17 Vectors; dot product

Vectors
Two views: First the geometric and then the analytic.

Geometric view
Vector = length and direction: (Discuss scaling, scalars)

 aparating

(same vector)

Length: denoted $|\mathbf{A}|$, also called magnitude or norm

Addition: (head to tail)

Subtraction: either tail to tail or $\mathbf{A} + (-\mathbf{B})$

Analytic or algebraic view
Place the tail of $\mathbf{A}$ at the origin $\Rightarrow$ the coordinates of the head determine $\mathbf{A}$:

$\mathbf{A} = (a_1, a_2) = a_1 \mathbf{i} + a_2 \mathbf{j}$.

You’ve seen the vectors $\mathbf{i}$ and $\mathbf{j}$ in physics. They have coordinates $\mathbf{i} = (1, 0)$, $\mathbf{j} = (0, 1)$

Notation and terminology
1. $(a_1, a_2)$ indicate as point in the plane.
2. $(a_1, a_2)$ indicates the vector from the origin to the point $(a_1, a_2)$. Of course, this vector can be translated anywhere and $(a_1, a_2) = a_1 \mathbf{i} + a_2 \mathbf{j}$.
3. For $\mathbf{A} = a_1 \mathbf{i} + a_2 \mathbf{j}$, $a_1$ and $a_2$ are called the $\mathbf{i}$ and $\mathbf{j}$ components of $\mathbf{A}$. Note: they are scalars.
4. $\overrightarrow{OP}$ is the vector from the origin to $P$.
5. In print we will often drop the arrow and just use the bold face to indicate a vector, i.e. $\mathbf{P} \equiv \overrightarrow{OP}$. 

7. Scalars: a real number is a scalar, you can use it to scale a vector.

Length: \(|\mathbf{A}| = \sqrt{a_1^2 + a_2^2}\)

Addition: \((a_1 \mathbf{i} + a_2 \mathbf{j}) + (b_1 \mathbf{i} + b_2 \mathbf{j}) = (a_1 + b_1) \mathbf{i} + (a_2 + b_2) \mathbf{j}\)

⇔ \((a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)\)

\(\overrightarrow{PQ} = \overrightarrow{Q} - \overrightarrow{P}\) (i.e. \(\overrightarrow{PQ}\) is the displacement from \(P\) to \(Q\)) – understand this geometrically and analytically

Dot product (scalar product)

Geometric definition: \(\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos(\theta)\)

Algebraic view

\(\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2\) (Hard to get geometrically)

Proof: Law of cosines: (won’t do in class)

\(|\mathbf{A} - \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}|\cos(\theta)\)

⇒ \((a_1^2 + a_2^2) + (b_1^2 + b_2^2) - ((a_1 - b_1)^2 + (a_2 - b_2)^2) = 2|\mathbf{A}||\mathbf{B}|\cos(\theta)\)

⇒ \(a_1b_1 + a_2b_2 = |\mathbf{A}||\mathbf{B}|\cos(\theta)\). QED

Algebraic law: \(\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}\).

(This follows from the algebraic view of dot product.)

Example 17.1. Find the dot product of \(\mathbf{A}\) and \(\mathbf{B}\).

(i) \(|\mathbf{A}| = 2, |\mathbf{B}| = 5, \theta = \pi/4\).

answer: (draw picture) \(\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos \theta = 10\sqrt{2}/2 = 5\sqrt{2}\).

(ii) \(\mathbf{A} = \mathbf{i} + 2\mathbf{j}, \mathbf{B} = 3\mathbf{i} + 4\mathbf{j}\).

answer: \(\mathbf{A} \cdot \mathbf{B} = 1 \cdot 3 + 2 \cdot 4 = 11\).

Unit vectors

Special vectors: \(\mathbf{i}\) and \(\mathbf{j}\). Note: \(\mathbf{i} \cdot \mathbf{i} = 1 = \mathbf{j} \cdot \mathbf{j}\) and \(\mathbf{i} \cdot \mathbf{j} = 0\).

Unit vector: \(\mathbf{u}: |\mathbf{u}| = 1\). Often indicate by \(\hat{\mathbf{u}}\).

Example 17.2. Are the following unit vectors?

(i) \(\mathbf{i} + \mathbf{j}\), \(\mathbf{ii}\) \(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\).

answer: (i) No. \(\mathbf{ii}\) Yes.

Example 17.3. Find two unit vectors parallel to \(2\mathbf{i} + 3\mathbf{j}\).

answer: \((2\mathbf{i} + 3\mathbf{j})/\sqrt{13}, -(2\mathbf{i} + 3\mathbf{j})/\sqrt{13}\)

Components or projection:

\(\mathbf{A} \cdot \mathbf{u} = |\mathbf{A}| \cos(\theta)\)

\(\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2\)
\[ A \perp B \iff A \cdot B = 0 \]

The component of \( A \) in the direction of \( \hat{u} \) is \( A \cdot \hat{u} \). (Note: it is a scalar.)

For a non-unit vector: the component of \( A \) in the direction of \( \hat{u} \) is the component of \( A \) in the direction of \( \hat{u} = \frac{B}{|B|} \).

**Example 17.4.** Find the component of \( A \) in the direction of \( B \).

(i) \(|A| = 2, |B| = 5, \theta = \pi/4\).

answer: (draw picture)

(ii) \( A = i + 2j, B = 3i + 4j \).

answer: Unit vector in direction of \( B \) is \( \hat{B} = \frac{B}{|B|} = \frac{3}{5}i + \frac{4}{5}j \) \( \Rightarrow \) component is \( A \cdot \hat{B} = \frac{3}{5} + \frac{8}{5} = \frac{11}{5} \).

**Trig identity**

\[ \cos(\beta - \alpha) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \]

Unit vectors: \( u = \cos \alpha i + \sin \alpha j, \ v = \cos \beta i + \sin \beta j \).

Angle between them is \( \theta = \beta - \alpha \)

Geometric: \( u \cdot v = |u||v| \cos \theta = \cos \theta = \cos(\beta - \alpha) \)

Analytic: \( u \cdot v = u_1v_1 + u_2v_2 = \cos \alpha \cos \beta + \sin \alpha \sin \beta \).

**Example 17.5.** \( P = (-5, 0), Q = (1, 3) \Rightarrow \vec{PQ} = 6i + 3j = (6, 3) \).

**Example 17.6.** Show \( \vec{PQ} + \vec{QR} + \vec{QP} = 0 \)

**Example 17.7.** Find 2 unit vectors parallel to \( v = 3i - 4j \).

\(|v| = 5: \ u_1 = \frac{3}{5}i - \frac{4}{5}j, u_2 = -u_1 \).

**Example 17.8.** Let \( A = (1, 2), B = (2, 3) \) and \( C = (2, -1) \). Find the cosine of \( \angle BAC \).

answer: Let \( \theta \) be the angle \( \Rightarrow \cos \theta = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}||\vec{AC}|} \).

\( \vec{AB} = (1, 1), \vec{AC} = (1, -3) \)

\[ \Rightarrow \cos \theta = \frac{1 - 3}{\sqrt{2} \sqrt{10}} = -\frac{2}{\sqrt{20}} = \frac{1}{\sqrt{5}} \]
Example 17.9. Velocities are vectors
A river flows at 3mph and a rower rows at 6mph. What heading should he use to get straight across a river?

**answer:** Need $\sin(\theta) = \frac{3}{6} \Rightarrow \theta = \pi/6$

Answer: Head at angle of $\pi/6$ radians upstream from straight across.

Example 17.10. Same question with river=2 mph, row=2$\sqrt{2}$ mph:

**answer:** $\sin(\theta) = \frac{2}{2\sqrt{2}} \Rightarrow \theta = \pi/4$.

Example 17.11. Same question with river=6 mph, row=3 mph:

**answer:** $\sin(\theta) = \frac{6}{3} \Rightarrow$ No such $\theta$

Example 17.12. Show the sum of the medians of a triangle = 0.

Medians of $\overrightarrow{AB} = P = \frac{1}{2}(A + B) \Rightarrow \overrightarrow{CP} = \frac{1}{2}(B + A) - C$.

Likewise: $\overrightarrow{BQ} = \frac{1}{2}(A + C) - B, \overrightarrow{AR} = \frac{1}{2}(B + C) - A$.

Thus, the sum of medians $= \overrightarrow{CP} + \overrightarrow{BQ} + \overrightarrow{AR} = 0$.

Three dimensions

Exactly the same except we have a third coordinate:

$$a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} = \langle a_1, a_2, a_3 \rangle$$

Example 17.13. Show that $A = (4,3,6), B = (-2,0,8), C = (1,5,0)$ are the vertices of a right triangle.

**answer:** Two legs of the triangle are $\overrightarrow{AC} = \langle -3, 2, -6 \rangle$ and $\overrightarrow{AB} = \langle -6, -3, 2 \rangle$

Taking their dot product we have

$$\overrightarrow{AC} \cdot \overrightarrow{AB} = 18 - 6 - 12 = 0.$$  

The dot product is 0, implies the vectors are orthogonal, i.e. they are the legs of a right triangle.