j=\text{column}: \; A = \begin{vmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
a_{3,1} & a_{3,2} & a_{3,3}
\end{vmatrix}\]

\[i, \; j \text{ minor } = \text{det after removing the } i\text{th row and } j\text{th column (i.e. the row and col. with } a_{i,j}.\]

\[i, \; j \text{ cofactor } = (-1)^{i+j}(i, \; j \text{ minor}). \text{ I.e. sign } = \begin{vmatrix} + & - & + \\
+ & + & - \\
+ & - & + \end{vmatrix}.\]

\[\text{det}(A) = \sum \text{ entry } \times \text{ cofactor}\]

**Example 18.3.** Compute \[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{vmatrix}
\]

along the top row

**Answer:** determinant = \[
1 \cdot \begin{vmatrix}
5 & 6 \\
8 & 9
\end{vmatrix} - 2 \cdot \begin{vmatrix}
4 & 6 \\
7 & 9
\end{vmatrix} + 3 \begin{vmatrix}
4 & 5 \\
7 & 8
\end{vmatrix} = 0
\]

**Example 18.4.** Compute \[
\begin{vmatrix}
1 & 2 & 3 \\
5 & 0 & 7 \\
8 & 0 & 9
\end{vmatrix}
\]

**Answer:** Use second column: \[\text{det } = -2 \cdot \begin{vmatrix}
5 & 7 \\
8 & 9
\end{vmatrix} + 0 \cdot -0 \cdot = 22\]

**Geometry**

\[\text{abs} \left(\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right) = \text{area of} \]

\[
\begin{array}{c}
(a_1, a_2) \\
\theta \\
(b_1, b_2)
\end{array}
\]

(0, 0)

So the area = |A||B| \sin \theta

\[\text{abs} \left(\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\right) = \text{volume parallelepiped}\]
Example 18.5. Rotate $B$ by $\pi/2$ clockwise $\Rightarrow B' = (b_2, -b_1)$

Area $= |A||B| \sin \theta = |A||B| \cos(\pi/2 - \theta) = A \cdot B' = a_1b_2 - a_2b_1$

Example 18.6. Volume of parallelepiped with vertices $(0,0,0), (1,2,3), (5,0,7), (8,0,9)$ $= 22$.

Cross product

Geometric definition of $A \times B$

length $= \text{area} = |A||B| \sin \theta$

direction $= \text{right hand rule (draw picture)}$

Algebraic facts:

$A \times A = 0, \ A \times B = -B \times A, \ A \times (B + C) = A \times B + A \times C$ (not obvious)

Non-associative: $(A \times B) \times C \neq A \times (B \times C)$ (example in a moment)

$i \times j = k, j \times k = i, k \times i = j$.

(i.e. cycle: $i \rightarrow j \rightarrow k$) (in diagram loop $k$ back to $i$, also add pictures)

Example 18.7. $(i \times j) \times j = -i$ but $i \times (j \times j) = 0$.

$(2i + 3j) \times (3i - 2j) = -13k$

(Picture - geometry, algebra - distribute)

Determinants for cross product

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

Example 18.8. $\begin{vmatrix} i & j & k \\ 2 & 3 & 0 \\ 3 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} k = -13k$

DON’T FORGET THE GEOMETRY - it will be used to solve problems.
19 Matrices; inverses

19.1 Matrix Notation

Matrices = rectangular array of numbers.

Example 19.1. The following are all matrices:

(i) \[
\begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix},
\]
(ii) \[
\begin{bmatrix}
3 \\
4
\end{bmatrix},
\]
(iii) \[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix},
\]
(iv) \[
\begin{bmatrix}
1 & 0 \\
2 & 0 \\
0 & 1
\end{bmatrix},
\]
(v) \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]
(vi) \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

The size of a matrix is given the number of rows \(\times\) the number of columns.

Example 19.2. In the examples above the sizes are

(i) 2 \(\times\) 2, (ii) 2 \(\times\) 1, (iii) 3 \(\times\) 3, (iv) 3 \(\times\) 2, (v) 3 \(\times\) 3, (vi) 2 \(\times\) 2.

We read these as: (i) two-by-two, (ii) two-by-one, (iii) three-by-three etc.

The number in the \(i^{th}\) row and \(j^{th}\) column is called the \(i,j^{th}\) entry of the matrix.

Example 19.3. For the matrix \[
\begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix}
\] the 1,1-entry is 6, the 1,2-entry is 5, the 2,1-entry is 1, and the 2,2-entry is 2.

A matrix with only one column is also called a column vector, likewise if it has only one row it is called a row vector.

19.2 Matrix Multiplication

We will do this extensively in class. You should make sure you know it cold.

We’ll start with some examples. The rules are explained below. After you’ve read the rules come back here and make sure you can compute the examples.

Example 19.4.

\[
\begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix} \begin{bmatrix}
3 \\
4
\end{bmatrix} = \begin{bmatrix}
6 \cdot 3 + 5 \cdot 4 \\
1 \cdot 3 + 2 \cdot 4
\end{bmatrix} = \begin{bmatrix}
38 \\
11
\end{bmatrix}.
\]

Example 19.5.

(i) A 4 \(\times\) 2 times a 2 \(\times\) 3 is okay. It gives a 4 \(\times\) 3 matrix:

\[
\begin{bmatrix}
6 & 5 \\
1 & 2 \\
7 & 8 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
6 & 5 & 3 \\
1 & 2 & 4
\end{bmatrix} = \begin{bmatrix}
41 & 40 & 38 \\
8 & 9 & 11 \\
50 & 51 & 53 \\
1 & 2 & 4
\end{bmatrix}
\]

(ii) A 2 \(\times\) 2 times a 2 \(\times\) 3 is okay. It gives a 2 \(\times\) 3 matrix:

\[
\begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix} \begin{bmatrix}
1 & 3 & 5 \\
2 & 4 & 6
\end{bmatrix} = \begin{bmatrix}
16 & 38 & 60 \\
5 & 11 & 17
\end{bmatrix}
\]
(iii) A $2 \times 3$ times a $3 \times 2$ is okay. It gives a $2 \times 2$ matrix:

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix} \cdot \begin{bmatrix}
7 & 0 \\
8 & 0 \\
9 & 0
\end{bmatrix} = \begin{bmatrix}
50 & 0 \\
122 & 0
\end{bmatrix}
\]

(iv) A $2 \times 2$ times a $3 \times 2$ is not okay. Don’t try this at home.

\[
\begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix} \cdot \begin{bmatrix}
3 & 4 \\
5 & 6 \\
7 & 8
\end{bmatrix} \quad \text{NOT A VALID EXPRESSION}
\]

Written out formally the $i,j$-entry of $AB$ is given by the dot product of the $i^{th}$ row of $A$ dotted with the $j^{th}$ column of $B$. That is

\[
i,j\text{-entry of } AB = \langle i^{th} \text{ row of } A \rangle \cdot \langle j^{th} \text{ column of } B \rangle
\]

This is illustrated in the following example.

**Example 19.6.** In the matrix equation below we’ve put a line through the $3^{rd}$ row of first matrix and the $2^{nd}$ column of the second matrix. The dot product of this row and column is $3,2$-entry of the product, in this case it’s 51.

\[
\begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix} \cdot \begin{bmatrix}
6 & \frac{5}{2} \\
7 & 8 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
* & * \\
* & 51 \\
* & *
\end{bmatrix}
\]

We summarize the rule of compatibility: **Only compatibly sized matrices can be multiplied.**

For matrices $A$ and $B$: the product $AB$ only makes sense if the number of columns of $A$ equals the number of rows of $B$.

Said with symbols: the product $AB$ only makes sense if the $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix. The product $AB$ is an $m \times p$ matrix.

### 19.3 Algebraic rules

1. **Matrix multiplication is NOT commutative.** That is $AB \neq BA$.

**Example:** Let $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ then the following multiplications show that $AB \neq BA$

\[
\begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 27 \\ 5 & 8 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 9 \\ 3 & 6 \end{bmatrix}
\]

Indeed sometimes the matrices are only compatible for one order of multiplication.

**Example:** Let $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ then the product $AB$ is legitimate, but the product $BA$ does not make sense.
Lesson: You need to be careful and precise when doing matrix algebra. Make sure you multiply in the correct order.

2. Identity: The following matrices are called identity matrices:

\[
I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

They are called the identity for the same reason the scalar 1 is called the multiplicative identity. That is if you multiply the identity times anything you get back that anything. For example

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}
\]

Identity matrices are always square matrices. That is they have the same number of rows and columns. We use the subscripts in \( I_2 \), \( I_3 \) etc. to indicate the size of the identity. If the size is clear from the context we drop the subscript and just write \( I \) for the identity matrix.

3. Addition: To add two matrices they must be the same size. In this case you add corresponding terms (we say you add the matrices termwise):

\[
\begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 8 & 16 \end{bmatrix} = \begin{bmatrix} 8 & 9 \\ 9 & 18 \end{bmatrix}
\]

4. Scalar multiplication: To multiply a matrix by a scalar you multiply every term in the matrix by the scalar:

\[
7 \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 42 & 35 \\ 7 & 14 \end{bmatrix}
\]

5. Distributive law: For compatibly sized matrices

\[(A + B) \cdot C = A \cdot C + B \cdot C\]

This is easy to show, but we won’t take the time.

6. Associative law: \((AB)C = A(BC)\).

19.4 Matrices and systems of equations

We can use matrices to write systems of equations in a compact form and to solve these equations using the algebraic rules for matrices.

Example 19.7. Halloween candy: Halloween candy will rot kids teeth, but that doesn’t matter in my neighborhood we give out bags of it to all comers. I like to construct different bags for different people as follows.

<table>
<thead>
<tr>
<th>Candy</th>
<th>Bags</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Kids</td>
</tr>
<tr>
<td>Lollipops</td>
<td>2</td>
</tr>
<tr>
<td>Snickers</td>
<td>1</td>
</tr>
<tr>
<td>Candy corn</td>
<td>1</td>
</tr>
</tbody>
</table>
The table shows the amount of candy I put in each type of bag. For example the kids’ bags each get 2 lollipops, 1 Snickers and 1 candy corn.

After I buy a pile of candy I want to figure out how many bags of each type I can make. To do this let’s use letters to indicate the amounts of candy I started with and the numbers of bags I end up with.

\[
\begin{align*}
L &= \text{number of lollipops} \\
S &= \text{number of snickers} \\
C &= \text{number of candy corns} \\
K &= \text{number of kids bags} \\
T &= \text{number of teens bags} \\
A &= \text{number of adults bags}
\end{align*}
\]

There are equations that relate the amount of candy and the number of bags. For example,

\[S = K + T + 2A.\]

This reflects the fact that for snickers, kids get 1, teens get 1, and adults get 2.

Writing out similar equations for the other candies we get the system of equations

\[
\begin{align*}
2K + T &= L \\
K + T + 2A &= S \\
K + 2T + 2A &= C
\end{align*}
\]

We can write this system very nicely using matrix notation. Multiply out the following matrix equation and see that it gives the identical system as the one above

\[
\begin{bmatrix}
2 & 1 & 0 \\
1 & 1 & 2 \\
1 & 2 & 2
\end{bmatrix}
\begin{bmatrix}
K \\
T \\
G
\end{bmatrix}
=
\begin{bmatrix}
L \\
S \\
C
\end{bmatrix}
\]

We can make this more compact by assigning names to each of the terms. Let

\[
A = \begin{bmatrix}
2 & 1 & 0 \\
1 & 1 & 2 \\
1 & 2 & 2
\end{bmatrix}, \quad \vec{x} = \begin{bmatrix}
K \\
T \\
G
\end{bmatrix}, \quad \vec{c} = \begin{bmatrix}
L \\
S \\
C
\end{bmatrix}
\]

With this notation equation [1] becomes

\[A\vec{x} = \vec{c}.
\]

The matrix \(A\) is called the coefficient matrix because it contains the coefficients of the variables in the equation. Look again at the system [1] you should see the coefficient matrix sitting there.

A standard problem is to solve these equations for the unknown \(\vec{x}\). In our example, after buying the candy we know \(\vec{c}\) the amount of each candy that we have, but we have to solve for \(\vec{x}\) the number of bags of each type we can make.

There are many ways to solve for \(\vec{x}\). In the next section we will learn about doing it using inverses.
19.5 Inverses

You’ve been using inverses to solve algebra problems for a long time.

**Example 19.8.** Solve $7x = 9$.

**answer:** Of course $x = 9/7$, but let’s do it carefully and precisely as practice for how we have to do algebra with matrices.

\[
\begin{align*}
7x &= 9 \quad \text{(original equation)} \\
7^{-1} \cdot 7x &= 7^{-1} \cdot 9 \quad \text{(multiply both sides of the equation by } 7^{-1}) \\
x &= 7^{-1} \cdot 9 \quad \text{(this is the solution)}
\end{align*}
\]

We can do the same thing with matrix inverses, but now we really do need to be precise with the algebra. We start with a fully numerical example.

**Example 19.9.** Solve the system of equations
\[
\begin{align*}
6x + 5y &= 3 \\
x + 2y &= 4
\end{align*}
\]

**answer:** First we write this in matrix form
\[
\begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
=
\begin{bmatrix}
3 \\
4
\end{bmatrix}
\]

You should check that multiplying this out gives the same system we started with.

In a moment we’ll explain where the following formula came from. For now you should just check that it is correct:
\[
\begin{bmatrix}
2/7 & -5/7 \\
-1/7 & 6/7
\end{bmatrix}
\begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
= I.
\]

Now we will show how to use this to solve the system of equations via matrix algebra. You should notice the algebra is exactly parallel to our solution in the previous example with the matrices above playing the roles of $7^{-1}$ and $7$.

\[
\begin{align*}
\begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
&=
\begin{bmatrix}
3 \\
4
\end{bmatrix} \quad \text{(original equation)} \\
\begin{bmatrix}
2/7 & -5/7 \\
-1/7 & 6/7
\end{bmatrix}
\begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
&=
\begin{bmatrix}
2/7 & -5/7 \\
-1/7 & 6/7
\end{bmatrix}
\begin{bmatrix}
3 \\
4
\end{bmatrix} \quad \text{(multiply both sides by the same thing; be careful to multiply on the left on both sides)} \\
I \cdot 
\begin{bmatrix}
x \\
y
\end{bmatrix}
&=
\begin{bmatrix}
-2 \\
3
\end{bmatrix} \\
\begin{bmatrix}
x \\
y
\end{bmatrix}
&=
\begin{bmatrix}
-2 \\
3
\end{bmatrix} \quad \text{(this is the solution)}
\end{align*}
\]

The matrix \[
\begin{bmatrix}
2/7 & -5/7 \\
-1/7 & 6/7
\end{bmatrix}
\] is called the **multiplicative inverse** of the matrix \[
\begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix}.
\]

We use the usual notation for inverses:
\[
\begin{bmatrix}
2/7 & -5/7 \\
-1/7 & 6/7
\end{bmatrix}
= \begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix}^{-1}
\]
In general, for a square matrix $A$ its \textbf{inverse is the matrix} $A^{-1}$ which multiplies to the identity:

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

\textbf{Algebraic caution 1:} Only square matrices can have inverses.

\textbf{Algebraic caution 2:} Not all square matrices have inverses. For example it is easy to see that the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has no inverse.

\textbf{Solving systems of equations using inverses.} We can do this following the same model as examples 19.8 and 19.9.

\textbf{Problem:} Suppose $A$ is an $n \times n$ matrix which has an inverse $A^{-1}$. Suppose also that $\vec{b}$ is a column vector. Solve the system of equations $A\vec{x} = \vec{b}$.

\textbf{answer:} To make sure you understand the setup to this problem: compare it with example 19.9 and line up the corresponding parts. The solution is just an abstract repetition of example 19.9

$$A\vec{x} = \vec{b} \quad \text{(original equation)}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b} \quad \text{(multiply both sides by the same thing; be careful to multiply on the left on both sides)}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b} \quad \text{(this is the solution)}$$

\textbf{A useful rule for inverses:} 

$$(AB)^{-1} = B^{-1}A^{-1}.$$  

In words this says that inverse reverses the order of a product. You can remember it by thinking about shoes and socks. In the morning you put on your socks and then your shoes –this is not commutative, try putting on your shoes and then your socks–. In the evening you reverse the process by first taking off your shoes and then your socks. The formal algebraic proof is also straightforward.

\textbf{Proof:} We have to show that $B^{-1}A^{-1}$ is the inverse of $AB$. So, we have to show that their product is $I$.

$$(AB) \cdot (B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I \quad \text{QED}$$

\subsection*{19.6 Computing inverses}

Before computing an inverse we should be sure it exists. We have the following rule to help us.

\textbf{For a matrix $A$, $A$ inverse exists only when $A$ is square and $\det(A) \neq 0$.}

Note that this is similar to scalars: for a scalar $a$, $a^{-1}$ exists only when $a \neq 0$.

\textbf{Two-by-two inverses.} You should memorize this formula. We will use it a lot.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(2)
In words, switch the entries on the main diagonal and change the signs of the other two entries (but don’t move them).

**Example 19.10.** Use formula \(2\) to find the inverse of \(A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}\).

**answer:** \(\det(A) = 7\), so \(A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -5 \\ -1 & 6 \end{bmatrix}\). If you look, you’ll see this is the same inverse we used in example 19.9.

**Proof of formula [2].**

We can verify the formula gives the inverse by directly checking that the product of the inverse and the matrix is the identity.

\[
\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} da - bc & db - bd \\ -ca + ac & -cb + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

**QED**

**Notice:** The formula has division by \(\det(A) = ad - bc\). If the determinant is 0 then the inverse of \(A\) doesn’t exist.

**Finding inverses via the adjoint method**

For matrices bigger than \(2 \times 2\) finding inverses is more involved.

The **step-by-step algorithm** is the following:

1. Start with \(A\).
2. Find the matrix of minors.
3. Find the matrix of cofactors.
4. Find the adjoint.
5. Divide by \(\det(A)\).

Of course, we have to explain what each of these things is. We will now do that one item at a time.

**Matrix of minors.** To compute a minor of a matrix you remove one row and one column and compute the determinant. You did this when you used Laplace expansion to compute determinants.

A matrix has lots of minors: one for every (row, column) pair, so we label them by the row and column.

**Example 19.11.** Find the 1,3-minor of \(A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 0 \end{bmatrix}\).

**answer:** For the 1,3-minor we have to remove the 1\(^{st}\) row and 3\(^{rd}\) column.

\[
\begin{bmatrix} 1 & 2 \color{red}{3} \\ 4 & 5 & 6 \\ 1 & 2 \color{red}{0} \end{bmatrix}
\]
The determinant of the remaining $2\times2$ matrix is \[
\begin{vmatrix}
4 & 5 \\
1 & 2
\end{vmatrix}
= 8 - 5 = 3. \] So the $1,3$-minor of $A = 3$.

The matrix of minors of $A$ is just the matrix made up of all the minors. The $i,j$-entry of the matrix of minors is the $i,j$-minor of $A$.

**Example:** Find the matrix of minors of $A = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 2 & 0
\end{bmatrix}$.

**answer:** $A$ is a $3 \times 3$ matrix so its matrix of minors is also $3 \times 3$. Here is the computation for each minor:

1,1 minor: \[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 2 & 0
\end{vmatrix}; \quad 1,1\text{-minor} = \begin{vmatrix}
5 & 6 \\
2 & 0
\end{vmatrix} = -12; \quad \text{matrix of minors} = \begin{bmatrix}
-12 & * & * \\
* & * & * \\
* & * & *
\end{bmatrix}
\]

1,2 minor: \[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 2 & 0
\end{vmatrix}; \quad 1,2\text{-minor} = \begin{vmatrix}
4 & 6 \\
1 & 0
\end{vmatrix} = -6; \quad \text{matrix of minors} = \begin{bmatrix}
-12 & -6 & * \\
* & * & * \\
* & * & *
\end{bmatrix}
\]

1,3 minor: \[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 2 & 0
\end{vmatrix}; \quad 1,3\text{-minor} = \begin{vmatrix}
4 & 5 \\
1 & 2
\end{vmatrix} = 3; \quad \text{matrix of minors} = \begin{bmatrix}
-12 & -6 & 3 \\
* & * & * \\
* & * & *
\end{bmatrix}
\]

2,1 minor: \[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 2 & 0
\end{vmatrix}; \quad 2,1\text{-minor} = \begin{vmatrix}
2 & 3 \\
2 & 0
\end{vmatrix} = -6; \quad \text{matrix of minors} = \begin{bmatrix}
-12 & -6 & 3 \\
-6 & * & * \\
* & * & *
\end{bmatrix}
\]

There are 5 more minors to compute. We show each of them, but without labels. You should practice by naming them and computing their value.

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 2 & 0
\end{bmatrix}; \quad \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 0 \\
1 & 2 & 0
\end{bmatrix}; \quad \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 2 & 0
\end{bmatrix}; \quad \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 0 \\
1 & 2 & 0
\end{bmatrix}; \quad \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 2 & 0
\end{bmatrix}
\]

The entire matrix of minors is therefore: \[
\begin{bmatrix}
-12 & -6 & 3 \\
-6 & -3 & 0 \\
-3 & -6 & -3
\end{bmatrix}
\]

**Matrix of cofactors.** Recall the checkerboard of signs we used for computing the determinant: \[
\begin{bmatrix}
+ & - & + \\
- & + & - \\
+ & - & +
\end{bmatrix}. \] To compute the matrix of cofactors of $A$ you change the signs in the matrix of minors according to the checkerboard.

**Example:** Find the matrix of cofactors for the matrix $A$ from the previous example.

**answer:** The matrix of minors is \[
\begin{bmatrix}
-12 & -6 & 3 \\
-6 & -3 & 0 \\
-3 & -6 & -3
\end{bmatrix} \] So the matrix of cofactors is \[
\begin{bmatrix}
-12 & 6 & 3 \\
6 & -3 & 0 \\
-3 & 6 & -3
\end{bmatrix}. \]

(Look carefully at how we changed signs to go from minors to cofactors.)
**Adjoint.** To make the adjoint matrix you switch the rows and columns of the cofactors matrix.

**Example:** Find the adjoint matrix for the matrix $A$ in the previous examples.

**answer:** The matrix of cofactors is $egin{bmatrix} -12 & 6 & 3 \\ 6 & -3 & 0 \\ -3 & 6 & -3 \end{bmatrix}$. So the adjoint is $egin{bmatrix} -12 & 6 & -3 \\ 6 & -3 & 6 \\ -3 & 0 & -3 \end{bmatrix}$.

Notice that the first column of the adjoint has the same entries as the first row of the cofactors matrix and likewise for the other rows.

**Finding the inverse.** We now know how to perform all the steps of the algorithm to find the inverse. The next example will show a good way to organize the computation.

**Example 19.12.** Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 0 \end{bmatrix}$.

**answer:** In order to have an inverse we need $\det(A) \neq 0$. So our first step is to compute the determinant. We do this by expansion along the first row:

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 0 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 2 & 0 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = 1(-12) - 2(-6) + 3(3) = 9.$$ 

Since $\det(A) \neq 0$ we can proceed on and use the algorithm to compute $A^{-1}$. Only the first step requires any real computation.

The algorithm says to first compute the matrix of minors. Notice that we found the minors for the first row when we computed the determinant. We can reuse those and only need to compute the other 6. (Actually we’ll just use the answers from the previous examples.)

1. Matrix of minors $= \begin{bmatrix} -12 & 6 & 3 \\ -6 & -3 & 0 \\ -3 & -6 & -3 \end{bmatrix}$ (compute each minor).
2. Matrix of cofactors $= \begin{bmatrix} -12 & 6 & 3 \\ 6 & -3 & 0 \\ -3 & 6 & -3 \end{bmatrix}$ (apply checkerboard).
3. Adjoint $= \begin{bmatrix} -12 & 6 & -3 \\ 6 & -3 & 6 \\ 3 & 0 & -3 \end{bmatrix}$ (swap rows and columns).
4. Inverse: $A^{-1} = \frac{1}{9} \begin{bmatrix} -12 & 6 & -3 \\ 6 & -3 & 6 \\ 3 & 0 & -3 \end{bmatrix}$ (divide by $\det(A)$).

We can check this by multiplying by multiplying $A^{-1} \cdot A$ and seeing that we get $I$. (You’ll have to do the actual computation.)

$$A^{-1} \cdot A = \frac{1}{9} \begin{bmatrix} -12 & 6 & -3 \\ 6 & -3 & 6 \\ 3 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 0 \end{bmatrix} = I.$$ 

**Fun note:** This algorithm works for the $2 \times 2$ case as well. You should try it out, it’s very fast.

## 20 Square matrices, planes

### Square Systems

We look at $3 \times 3$ (and $2 \times 2$) but this applies to all dimensions.

\[
\begin{bmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
= 
\begin{bmatrix}
d_1 \\
d_2 \\
d_3 \\
\end{bmatrix}
\leftrightarrow A \cdot \vec{X} = \vec{d}
\]

The goal is to solve for $\vec{X}$ when $A$ and $\vec{d}$ are known.

First we study the case where $\det(A) \neq 0$, i.e. where $A^{-1}$ exists.

Working carefully:

\[
A \cdot \vec{X} = \vec{d}
\]

\[
\Rightarrow A^{-1}(A \cdot \vec{X}) = A^{-1} \cdot \vec{d}
\]

\[
\Rightarrow (A^{-1}A) \vec{X} = A^{-1} \cdot \vec{d}
\]

\[
\Rightarrow I \cdot \vec{X} = A^{-1} \cdot \vec{d}
\]

\[
\Rightarrow \vec{X} = A^{-1} \cdot \vec{d}
\]

**Conclusion:** If $\det(A) \neq 0$ then there is exactly one solution: $\vec{X} = A^{-1} \cdot \vec{d}$.

#### Example 20.1.

Using the example from last time:

$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$, $\det(A) = -4$, $A^{-1} = \begin{bmatrix} -2 & -2 & 2 \\ 0 & 4 & -4 \\ 1 & -3 & 1 \end{bmatrix}$

1. $\vec{X} = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 2 \\ 0 & 4 & -4 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/4 \end{bmatrix}$

(\check this answer)

2. Solve $A \cdot \vec{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ \Rightarrow $\vec{X} = A^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ = $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(This one is easy to solve –but note the analysis above guarantees this is the only solution.)

Now we look at the case $\det(A) = 0$ (i.e. where $A^{-1}$ doesn’t exist).

1. $A \cdot \vec{X} = \vec{0}$ has infinitely many solutions. (**Homogeneous case**)
2. If $\vec{d} \neq \vec{0}$ then $A \cdot \vec{X} = \vec{d}$ has either 0 or many solutions depending on $\vec{d}$.

#### Example 20.2. (1 x 1 case)

$7x = 5$ one solution

$7x = 0$ one solution

$0x = 5$ no solutions

$0x = 0$ infinitely many solutions
Cramer’s rule: read the supplementary notes.

Here’s the reasoning in the $2 \times 2$ case.

**Example 20.3.**

$$
\begin{bmatrix}
1 & 2 \\
3 & 6
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
d_1 \\
d_2
\end{bmatrix} \Rightarrow 
x + 2y = d_1 \\
3x + 6y = d_2
$$

Each of these equations is the equation of a line. Geometrically solving the system of equations means finding the intersection of these two lines. In this case the lines are parallel, if $d_2 = 3d_1$ the parallel lines are, in fact, the same and there are lots of solutions. Otherwise the lines are parallel and don’t intersect, so there are no solutions.

In general, $\det(A) = 0$ means the two rows are multiples of each other, i.e. the two lines are parallel.

Two possibilities:

1. The lines are different $\Rightarrow$ no solutions:
2. The lines are the same $\Rightarrow$ infinitely many solutions:

In the homogeneous case the lines are automatically the same (parallel and through the origin –see above picture).

**Summary** (valid for any size square system)

<table>
<thead>
<tr>
<th></th>
<th>$\det(A) \neq 0$</th>
<th>$\det(A) = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Homogeneous</td>
<td>1 solution: $\mathbf{X} = \mathbf{0}$</td>
<td>many solutions</td>
</tr>
<tr>
<td>Inhomogeneous</td>
<td>1 solution: $\mathbf{X} = A^{-1}\mathbf{d}$</td>
<td>depends on $\mathbf{d}$</td>
</tr>
</tbody>
</table>

NOTE: For the 1 solution cases –*no matter* how it’s found the solution is unique.

**Lines in the plane** (this is a warmup for planes in space)

**Slope-intercept form:** Given the slope $m$ and the $y$-intercept $b$ the equation of a line can be written $y = mx + b$.

**Point-normal form:** Given a point $(x_0, y_0)$ on the line and a vector $\langle a, b \rangle$ normal to the line the equation of the line can be written $a(x - x_0) + b(y - y_0) = 0$.

**Planes in point-normal form**

$P = (x_0, y_0, z_0) = \text{point in plane}$

$\mathbf{N} = \langle a, b, c \rangle = \text{normal to plane}$

$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ is the equation of the plane.
Abstractly, let $X = (x, y, z)$ and we can write:

$$\overrightarrow{PX} \cdot \vec{N} = 0 \iff (\overrightarrow{X} - \overrightarrow{P}) \cdot \vec{N} = 0 \iff \overrightarrow{X} \cdot \vec{N} = \overrightarrow{P} \cdot \vec{N}$$

**Example 20.4.** Find the plane through the point $(1,4,9)$ with normal $\langle 2,3,4 \rangle$.

**answer:** Point-normal form of the plane is $2(x-1) + 3(y-4) + 4(z-9) = 0$.

**Example 20.5.** Find the plane with normal $\vec{N} = \hat{k}$ containing the point $(0,0,3)$

**answer:** Equation of plane: $\langle 0,0,1 \rangle \cdot \langle x,y,z-3 \rangle = 0 \iff z = 3$.

**Example 20.6.** Find the plane with $x$, $y$ and $z$ intercepts $a$, $b$ and $c$.

**answer:** Fast way: the plane has the points $(a,0,0)$, $(0,b,0)$ and $(0,0,c) \Rightarrow$ equation is $x/a + y/b + z/c = 1$

Slow way (but, works in general):

The 3 points give us 2 vectors in the plane, $\langle -a, b, 0 \rangle$ and $\langle -a, 0, c \rangle$.

$\Rightarrow \vec{N} = \langle -a, b, 0 \rangle \times \langle -a, 0, c \rangle = \langle bc, ac, ab \rangle$.

Point-normal form: $bc(x-a) + ac(y-0) + ab(z-0) = 0$

$\Leftrightarrow bc x + ac y + ab z = abc \Leftrightarrow x/a + y/b + z/c = 1$.

Various methods of solving systems

**Example 20.7.** Solve

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overrightarrow{0}$$
**Example 20.8.** For what \( c \) does the following have a non-zero solution?

\[
\begin{align*}
2x + cz &= 0 \\
-x - y + 2z &= 0 \\
-x - 2y + 2z &= 0
\end{align*}
\]

**answer:** In matrix form:

\[
\begin{pmatrix}
2 & 0 & c \\
1 & -1 & 2 \\
1 & -2 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

So we want \( \det = 4 - c = 0 \Rightarrow c = 4 \).

In this case, \( \vec{x}_0 = (2, 0, 4) \times (1, -1, 2) = (4, 0, -2) \) is a solution (as is \( a \cdot \vec{x}_0 \) for any \( a \)).

**Example 20.9.** For what \( d_1 \) and \( d_2 \) does the following have a solution?

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \begin{pmatrix}
d_1 \\
d_2
\end{pmatrix}
\]

**answer:** The equations can be written \( x = d_1 \) and \( 0 = d_2 \). So,

\[
\begin{cases}
\text{if } d_2 \neq 0 & \text{no solutions} \\
\text{if } d_2 = 0 & \begin{pmatrix}
d_1 \\
y
\end{pmatrix} \text{ a solution for any } y
\end{cases}
\]

**Distances:**

1. **Distance point to plane:**

   Ingredients:  
   i) A point \( P \),  
   ii) A plane with normal \( \vec{N} \) and point \( Q \).

   The distance from \( P \) to the plane is

   \[
   d = |\vec{PQ}| \cos \theta = |\vec{PQ} \cdot \frac{\vec{N}}{|N|}|.
   \]

**Example 20.10.** Let \( P = (1, 3, 2) \). Find the distance from \( P \) to the plane \( x + 2y = 3 \).

\[
\begin{pmatrix}
P \\
\vec{N} \\
\theta \\
Q
\end{pmatrix}
\]

\( Q \) = any point on the plane, we take \( Q = (3, 0, 0) \).

\( N \) = normal to plane = \( \hat{i} + 2\hat{j} = (1, 2, 0) \).

Distance = \( |\text{Proj}_N \vec{PQ}| = |\vec{PQ} \cdot \frac{\vec{N}}{|N|}| = |\vec{PQ} \cdot \frac{\vec{N}}{|N|}| = \frac{4}{\sqrt{5}}. \)
2. **Distance point to line:**

**Ingredients:** (i) A point \( P \), (ii) A line with direction vector \( v \) and point \( Q \).

The distance from \( P \) to the line is

\[
d = |QP| \sin \theta = \left| \overrightarrow{QP} \times \frac{v}{|v|} \right|.
\]

An alternate formula using projection is

\[
\overrightarrow{QR} = \text{Proj}_v \overrightarrow{QP} = \left( \overrightarrow{QP} \cdot \frac{v}{|v|} \right) \frac{v}{|v|},
\]

\[
d = |\overrightarrow{RP}| = |\overrightarrow{QP} - \overrightarrow{QR}|.
\]

**Example 20.11.** Let \( P = (1, 3, 2) \), find the distance from the point \( P \) to the line through \((1, 0, 0)\) and \((1, 2, 0)\).

\[\begin{array}{c}
P \\
\theta \\
Q \\
R
\end{array}\]

**answer:** \( Q = \) any point on the line, we take \( Q = (1, 0, 0) \).

\( v = \) direction vector of line = \( \langle 1, 2, 0 \rangle - \langle 1, 0, 0 \rangle = 2\hat{j} \).

\( R = \) point on line closest to \( P \) (unknown).

\( \overrightarrow{QP} = 3\hat{j} + 2\hat{k} \).

Method 1: cross product formula: \( d = |\overrightarrow{QP} \times \frac{v}{|v|}| = |(3\hat{j} + 2\hat{k}) \times \hat{j}| = 2 \).

Method 2: projection formula: \( \overrightarrow{QR} = \left( \overrightarrow{QP} \cdot \frac{v}{|v|} \right) \frac{v}{|v|} = 3\hat{j} \Rightarrow d = |\overrightarrow{QP} - \overrightarrow{QR}| = |2\hat{k}| = 2 \).

3. **Distance between parallel planes:** Reduce to the distance from a point to a plane.

**Example 20.12.** Find the distance between the planes \( x + 2y - z = 4 \) and \( x + 2y - z = 3 \).

**answer:** Both planes have normal \( N = \hat{i} + 2\hat{j} - \hat{k} \) so they are parallel. Take any point on the first plane, say, \( P = (4, 0, 0) \).

Distance between planes = distance from \( P \) to second plane.

Choose \( Q = (3, 0, 0) = \) point on second plane \( \Rightarrow d = |\overrightarrow{QP} \cdot \frac{N}{|N|}| = |\hat{i} \cdot (\hat{i} + 2\hat{j} - \hat{k})|/\sqrt{6} = 1/\sqrt{6} \).

4. **Distance between skew lines:** Put the lines in parallel planes.

Normal to planes = \( N = \mathbf{v}_1 \times \mathbf{v}_2 \), where \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are the dir. vectors of the lines.

**Geometric view of the 3 by 3 case**

For 3 by 3 system of equations there are more possibilities than for a 2 by 2 system. Geometrically, solving means finding the intersection of three planes.

As always, we write the system as \( AX = d \).

If \( \det(A) = 0 \) then the 3 planes are all perpendicular to one plane (volume = 0 \( \Rightarrow \) rows of \( A \) are all in a plane \( \Rightarrow \) normals all in this plane). A head on view gives the following possibilities:
21 Parametric equations and vector derivatives

Parametric Equations

General parametric equations:

Notation: \( \vec{r} (t) = (x(t), y(t)) = x(t) \hat{i} + y(t) \hat{j} = \text{position vector} \)

We can view \((x(t), y(t))\) as the coordinates of a particle moving in the plane or we can view \(\vec{r} (t) = (x(t), y(t))\) as its position vector.

Example 21.1. A rocket takes off from the origin with initial \(x\)-velocity \(v_{0,x}\) and initial \(y\)-velocity \(v_{0,y}\). Find the parametric equations for its path.

Physics \(\Rightarrow\) \(x(t) = v_{0,x} t, \quad y(t) = -\frac{1}{2}gt^2 + v_{0,y} t.\)

At time \(t\) the rocket is at point \(P = (x(t), y(t))\). We call the vector \(\vec{r}(t) = \overrightarrow{OP} = x(t) \hat{i} + y(t) \hat{j}\) the position vector.

![Diagram of a rocket's path](image)

Lines

parametric form \((t=\text{param.})\) \quad symmetric form
\[
\vec{X} = \vec{A} + t \vec{V} \\
\Leftrightarrow \begin{cases} 
\quad x = a_1 + v_1 t \\
\quad y = a_2 + v_2 t 
\end{cases} \quad \Leftrightarrow \frac{x-a_1}{v_1} = \frac{y-a_2}{v_2}
\]

Example 21.2. Find the line through \((0, -2, 1)\) and \((1, 0, 2)\)
answer: \( \vec{v} = \overrightarrow{PQ} = \overrightarrow{Q} - \overrightarrow{P} = (1, 2, 1) \)

\[ \vec{X} = \langle t, -2 + 2t, 1 + t \rangle = t \hat{i} + (-2 + 2t) \hat{j} + (1 + t) \hat{k} \]

We can also write this in column vector form as

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
0 \\
-2 \\
1
\end{bmatrix} +
\begin{bmatrix}
t \\
2 \\
t
\end{bmatrix}.
\]

**Velocity, speed and arclength**

Now we see the benefits of using the position vector.

\[
\frac{\text{d} \mathbf{r}}{\text{d} t} = \text{change in position} \quad \text{change in time} = \text{velocity} = \mathbf{v}(t) = x'(t) \hat{i} + y'(t) \hat{j} = \langle x', y' \rangle
\]

Geometrically: the picture shows a parametric curve for a moving point. Over a small time \( \text{d}t \), the point moves a \( \text{d} \mathbf{r} = \text{dx} \hat{i} + \text{dy} \hat{j} \) and its (instantaneous) velocity

\[
\frac{\text{d} \mathbf{r}}{\text{d} t} = \frac{\text{dx}}{\text{d} t} \hat{i} + \frac{\text{dy}}{\text{d} t} \hat{j} = \langle x'(t), y'(t) \rangle
\]

The picture for arclength, \( s \), shows

\[
\text{ds} = |\text{d} \mathbf{r}| = \sqrt{(\text{dx})^2 + (\text{dy})^2}
\]

\[ \Rightarrow \text{speed} = \frac{\text{d}s}{\text{d}t} = \frac{\text{distance}}{\text{time}} = \left| \frac{\text{d} \mathbf{r}}{\text{d} t} \right| = \sqrt{\left(\frac{\text{dx}}{\text{d} t}\right)^2 + \left(\frac{\text{dy}}{\text{d} t}\right)^2}. \]

**Acceleration** (change in velocity/change in time)

\[ \mathbf{a}(t) = \frac{\text{d} \mathbf{v}}{\text{d} t} = \frac{\text{d}^2 \mathbf{r}}{\text{d} t^2} = x''(t) \hat{i} + y''(t) \hat{j} = \langle x'', y'' \rangle \]

**Example 21.3.** Find velocity, speed, acceleration and arclength for the rocket example.

**answer:** \( \mathbf{r}(t) = x(t) \hat{i} + y(t) \hat{j} = v_{0,x} t \hat{i} + (-\frac{1}{2}gt^2 + v_{0,y}) \hat{j} \).

\[ \Rightarrow \mathbf{v}(t) = \frac{\text{d} \mathbf{r}}{\text{d} t} = v_{0,x} \hat{i} + (-gt + v_{0,y}) \hat{j}, \quad \Rightarrow \mathbf{a}(t) = \frac{\text{d} \mathbf{v}}{\text{d} t} = -g \hat{j}. \]

\[ \text{Speed} = \frac{\text{d}s}{\text{d}t} = |\mathbf{v}(t)| = \sqrt{v_{0,x}^2 + (-gt + v_{0,y})^2}. \]

To avoid too many symbols. Let \( v_{0,x} = 10, \; v_{0,y} = 10 \; g = 10 \) and find the arclength of the path from \( t = 0 \) to \( t = 1 \).

\[ \text{Arclength} \; L = \int_0^1 \frac{\text{d}s}{\text{d}t} \text{d}t = 10 \int_0^1 \sqrt{1 + (-t + 1)^2} \text{d}t \]

Make the change of variables \( u = -t + 1 \)

\[ \Rightarrow \text{du} = -\text{dt}, \; t = 0 \rightarrow u = 1, \; t = 1 \rightarrow u = 0. \quad \Rightarrow \; L = 10 \int_0^1 \sqrt{1 + u^2} \text{du}. \]

Use trig. subst. \( u = \tan \theta \)
\[ L = 10 \int_0^{\pi/4} \sec^3 \theta \, d\theta = 5 \sec \theta \tan \theta + \ln(\sec \theta + \tan \theta) \bigg|_0^{\pi/4} = 5(\sqrt{2} + \ln(\sqrt{2} + 1)). \]

**Tangent vector:** (same thing as velocity) In the picture above, we see that as \( \Delta t \) shrinks to 0 the vector \( \frac{\Delta \mathbf{r}}{\Delta t} \) becomes tangent to the curve.

When the parameter is \( t \) we can refer to \( \mathbf{r}'(t) \) as the *velocity*. In general, the derivative is given its geometric name: the *tangent vector*.

**Unit tangent vector**

The unit vector in the direction of the *tangent vector* is denoted \( \mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}}{v} \).

It’s called the *unit tangent vector*.

Note \( \frac{d\mathbf{T}}{dt} = \mathbf{r}'(t) \).

**Reparameterization** = Changing variables.

**Example 21.4.** Circle = \( \mathbf{r}(\theta) = 2(\cos \theta, \sin \theta) \)

Suppose you know \( \theta = 2\pi t \) then \( \mathbf{r}(t) = 2(\cos(2\pi t), \sin(2\pi t)) \)

These represent the same trajectory, with different parameterizations.

When dealing with more than one variable we use Liebnitz notation to avoid confusing.

The chain rule gives: \( \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta} \frac{d\theta}{dt} \)

NOTES: 1. This shows the two tangent vectors \( \frac{d\mathbf{r}}{dt} \) and \( \frac{d\mathbf{r}}{d\theta} \) are parallel – as geometric considerations tell us they must be.

2. Likewise \( \frac{ds}{d\theta} = \frac{|d\mathbf{r}|}{|d\theta|} \)

**Curvature:** How sharply curved is the trajectory?

That is, how fast does the tangent vector turn per unit arc length?

This is tricky so pay attention. Curvature is the rate \( \mathbf{T} \) is turning per unit arc length.

That is, \( \kappa = \left| \frac{d\mathbf{T}}{ds} \right| \).

(Smaller circle = faster turning = greater curvature.)

\[
\begin{align*}
\Delta T & \quad \Delta s \\
T & \quad T
\end{align*}
\]

Note well, curvature is a geometric idea – we measure the rate with respect to arc length.

The speed the point moves over the trajectory is irrelevant.

\( \mathbf{T} \) is a unit vector \( \Rightarrow \mathbf{T} = \langle \cos \phi, \sin \phi \rangle \) where \( \phi \) is the tangent angle.

\[ \Rightarrow \frac{d\mathbf{T}}{ds} = \frac{d}{ds} \langle \cos \phi, \sin \phi \rangle = \frac{d\phi}{ds} \langle -\sin \phi, \cos \phi \rangle. \]
Both magnitude and direction of $\frac{dT}{ds}$ are useful:

Curvature $\kappa = \frac{|dT|}{ds} = \frac{|d\phi|}{ds}$.

Direction of $\frac{dT}{ds} = N = \text{unit normal} \perp T$.

Note: the book doesn’t use the absolute value in its definition of $\kappa$, but it’s more standard to include it.

**Radius of curvature** $= \frac{1}{\kappa}$.

**The center of curvature and the osculating circle:**

The osculating (kissing) circle is the best fitting circle to the curve.

Radius $= \text{radius of curvature}$.

Center along normal direction.

**Nomenclature summary:**

Here are a list of names and formulas. We will motivate and derive them below.

$r(t) = \text{position}$.

$s = \text{arclength, speed} = v = \frac{ds}{dt}$.

$v(t) = r'(t) = \frac{ds}{dt} \ T = \text{tangent vector, velocity}$.

$a(t) = \frac{dv}{dt} = \frac{d^2r}{dt^2} = \text{acceleration}$.

$T = \text{unit tangent vector, } N = \text{unit normal vector}$.

$\kappa = \text{curvature, } R = \frac{1}{\kappa} = \text{radius of curvature}$.

$\phi = \text{tangent angle}$.

$C = \text{Center of curvature} = \text{center of best fitting circle (has radius} = \text{radius of curvature)}$.

**Formulas:** (explained in the following pages)

1. Speed $= \frac{ds}{dt} = |v(t)| = \sqrt{(x')^2 + (y')^2}$.

2. $v = \frac{ds}{dt} T, \ T = \frac{v}{ds/dt}$.
3. \( \mathbf{a}(t) = \frac{d^2 s}{dt^2} \mathbf{T} + \kappa \left( \frac{ds}{dt} \right)^2 \mathbf{N} = \frac{d^2 s}{dt^2} \mathbf{T} + \frac{v^2}{R} \mathbf{N} \)

4. \( \kappa = \frac{|d\mathbf{T}|}{ds} = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} \) (check dimensions), \( \frac{dT}{ds} = \kappa \mathbf{N} \).

4a. For plane curves \( \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} \):
\[
\kappa = \left| \begin{vmatrix} x'' & y' \\ x' & y'' \end{vmatrix} \right| = \left| a \times v \right| \left| v \right| ^3.
\] (Maximum curvature at \( t = 0 \) as expected.)

\[
R = \frac{1}{\kappa} = \frac{(1+4t^2)^{3/2}}{2}.
\]

5. \( \mathbf{v} \times (\mathbf{a} \times \mathbf{v}) = \kappa \mathbf{v}^4 \mathbf{N} \).

6. \( C = \mathbf{r} + R \mathbf{N} = \mathbf{r} + \frac{1}{\kappa} \mathbf{N} \).

**Example 21.5.** For the parabola \( \mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} \) find \( \mathbf{v}, \mathbf{a}, \mathbf{T}, ds/dt, \kappa, R, \mathbf{N} \) and \( C \) for arbitrary \( t \).

\( \mathbf{v} = \mathbf{i} + 2t \mathbf{j}, \quad \mathbf{a} = 2 \mathbf{j} \).

\[
\Rightarrow ds/dt = |\mathbf{v}| = \sqrt{1 + 4t^2}, \quad \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+4t^2}} \mathbf{i} + \frac{2t}{\sqrt{1+4t^2}} \mathbf{j}.
\]

Formula 4: \( \mathbf{a} \times \mathbf{v} = -2 \mathbf{k} \). \( \Rightarrow \kappa = \frac{2}{(1+4t^2)^{3/2}} \).

Formula 5: \( \mathbf{v} \times (\mathbf{a} \times \mathbf{v}) = -4t \mathbf{i} + 2 \mathbf{j} \). \( \Rightarrow \mathbf{N} = \frac{1}{2\sqrt{1+4t^2}} (-4t \mathbf{i} + 2 \mathbf{j}) \).

Formula 6: \( C = \mathbf{r} + R \mathbf{N} = (t \mathbf{i} + t^2 \mathbf{j}) + \frac{1+4t^2}{4} (-4t \mathbf{i} + 2 \mathbf{j}) \).

**Example 21.6.** For the line through \((0, -2, 1)\) and \((1, 0, 2)\) (this is from a previous example) find
\( x(t), y(t), z(t), \mathbf{v}, \mathbf{T}, \mathbf{a}, \kappa, \mathbf{N}, R, C \).

**Answer:** \( x(t) = t, \quad y(t) = -2 + 2t, \quad z(t) = 1 + t \).

\( \mathbf{v}(t) = \langle x', y', z' \rangle = \langle 1, 2, 1 \rangle \) (constant velocity). \( \Rightarrow \mathbf{T}(t) = \mathbf{v}/|\mathbf{v}| = \frac{1}{\sqrt{6}} \langle 1, 2, 1 \rangle \).

\( \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = 0 \).

\( \kappa = 0 \) (easy using any of the formulas for \( \kappa \)). \( \Rightarrow R = \infty, \quad \mathbf{N}, \quad C \) are undefined.

**Proofs of formulas 3-5:**

Note: we will repeatedly use that \( v = \frac{ds}{dt} \).

Formula 3 is an application of the product and chain rules:

Start with \( v = \frac{ds}{dt} \mathbf{T} \).
⇒ \( \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2 s}{dt^2} \mathbf{T} + \frac{ds}{dt} \frac{d\mathbf{T}}{ds} \frac{ds}{dt} = \frac{ds}{dt} \frac{d^2 s}{ds^2} \mathbf{T} + \frac{ds}{ds} \frac{d^2 s}{ds^2} \mathbf{T} \mathbf{N} \).

In physics this is the decomposition of acceleration into tangential and radial components.

Formula 4 now follows from formula 3 since \( \mathbf{T} \) and \( \mathbf{N} \) are orthogonal unit vectors:

\[
\mathbf{a} \times \mathbf{v} = \left( \frac{d^2 s}{dt^2} \mathbf{T} + \left( \frac{ds}{dt} \right)^2 \kappa \mathbf{N} \right) \times \frac{ds}{dt} \mathbf{T} = \left( \frac{ds}{dt} \right)^3 \kappa (\mathbf{N} \times \mathbf{T}).
\]

Since \( \mathbf{N} \) and \( \mathbf{T} \) are orthogonal unit vectors \( \mathbf{N} \times \mathbf{T} \) is a unit vector.

⇒ \( |\mathbf{a} \times \mathbf{v}| = \left( \frac{ds}{dt} \right)^3 \kappa = v^3 \kappa \).

The second part of formula 4 is just the first in coordinates:

\[
\mathbf{v} = x' \hat{i} + y' \hat{j} \text{ and } \mathbf{a} = x'' \hat{i} + y'' \hat{j}
\]

⇒ \( \mathbf{a} \times \mathbf{v} = (x'' y' - x' y'') \hat{k} \text{ and } v = \sqrt{(x')^2 + (y')^2} \)

⇒ what we want.

Formula 5 now follows from what we just did. We found

\[
\mathbf{a} \times \mathbf{v} = v^3 \kappa (\mathbf{N} \times \mathbf{T}). \Rightarrow \mathbf{v} \times (\mathbf{a} \times \mathbf{v}) = v^4 \kappa \mathbf{T} \times (\mathbf{N} \times \mathbf{T}) = v^4 \kappa \mathbf{N}.
\]

The last equality is easy using your right hand (since \( \mathbf{T} \) and \( \mathbf{N} \) are orthogonal unit vectors).

Next time we’ll do a number of examples of all these concepts.

22 Parametric equations continued

22.1 Circles and ellipses

We will use the following parametrization of the circle all the time.

\[
x(t) = a \cos(\theta) \quad \Rightarrow \quad \text{circle } x^2 + y^2 = a^2.
\]

A small modification gives an ellipse.

\[
x(t) = a \cos(t) \quad \Rightarrow \quad \text{ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
\]
Note that for the circle, the parameter $\theta$ is the usual angle measured from the positive $x$-axis.

### 22.1.1 Parametric and symmetric forms

$x(t) = a \cos(\theta)$, $y(t) = a \sin(\theta)$ is called **parametric form**

$x^2 + y^2 = a^2$ is called **symmetric form**.

**Example 22.1.** For the circle $\mathbf{r}(t) = b(\cos(t) \mathbf{i} + \sin(t) \mathbf{j})$ find the curvature.

**Answer:**

\[
\mathbf{v} = \mathbf{r}'(t) = b(-\sin(t) \mathbf{i} + \cos(t) \mathbf{j}), \quad \Rightarrow \quad |\mathbf{v}| = \frac{ds}{dt} = |\mathbf{r}'(t)| = b.
\]

Using formula 4 (in topic 21): $\mathbf{a} = -b(\cos(t) \mathbf{i} + \sin(t) \mathbf{j}) \Rightarrow \kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = b^2/b^3 = 1/b$, i.e. the curvature of a circle = $1$/radius (bigger circle = smaller curvature).

Alternate method: The unit tangent vector is

\[
\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}|} = (-\sin(t), \cos(t)).
\]

As usual, we can write $\mathbf{T}(t) = (\cos(\phi), \sin(\phi))$. In this case, $\phi(t) = t + \pi/2$. Thus, $\kappa = \frac{d\phi}{ds} = \frac{d\phi/dt}{ds/dt} = \frac{1}{b}$.

**Example 22.2.** Here’s a circle with different parameterization: $\mathbf{r}(t) = b(\cos(t^2) \mathbf{i} + \sin(t^2) \mathbf{j})$.

Since curvature is geometric it should be independent of parameterization. Use formula 4 from topic 21 to verify this for this example.

**Answer:** First we compute velocity and acceleration.

\[
\mathbf{v} = (-2bt \sin(t^2), 2bt \cos(t^2)), \quad \mathbf{a} = (-2b \sin(t^2) - 4bt^2 \cos(t^2), 2b \cos(t^2) - 4bt^2 \sin(t^2)).
\]

A little algebra gives: $\mathbf{a} \times \mathbf{v} = -8b^2t^3 \mathbf{k}$ and $|\mathbf{v}| = 2bt$. Formula 4 now gives $\kappa = \frac{8b^2t^3}{8b^3t^3} = \frac{1}{b}$.

**Example 22.3.** For the circle in Example 22.1, find $x(t)$, $y(t)$, $T$, $N$, $R$ (radius of curvature) and $C$ (center of curvature).
answer: \( x(t) = b \cos(t), \quad y(t) = b \sin(t). \)

\[
T = \frac{v}{|v|} = -\sin(t) \mathbf{i} + \cos(t) \mathbf{j}
\]

\[
\kappa \mathbf{N} = \frac{dT}{ds} = \frac{dT}{dt} = \frac{-\cos(t) \mathbf{i} - \sin(t) \mathbf{j}}{b} \Rightarrow \kappa = \frac{1}{b}, \quad \text{and} \quad \mathbf{N} = -(\cos(t) \mathbf{i} + \sin(t) \mathbf{j}).
\]

\[ R = \frac{1}{\kappa} = b = \text{the radius of the circle}. \]

\[ C = r(t) + RN = b(\cos(t) \mathbf{i}, \sin(t)) + b(-\cos(t), \sin(t)) = 0 = \text{center of circle}. \]

This example shows that the terms 'radius of curvature' for \( 1/\kappa \) and 'center of curvature' for \( C \) are well chosen!

### 22.2 The cycloid

The cycloid is defined by the following description. Roll a wheel (circle of radius \( a \)) along the \( x \)-axis and follow the trajectory of a point on the wheel. This trajectory is called a cycloid.

The cycloid plays a roll in many important problems: For example, the brachistochrone - Bernouilli, Newton and the tautochrone – Huygens. Wikipedia will tell you all about these things.

There is a nice mathlet that gives a dynamic view of the cycloid. See [http://mathlets.org/mathlets/wheel/](http://mathlets.org/mathlets/wheel/)

#### 22.2.1 Parametric equations for the cycloid

Our next goal is to find a parametrization of the cycloid.

To do this we use vectors. In the figure below on the left we see that \( \mathbf{r}(\theta) = \overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QC} + \overrightarrow{CP} \)

Each of the individual vectors are easy to compute:

\( \overrightarrow{OQ} = (a\theta, 0) = \text{amount rolled} \ (a=\text{radius}) \)

\( \overrightarrow{QC} = (0, a) \)

\( \overrightarrow{CP} = (-a \sin(\theta), -a \cos(\theta)) \) (See the right hand figure just below.)
So,
\[
\vec{r}(\theta) = \langle a\theta - a\sin(\theta), a - a\cos(\theta)\rangle \\
= a\langle \theta - \sin(\theta), 1 - \cos(\theta)\rangle \\
= a((\theta - \sin(\theta)) \mathbf{i} + (1 - \cos(\theta)) \mathbf{j}).
\]

We can also write this as
\[
x(\theta) = a(\theta - \sin(\theta)), \quad y(\theta) = a(1 - \cos(\theta)).
\]

Notes. (1) The symmetric form of the cycloid equations is hard to write down
(2) Physical units like speed and acceleration will depend on $\theta(t)$.

Example 22.4. For the cycloid in the previous example find the curvature $\kappa$ and the arclength of one arch.

**Answer:**
\[
\frac{d\vec{r}}{d\theta} = a\langle 1 - \cos(\theta), \sin(\theta)\rangle = 2a\langle \sin^2(\theta/2), \sin(\theta/2)\cos(\theta/2)\rangle.
\]
\[
\frac{|d\vec{r}|}{d\theta} = \frac{ds}{d\theta} = 2a\sqrt{\sin^2(\theta/2) = 2a|\sin(\theta/2)|}.
\]

Note: $T = \frac{2a\langle \sin^2(\theta/2), \sin(\theta/2)\cos(\theta/2)\rangle}{2a|\sin(\theta/2)|} = \pm\langle \sin(\theta/2), \cos(\theta/2)\rangle$ (this is a unit vector).

At the cusp $ds/d\theta = 0$, i.e., physically, you must stop to make a sudden 180 degree turn.

For one arch, $0 < \theta < 2\pi$, $\frac{ds}{d\theta} = 2a\sin(\theta/2)$. (We can drop the absolute value because $\sin(\theta/2) > 0$ for this range of $\theta$.)

\[
\text{arclength of arch} = \int_0^{2\pi} \frac{ds}{d\theta} d\theta = \int_0^{2\pi} 2a\sin(\theta/2) d\theta = -4a\cos(\theta/2)|_0^{2\pi} = 4a.
\]

This is called Wren’s theorem: the arclength of one arch of the cycloid = $4a$.

To find the curvature $\kappa$ we compute velocity and acceleration:
\[
\vec{r}(\theta) = a(\theta - \sin(\theta)) \mathbf{i} + a(1 - \cos(\theta)) \mathbf{j}
\]
\[
\vec{v} = a(1 - \cos(\theta)) \mathbf{i} + a\sin(\theta) \mathbf{j} \quad \text{and} \quad \vec{a} = a\sin(\theta) \mathbf{i} + a\cos(\theta) \mathbf{j}.
\]

Now we compute the formula $\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$:
\[
\vec{a} \times \vec{v} = a^2((1 - \cos(\theta))\cos(\theta) - \sin(\theta)\sin(\theta)) \mathbf{k} = a^2(\cos(\theta) - 1) \mathbf{k} = -a^2\sin^2(\theta/2) \mathbf{k}
\]
We know that $|v| = 2a|\sin(\theta/2)|$. So,

$$
\kappa = \frac{|a \times v|}{|v|^3} = \frac{a^2(1 - \cos(\theta))}{a^3 2^{3/2} (1 - \cos(\theta))^{3/2}} = \frac{1}{a 2^{3/2} \sqrt{(1 - \cos(\theta))}} = \frac{1}{4a|\sin(\theta/2)|}.
$$

Note, $\kappa$ is undefined when $\theta = 2n\pi$, i.e. at the cusps of the cycloid.

### 22.3 The symmetric form can lose information

**Example 22.5.** Where symmetric form loses information.

Find the symmetric form for $x = \cos^2(t)$, $y = \sin^2(t)$.

**answer:** Easily we see that $x + y = 1$, with $x$, $y$ non-negative.

The symmetric form shows a line, while the parametric equations show which part of the line is actually traced out.

![Symmetric Form Example](image)

### 22.4 Intersection of two planes

Suppose we have two planes with respective normals $\vec{N}_1$ and $\vec{N}_2$. The two planes intersect in a line. This line is perpendicular to both normals, i.e. it is in the same direction as $\vec{N}_1 \times \vec{N}_2$.

**Example 22.6.** Find the intersection of the planes $x + y + z = 1$ and $y = 2$.

**answer:** Normals: $\vec{N}_1 = (1, 1, 1)$ and $\vec{N}_2 = (0, 1, 0)$.

Direction vector: $\vec{v} = \vec{N}_1 \times \vec{N}_2 = -\hat{i} + \hat{k} = (-1, 0, 1)$.

One point of intersection: (by elimination) $y = 2 \Rightarrow x + 2 + z = 3 \Rightarrow P = (1, 2, 0)$.

**answer:** The intersection is the line $\langle 1, 2, 0 \rangle + t(-1, 0, 1)$. 
22.5 Three dimensional example

Example 22.7. For the helix \( r(t) = \cos(t) \mathbf{i} + \sin(t) \mathbf{j} + at \mathbf{k} \) find the radius of curvature and center of curvature for arbitrary \( t \).

**Answer:** We will use the Formulas (2), (3) and (4) from topic 21, 
\[ v = -\sin(t) \mathbf{i} + \cos(t) \mathbf{j} + a \mathbf{k}; \quad a = -\cos(t) \mathbf{i} - \sin(t) \mathbf{j}. \]
\[ \Rightarrow |v| = \sqrt{1 + a^2}; \quad a \times v = -a \sin(t) \mathbf{i} + a \cos(t) \mathbf{j} - \mathbf{k}. \]
Formula (4) \[ \Rightarrow \kappa = \frac{|a \times v|}{|v|^3} = \frac{\sqrt{1 + a^2}}{(1 + a^2)^{3/2}} = \frac{1}{1 + a^2}. \]
\[ \Rightarrow \text{radius of convergence} = R = 1 + a^2. \]
The center of curvature \( C = r(t) + RN \Rightarrow \) we have to find \( RN \).
Since we already have \( a \times v \) we use Formula (5) from topic 21, i.e. \( v \times (a \times v) = \kappa v^4 N \).
\[ v \times (a \times v) = \langle -\sin(t), \cos(t), a \rangle \times \langle -a \sin(t), a \cos(t), -1 \rangle \]
\[ = \langle -\cos(t) - a^2 \cos(t), -\sin(t) - a^2 \sin(t), 0 \rangle \]
\[ = (1 + a^2) \langle -\cos(t), -\sin(t), 0 \rangle \]
Now, Formula 5 implies \( (1 + a^2) \langle -\cos(t), -\sin(t), 0 \rangle = \kappa v^4 N \)
Since \( v = \sqrt{1 + a^2} \) this gives
\[ N = \langle -\cos(t), -\sin(t), 0 \rangle \text{ and } \kappa = 1/(1 + a^2) \Rightarrow R = 1 + a^2. \]
Thus, \( C = (\cos(t), \sin(t), at) + RN = (-a^2 \cos(t), -a^2 \sin(t), at) \).

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23 Kepler’s second law

What I have here is just a brief version of the notes §K. I suggest you read that instead.

**Claim:** If a body moves under a central force then it sweeps out equal areas in equal time.

**Proof:**
Note a central force means \( \overrightarrow{F} \) is parallel to \( \overrightarrow{a} \).
In a short time $\Delta t$ the position vector sweeps out an area $\Delta A$.

Using vectors we see $\Delta A \approx \frac{1}{2} |\vec{r} \times \Delta \vec{r}|$. So, $\frac{dA}{dt} = \frac{1}{2} |\vec{r} \times \frac{d\vec{r}}{dt}|$.

Equal areas in equal time means $\frac{dA}{dt} = \text{constant}$.

Consider $\vec{w} = \vec{r} \times \frac{d\vec{r}}{dt}$. By the product rule
\[
\frac{d\vec{w}}{dt} = \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + \vec{r} \times \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + \vec{r} \times \vec{a}
\]

Both terms are 0 since $\vec{a}$ is parallel to $\vec{r}$. Thus
\[
\frac{d\vec{w}}{dt} = 0 \Rightarrow \frac{dA}{dt} = \text{constant}.
\]

\[\blacksquare\]