11 Improper Integrals

11.1 Integrals with $\infty$ as a limit

**Definition:** An integral of the form $\int_a^\infty f(x) \, dx$ is called an improper integral.

The upper limit of $\infty$ is what makes this integral improper. We will see other types of improper integrals below.

We say that the improper integral converges if the limit $\lim_{b \to \infty} \int_a^b f(x) \, dx$ exists. In this case, we define

$$\int_a^\infty f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx.$$ 

If the limit does not exist we say that the integral diverges.

**Example 11.1.** Compute $\int_1^\infty \frac{1}{x^2} \, dx$.

**answer:** This is an improper integral, so it equals

$$\lim_{b \to \infty} -x^{-1}\bigg|_1^b = \lim_{b \to \infty} 1 - 1/b = 1.$$

Thus, the integral converges to 1.

**Example 11.2.** Compute $\int_1^\infty \frac{1}{x^{1/2}} \, dx$.

**answer:** This improper integral equals

$$\lim_{b \to \infty} 2x^{1/2}\bigg|_1^b = \lim_{b \to \infty} 2b^{1/2} - 2.$$

We see this diverges as $b \to \infty$, so the integral diverges.

**Example 11.3.** Compute $\int_1^\infty \frac{1}{x^p} \, dx$. (To avoid an annoying case, assume $p \neq 1$.)

**answer:** This improper integral equals

$$\lim_{b \to \infty} x^{p+1}\bigg|_1^b = \lim_{b \to \infty} b^{-p+1} + \frac{1}{p-1} = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p < 1. \end{cases}$$

So the integral converges if $p > 1$ and diverges to $\infty$ if $p < 1$.

In the case $p = 1$, we have

$$\lim_{b \to \infty} \int_1^b \frac{1}{x} \, dx = \lim_{b \to \infty} \ln(b) = \infty.$$

Thus, when $p = 1$ the integral diverges.
11.1.1 The $p$-test

The previous example can be summarized as the $p$-test.

$$
\int_1^\infty \frac{1}{x^p} \, dx \begin{cases}
  p > 1 & \text{converges} \\
  p \leq 1 & \text{diverges}
\end{cases}
$$

(1)

You should know this. But realize that you will need it in settings where the integrals are not presented in exactly this form.

**Example 11.4.** Compute the work (energy) needed to move a mass $m$ from the surface of the earth to a distance $x$ from the center of the earth. (Assume $x > R$, the radius of the earth.)

**answer:** We have the following quantities:

$r =$ distance from center of earth.

$R =$ radius of earth.

$F = \frac{GmM}{r^2} = \frac{C}{r^2} =$ force on $m$ at distance $r$.

The work needed to move the mass from $r$ to $r + dr$ is $\frac{C}{r^2} dr$. We use an integral to find the total work needed:

$$
W = \int_R^x \frac{C}{r^2} \, dr = \left. \frac{-C}{r} \right|_R^x = \frac{C}{R} - \frac{C}{x}.
$$

**Example 11.5.** In the previous example, how much energy (work) would it take for the mass to escape earth’s gravity.

**answer:** We use the previous computation and let $x$ go to $\infty$. This implies the work needed to escape is

$$
W = \lim_{x \to \infty} \frac{C}{R} - \frac{C}{x} = \frac{C}{R},
$$

i.e. we need at least $\frac{C}{R}$ units of energy to escape.

**Example 11.6.** Consider $f(x) = \frac{1}{x}$ for $1 \leq x < \infty$. Find the area under this curve. Also, find the volume of this curve revolved around the $x$-axis.

**answer:** The area under curve $= \int_1^\infty \frac{1}{x} \, dx = \ln x|_1^\infty = \infty$.

Note: we dispensed with explicitly writing a limit. Instead we took the limit by ‘plugging’ $\infty$ into $\ln(x)$.

The volume of revolution around the $x$-axis is

$$
\int_1^\infty \pi \left(\frac{1}{x}\right)^2 \, dx = \pi \left. \frac{1}{x} \right|_1^\infty = \pi.
$$
This is finite, so the volume converges to $\pi$.

\[ y \]
\[ 1 \]
\[ x \]

### 11.2 Comparison test

When we can compare two positive functions we have tests relating the convergence of their integrals.

**Comparison test.** Suppose $f(x)$ and $g(x)$ are two positive functions with $0 \leq f(x) \leq g(x)$. Then we have the following statements about convergence of integrals.

- If $\int_a^\infty g(x) \, dx$ converges then so does $\int_a^\infty f(x) \, dx$.
- If $\int_a^\infty f(x) \, dx$ diverges then so does $\int_a^\infty g(x) \, dx$.

The following picture should make this clear. If the area under $g(x)$ is finite then so is the area under $f(x)$, i.e. if the integral in $g$ converges then so does the integral in $f$.

Likewise, if the area under $f(x)$ is infinite then so is the area under $g(x)$, i.e. if the integral in $f$ diverges then so does the integral in $g$.

\[ y = g(x), \quad y = f(x) \]

Two positive functions with $f(x) < g(x)$.

**Example 11.7.** Use the comparison test to decide whether each of the following integrals converge.

(i) $\int_1^\infty \frac{1}{\sqrt{x^3+1}} \, dx$

(ii) $\int_1^\infty e^{-x^2} \, dx$

(iii) $\int_1^\infty e^{-x^2} \, dx$

**answer:** (i) We know $\frac{1}{\sqrt{x^3+1}} < \frac{1}{x^{3/2}}$. Also, $\int_1^\infty \frac{1}{x^{3/2}} \, dx$ converges by the $p$-test. Therefore, the original integral converges by the comparison test.
(ii) We can’t compute this directly, so we will need to do a comparison. For \( x > 1 \) we know that \( e^{-x^2} < e^{-x} \) (because \( x^2 > x \)). We can compute directly

\[
\int_{1}^{\infty} e^{-x} \, dx = -e^{-x} \bigg|_{1}^{\infty} = 1.
\]

Since this integral converges, the comparison test shows that so does \( \int_{1}^{\infty} e^{-x^2} \, dx \).

(iii) This is similar to part (ii) except the lower limit is 0. Between 0 and 1 \( e^{-x^2} > e^{-x} \), so it may appear that comparison to \( e^{-x} \) won’t work. The fix is to split the integral into two pieces.

\[
\int_{0}^{\infty} e^{-x^2} \, dx = \int_{0}^{1} e^{-x^2} \, dx + \int_{1}^{\infty} e^{-x^2} \, dx.
\]

The integral from 0 to 1 is an ordinary definite integral, so it converges. The second integral converges by part (ii).

Since both pieces converge, the original integral from 0 to \( \infty \) converges.

Direct comparison is a pain. We can do better.

### 11.3 Limit comparison test

**Limit comparison test.** Assume \( f, g \) are positive functions.

Suppose that

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1
\]

then the improper integrals

\[
\int_{a}^{\infty} f(x) \, dx \text{ and } \int_{a}^{\infty} g(x) \, dx
\]

either both converge or both diverge.

**Notes:**

1. If Equation 2 holds, then we say that \( f \) is asymptotic with \( g \) and write \( f \sim g \).
2. Limit comparison is also called asymptotic comparison.

**Example 11.8.** Use the limit comparison test to redo Example 11.7. That is, use limit comparison to decide whether each of the following integrals converge.

(i) \( \int_{1}^{\infty} \frac{1}{\sqrt{x^3 + 1}} \, dx \)

(ii) \( \int_{1}^{\infty} e^{-x^2} \, dx \)

(iii) \( \int_{1}^{\infty} e^{-x^2} \, dx \)

**Answer:** (i) We know \( \lim_{x \to \infty} \frac{1}{\sqrt{x^3} + 1} = 1 \). Since \( \int_{1}^{\infty} \frac{1}{x^{3/2}} \, dx \) converges by the \( p \)-test, we also know, the original integral converges by the limit comparison test.
(ii) We know \( \lim_{x \to \infty} e^{-x^2} = \lim_{x \to \infty} e^{-x^2+x} = 0 \). Therefore \( e^{-x^2} \) is asymptotically smaller than \( e^{-x} \). (This idea is explained in more detail below.) Since the integral \( \int_1^\infty e^{-x} \, dx \) converges, so does \( \int_1^\infty e^{-x^2} \, dx \) by limit comparison.

(iii) The argument is identical to part (ii), because the lower limit is not important to the limit comparison test. (See the answer to Example 11.7 if it is not clear why this is so.)

Example 11.9. Use limit comparison to show that
\[
\int_2^{\infty} \frac{1}{\sqrt{x^3-1}} \, dx
\]
converges.

\textbf{answer:} Let
\[
f(x) = \frac{1}{\sqrt{x^3-1}}, \quad g(x) = \frac{1}{x^{3/2}} \quad \text{then} \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.
\]

Thus, by the limit comparison test, since \( \int_2^{\infty} \frac{1}{x^{3/2}} \, dx \) converges so does \( \int_2^{\infty} \frac{1}{\sqrt{x^3-1}} \, dx \).

Note: We need the lower limit in the integral to be greater than 1 so that \( g(x) \) is defined on the entire interval we integrate over.

11.3.1 "Proof" of the limit comparison test

We show this by appealing to the regular comparison test. Since \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1 \) for large \( x \), say \( x > c \) for some \( c \), we must have \( \frac{1}{2}g(x) < f(x) < 2g(x) \).

This implies
\[
\frac{1}{2} \int_c^{\infty} g(x) \, dx < \int_c^{\infty} f(x) \, dx < 2 \int_c^{\infty} g(x) \, dx.
\]

Now the regular comparison test shows that if \( \int_c^{\infty} f(x) \, dx < \infty \) then so is \( \frac{1}{2} \int_c^{\infty} g(x) \, dx \).

That is, if the integral in \( f \) converges then so does the integral in \( g \).

Likewise, if \( \int_c^{\infty} f(x) \, dx = \infty \) then so does \( 2 \int_c^{\infty} g(x) \, dx \). That is, if the integral in \( f \) diverges then so does the integral in \( g \).

11.3.2 Extension of limit comparison?

As usual, suppose \( f(x) \) and \( g(x) \) are positive functions and consider the integrals
\[
I_1 = \int_a^{\infty} f(x) \, dx \quad \text{and} \quad I_2 = \int_a^{\infty} g(x) \, dx.
\]

Further suppose that \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = c \).

1. If \( c \neq 0 \) and \( c \neq \infty \) then both integrals converge of both diverge.
2. If \( c = 0 \) then \( f \) is asymptotically smaller than \( g \) so \( I_2 \) converges implies \( I_1 \) converges and \( I_1 \) diverges implies \( I_2 \) diverges.
3. If \( c = \infty \) then \( g \) is asymptotically smaller than \( f \) and similar conclusions hold.
Example 11.10. Show that \( \int_{1}^{\infty} \frac{\ln x}{x^3} \, dx \) converges by asymptotic comparison with \( 1/x^2 \)

**answer:** It’s straightforward to see that \( \lim_{x \to \infty} \frac{\ln(x)/x^3}{1/x^2} = 0 \). Since \( \int_{1}^{\infty} \frac{1}{x^2} \) converges and \( \ln(x)/x^3 \) is asymptotically smaller than \( 1/x^2 \), the integral in \( \ln(x)/x^3 \) converges also.

### 11.4 Other improper integrals

If the limits are finite, but the integrand becomes infinite at one of the endpoints then we also call the integral improper.

Consider the integral \( \int_{0}^{1} \frac{1}{x^{1/3}} \, dx \). This is improper because \( \frac{1}{x^{1/3}} = \infty \) when \( x = 0 \).

As before we say that the improper integral converges if the limit \( \lim_{a \to 0^+} \int_{a}^{1} \frac{1}{x^{1/3}} \, dx \) exists. In this case, we define

\[
\int_{0^+}^{1} \frac{1}{x^{1/3}} \, dx = \lim_{a \to 0^+} \int_{a}^{1} \frac{1}{x^{1/3}} \, dx.
\]

**Example 11.11.** Show that \( \int_{0}^{1} \frac{1}{x^{1/3}} \, dx \) converges.

**answer:** We can compute the integral directly:

\[
\lim_{a \to 0^+} \int_{a}^{1} \frac{1}{x^{1/3}} \, dx = \lim_{a \to 0^+} \left[ \frac{3}{2} x^{2/3} \right]_{a}^{1} = \lim_{a \to 0^+} \left( \frac{3}{2} - \frac{3}{2} a^{2/3} \right) = \frac{3}{2}.
\]

Thus, integral converges to the value \( 3/2 \).

Of course, we don’t usually write the limit down explicitly. Normally, we would just write

\[
\int_{0}^{1} \frac{1}{x^{1/3}} \, dx = \frac{3}{2} x^{2/3} \bigg|_{0}^{1} = \frac{3}{2}.
\]

### 11.5 The full \( p \)-test

The integrand in Example 11.11 has a power \( 1/x^{1/3} \). We can do the computation for any power \( p \). Here’s the full version of the \( p \)-test (including our previous case where the integral goes to infinity – see Equation 1).

\[
\begin{align*}
\int_{1}^{\infty} \frac{1}{x^{p}} \, dx & \begin{cases} p > 1 & \text{converges} \\ p \leq 1 & \text{diverges} \end{cases} \\
\int_{0}^{1} \frac{1}{x^{p}} \, dx & \begin{cases} p < 1 & \text{converges} \\ p \geq 1 & \text{diverges} \end{cases}
\end{align*}
\]

Sometimes the power of \( x \) is in the numerator, so we also write the \( p \)-test in that form.

\[
\begin{align*}
\int_{1}^{\infty} x^{p} \, dx & \begin{cases} p < -1 & \text{converges} \\ p \geq -1 & \text{diverges} \end{cases} \\
\int_{0}^{1} x^{p} \, dx & \begin{cases} p > -1 & \text{converges} \\ p \leq -1 & \text{diverges} \end{cases}
\end{align*}
\]
11.6 A pitfall


The integral \( \int_{-1}^{1} \frac{1}{x^2} \, dx \) is improper because the integrand \( 1/x^2 \) becomes infinite at \( x = 0 \) which is inside the interval of integration.

In this case, the integral must be split into pieces so that the integrand is only improper at the endpoints. In this example we would write

\[
\int_{-1}^{1} \frac{1}{x^2} \, dx = \int_{-1}^{0} \frac{1}{x^2} \, dx + \int_{0}^{1} \frac{1}{x^2} \, dx
\]

The \( p \)-test tells us that neither of the two integrals on the right converges, so the original integral does not converge.

11.7 A few more examples

Comparison also works for the following types of improper integrals.

Example 11.13. \( \int_{0}^{1} \frac{1}{\sqrt{x(x^2 + 1)}} \, dx \) converges since near \( x = 0 \), \( \frac{1}{\sqrt{x(x^2 + 1)}} \leq \frac{1}{\sqrt{x}} \).

Be careful –at first glance you might think the appropriate \( p \) for comparison is \( p = 3/2 \), but the quadratic term \( x^2 + 1 \) is not 0 when \( x = 0 \), so only the linear term \( x \) matters for convergence.

Sometimes it is easier to deal with questions of convergence if we change variables so the problem limit is at \( x = 0 \).

Example 11.14. (Change of variable.) Does \( \int_{0}^{1-} \frac{1}{\sqrt{1-x^3}} \, dx \) converge?

Answer: The integral is improper at 1. To see what is happening we change variables so that it’s improper at 0.

Let \( u = 1 - x \). The integral becomes \( \int_{0}^{1} \frac{1}{\sqrt{u(1-u^2 - 3u + 3)}} \, du \).

This clearly converges by comparison with \( 1/\sqrt{u} \).