1 Linear and quadratic approximation

**Goal:** To approximate hard to compute functions by easier functions.

**Reading:** More details are given in the reading from the supplementary notes: [http://math.mit.edu/suppnotes/suppnotes01-01a/01a.pdf](http://math.mit.edu/suppnotes/suppnotes01-01a/01a.pdf)

**Vocabulary:**

- Linear approximation = linearization
- Quadratic approximation
- Geometric series
- Binomial theorem

**Basic idea:** If \( h \) is small then \( h^2 \) is really small and \( h^3 \) is really, really small.

**Example 1.1.** Approximate \( f(x) = 3 + 4x + 5x^2 + 7x^3 \) for \( x \) near 0.

The simplest approximation is the **best linear approximation**. For \( x \) small we can ignore the higher powers of \( x \):

\[
\text{For } x \text{ small we have } f(x) \approx 3 + 4x.
\]

Here the wavy equal sign '\( \approx \)' is read as 'approximately equals'.

If we want a more accurate approximation we can use the **best quadratic approximation**:

\[
f(x) \approx 3 + 4x + 5x^2.
\]

Here we kept the first two powers of \( x \) and dropped the others.

To see why these are the best approximations we turn to calculus and draw some pictures. While we’re at it we’ll work near an arbitrary base point \( x = a \).

### 1.1 Basic linear formulas

**A1.** \( f(x) \approx f(a) + f'(a)(x - a) \) for \( x \approx a \).

**A2.** \( \frac{1}{1 - x} \approx 1 + x \) for \( x \approx 0 \).

**A3.** \( (1 + x)^r \approx 1 + rx \) for \( x \approx 0 \).

**A4.** \( \sin x \approx x \) for \( x \approx 0 \).
A1 is a theoretical statement valid for all \( f(x) \). A2 – 4 are statements about specific functions. These will be some of our building blocks for more complicated functions.

To be perfectly rigorous we should say, all \( f(x) \) that have a continuous first derivative near \( x = a \). In 18.01 that will be all functions that aren’t a disaster at \( x = a \), e.g. \( 1/(x - 1) \) at \( x = 1 \).

In class we will prove A2-4 using A1. We give two proofs of A1 right now.

**Algebraic proof of A1**

The definition of the derivative tells us that if \( y = f(x) \) then

\[
f'(a) \approx \frac{\Delta y}{\Delta x}, \quad \text{where} \quad \Delta x = x - a, \; \Delta y = f(x) - f(a) \; \text{and} \; x \approx a.
\]

Rearranging this we get

\[
\Delta y \approx f'(a) \Delta x.
\]

Using \( \Delta y = f(x) - f(a) \) and \( \Delta x = x - a \) this becomes

\[
f(x) - f(a) \approx f'(a)(x - a).
\]

One more step (moving \( f(a) \) from the left-side to the right-side gives A1:

\[
f(x) \approx f(a) + f'(a)(x - a).
\]

**Geometric proof of A1** (The tangent line approximates the graph.)

The formula for the tangent line is shown in the figure above. Geometrically we see that near the point \((a, f(a))\) the blue tangent line approximates the graph. That is

\[
y = f(x) \approx y_{\text{tangent}} = f(a) + f'(a)(x - a)
\]

This is exactly the approximation formula A1.

### 1.2 Examples of linear approximation

**Example 1.2.** Find the best linear approximation of \( f(x) = (1 + x)^{99}(1 + 3x)^{77} \) for \( x \approx 0 \).

**answer:** This example is intended to convince you that formulas A2-4 and some algebra are often easier than trying to apply A1 directly. Since linear approximation ignores the higher order terms we can replace each of the factors of \( f(x) \) by its linear approximation given by A3. (More details will be given in class.)

\[
(1 + x)^{99} \approx 1 + 99x; \; (1 + 3x)^{77} \approx 1 + 77.3x; \; \text{so} \; f(x) \approx (1 + 99x)(1 + 77.3x) = 1 + 330x + 99 \cdot 231x^2 \approx 1 + 330x.
\]
In the last step we again dropped the higher order term $231x^2$.

**Example 1.3.** Approximate $f(x) = 1/(1 - \sin x)^2$ for $x \approx 0$.

**Answer:** $\sin x \approx x \Rightarrow f(x) \approx 1/(1 - x)^2 \approx (1 + x)^2 = 1 + 2x + x^2 \approx 1 + 2x$.

**Example 1.4.** Approximate $f(x) = e^x$ for $x$ near 0.

**Answer:** None of the formulas A2-4 helps here, so we use A1 directly:

$$f'(x) = e^x,$$

so $f(0) = 1, f'(0) = 1$. Thus $e^x \approx 1 + x$.

**Example 1.5.** Approximate $f(x) = e^{x^2}$ near $x = 0$.

**Answer:** $f(x) \approx 1 + 2x$.

### 1.3 Basic quadratic formulas

A5. $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$ for $x \approx a$.

A6. $\frac{1}{1-x} \approx 1 + x + x^2$ for $x \approx 0$.

A7. $(1 + x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2$ for $x \approx 0$.

A8. $\sin x \approx x$ for $x \approx 0$.

A9. $\cos x \approx 1 - \frac{x^2}{2}$ for $x \approx 0$.

Similar to linear approximations, A5 is theoretical and can be used to prove the explicit formulas A6-9.

**Example 1.6.** Find the best quad. approximation to $\sqrt{1+3x}$ near 0.

**Answer:** $\sqrt{1+3x} = (1 + 3x)^{1/2} = 1 + \frac{1}{2}(3x) - \frac{1}{8}(3x)^2$.

**Example 1.7.** Find the best quad. approximation to $f(\theta) = \tan \theta = \frac{\sin \theta}{\cos \theta}$ near 0.

**Answer:** $f(\theta) \approx \frac{\theta}{1 - \theta^2/2} \approx \theta(1 + \theta^2/2) \approx \theta$.

More examples:

**Example 1.8.** Find the best linear approximation of $\sqrt{a + bx}$ in two ways. First by using formula (A1) and second using the basic formulas and algebra.

**Answer:** (i) Give the function a name: $f(x) = \sqrt{a + bx}$ and then find the pieces of (A1).

$$f(0) = \sqrt{a}; \quad f'(x) = \frac{b}{2\sqrt{a+bx}} \Rightarrow f'(0) = \frac{b}{2\sqrt{a}}.$$  

Using (A1): $f(x) \approx \sqrt{a} + \frac{b}{2\sqrt{a}}x,$ for $x \approx 0$.

(ii) $f(x) = \sqrt{a} \left(1 + \frac{bx}{a}\right)^{1/2} \approx \sqrt{a} \left(1 + \frac{1}{2}\frac{bx}{a}\right)$ (same as (i)).
Example 1.9. Find the best quadratic approximation for $e^x$ for $x \approx 0$.

**answer:** $f(0) = 1; \ f'(x) = e^x \Rightarrow f'(0); \ f''(x) = e^x \Rightarrow f''(0) = 1.

$e^x \approx 1 + x + \frac{x^2}{2}$.

Example 1.10. Find the quadratic approximation for $f(x) = \frac{1}{1-x}$ for $x \approx 1/2$.

**answer:** Find the pieces for (A5) (here, $a = \frac{1}{2}$).

$f(\frac{1}{2}) = 2; \ f'(x) = \frac{1}{(1-x)^2} \Rightarrow f'(\frac{1}{2}) = 4; \ f''(x) = \frac{2}{(1-x)^3} \Rightarrow f''(\frac{1}{2}) = 16$.

Using (A5): $f(x) \approx 2 + 4(x - \frac{1}{2}) + 8(x - \frac{1}{2})^2$.

Example 1.11. Same problem as above, finding the answer using algebra:

**answer:** Let $y = f(x)$.

Let $u = x - \frac{1}{2}$, (so $x \approx \frac{1}{2} \Leftrightarrow u \approx 0$).

$\Rightarrow y = \frac{1}{1/2-u} = \frac{2}{1-2u} \approx 2(1 + 2u + 4u^2) = 2 + 4(x - \frac{1}{2}) + 8(x - \frac{1}{2})^2$.

(The first approximation comes using (A4).)

Example 1.12. (Special relativity: example 3 in notes §A.)

Special relativity tells us that the mass $m$ of an object moving with respect to an inertial frame of reference is bigger than its rest mass $m_0$. The formula relating the two is

$$m = m_0 \frac{c}{\sqrt{c^2 - v^2}}$$

where $v$ is the speed of the mass and $c$ is the speed of light. What $v$ is needed to produce 1% increase in mass?

**answer:** An increase of 1% means we want $m/m_0 = 1.01$. Here’s a case where we need to prepare by using some algebra to find the right way to express $m/m_0$ so we can use our approximation formulas.

$$\frac{m}{m_0} = \frac{c}{\sqrt{c^2 - v^2}} = (1 - (v/c)^2)^{-1/2} \approx 1 + \frac{1}{2} (v/c)^2.$$  

Let $u = v/c$, so we want $1.01 = 1 + \frac{1}{2} u^2 \Rightarrow .02 = u^2 \Rightarrow u \approx \frac{1}{7} \Rightarrow v \approx c/7 \approx 27000 \text{ mi/sec}.$

Example 1.13. Suppose you have $1000 in bank at 2% continuous interest. Approximately how much money is in the bank after 1 year? After 2 years?

**answer:** For this we need to know that continuous interest leads to exponential growth in your bank balance. So if we let $f(t)$ be the balance we have

$$f(t) = 1000e^{.02t} \approx 1000(1 + .02t + (.02t)^2/2).$$

Plugging in $t = 1$ and $t = 2$ gives:

$f(1) \approx 1000(1 + .02 + .0002) = 1020.20 \quad \text{(exact: } f(1) = 1020.2013).$
\( f(2) \approx 1000(1 + .04 + .0008) = 1040.80 \) (exact: \( f(2) = 1040.8108 \)).

**Example 1.14.** Find the best quadratic approximation of \( f(x) = \ln(1 + x) \) near \( x = 0. \)

**Answer:**
\[
\begin{align*}
&f(0) = 0; \quad f'(x) = 1/(1 + x) \Rightarrow f'(0) = 1; \quad f''(x) = -1/(1 + x)^2 \Rightarrow f''(0) = -1. \\
\Rightarrow \ln(1 + x) &\approx x - \frac{x^2}{2}.
\end{align*}
\]

### 1.4 Algebraic substitution rules

1. Can substitute a linear (quadratic) approx for any factor or divisor as long the divisor has a non-zero constant term.

2. Once you make a linear substitution you can never recover the best quadratic approximation.

**Example 1.15.** (Why we need to have a constant term) In each of the following examples the denominator has no constant term. If we don’t cancel the extra factors of \( x \) in the numerator and denominator we get spurious results.

1. \( \frac{x(1+x)}{x(2+x)} \not\approx \frac{x}{2x}. \)

2. \( \frac{\ln(1 + x)}{xe^x} \not\approx \frac{x}{x} = 1. \) Instead \( = \frac{\ln(1 + x)/x}{e^x} \approx \frac{1-x/2}{1+x} \approx (1 - x/2)(1 - x) \approx 1 - 3x/2. \)

(Note: this would be hard to do by differentiation.)

**Example 1.16.** (Why we can’t get the best quadratic approximation after a linear substitution.) Consider the function

\[
f(x) = (1 + x + x^2 + x^3)(1 + 2x + 3x^2).
\]

Multiplying this out and keeping just terms up to order 2 we get the quadratic approximation near 0:

\[
f(x) \approx (1 + x + x^2)(1 + 2x + 3x^2) \approx 1 + 3x + 6x^2
\]

If first made linear approximations of each factor we get:

\[
f(x) \approx (1 + x)(1 + 2x) = 1 + 3x + 2x^2
\]

which is not **THE BEST** quadratic approximation of \( f(x) \). What happened is that by throwing away the quadratic terms in each factor they are not included in the product the way they should be.