21 Parametric equations and vector derivatives

Parametric Equations

General parametric equations:
Notation: \( \overrightarrow{r}(t) = (x(t), y(t)) = x(t)\hat{i} + y(t)\hat{j} = \text{position vector} \)

We can view \((x(t), y(t))\) as the coordinates of a particle moving in the plane or we can view \(\overrightarrow{r}(t) = (x(t), y(t))\) as its position vector.

Example 21.1. A rocket takes off from the origin with initial \(x\)-velocity \(v_{0,x}\) and initial \(y\)-velocity \(v_{0,y}\). Find the parametric equations for its path.

Physics \(\Rightarrow x(t) = v_{0,x}t, \quad y(t) = -\frac{1}{2}gt^2 + v_{0,y}t.\)

At time \(t\) the rocket is at point \(P = (x(t), y(t))\). We call the vector \(\overrightarrow{r}(t) = \overrightarrow{OP} = x(t)\hat{i} + y(t)\hat{j}\) the position vector.

\[
\begin{align*}
\frac{dx}{dt} &= \text{velocity} = v(t) = x'(t)\hat{i} + y'(t)\hat{j} = \langle x', y' \rangle.
\end{align*}
\]
Geometrically: the picture shows a parametric curve for a moving point. Over a small time $dt$ the point moves a (vector) $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$ and its (instantaneous) velocity

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = (x'(t), y'(t))$$

The picture for arclength, $s$, shows

$$ds = |d\mathbf{r}| = \sqrt{(dx)^2 + (dy)^2}$$

$$\Rightarrow \text{speed} = \frac{ds}{dt} = \frac{\text{distance}}{\text{time}} = \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

**Acceleration** (change in velocity/change in time)

$$a(t) = \frac{dv}{dt} = \frac{d^2\mathbf{r}}{dt^2} = x''(t)\mathbf{i} + y''(t)\mathbf{j} = (x'', y'')$$

**Example 21.3.** Find velocity, speed, acceleration and arclength for the rocket example.

**answer:** $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = v_{0,x}t\mathbf{i} + (-\frac{gt^2}{2} + v_{0,y}t)\mathbf{j}$.

$$\Rightarrow \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = v_{0,x}\mathbf{i} + (-gt + v_{0,y})\mathbf{j}, \quad \Rightarrow a(t) = \frac{dv}{dt} = -g\mathbf{j}.$$

Speed $= \frac{ds}{dt} = |\mathbf{v}(t)| = \sqrt{v_{0,x}^2 + (-gt + v_{0,y})^2}$.

To avoid too many symbols. Let $v_{0,x} = 10, \ v_{0,y} = 10 \ \ g = 10$ and find the arclength of the path from $t = 0$ to $t = 1$.

Arclength $L = \int_0^1 \frac{ds}{dt} dt = 10 \int_0^1 \sqrt{1 + (-t + 1)^2} dt$.

Make the change of variables $u = -t + 1$

$$\Rightarrow du = -dt, \ t = 0 \rightarrow u = 1, \ t = 1 \rightarrow u = 0. \Rightarrow L = 10 \int_0^1 \sqrt{1 + u^2} du.$$

Use trig. subst. $u = \tan \theta$

$$\Rightarrow L = 10 \int_0^{\pi/4} \sec^3 \theta \, d\theta = 5[\sec \theta \tan \theta + \ln(\sec \theta + \tan \theta)]_0^{\pi/4} = 5(\sqrt{2} + \ln(\sqrt{2} + 1))$$

**Tangent vector:** (same thing as velocity) In the picture above, we see that as $\Delta t$ shrinks to 0 the vector $\Delta \mathbf{r} / \Delta t$ becomes tangent to the curve

When the parameter is $t$ we can refer to $\mathbf{r}'(t)$ as the **velocity**. In general, the derivative is given its geometric name: the **tangent vector**.

**Unit tangent vector**

The unit vector in the direction of the tangent vector is denoted $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}}{|\mathbf{v}|}$. 
It’s called the unit tangent vector.

Note $\frac{ds}{dt} \mathbf{T} = r'(t)$.

**Reparameterization** = Changing variables.

**Example 21.4.** Circle $= r(\theta) = 2(\cos \theta, \sin \theta)$

Suppose you know $\theta = 2\pi t$ then $r(t) = 2(\cos(2\pi t), \sin(2\pi t))$

These represent the same trajectory, with different parameterizations.

When dealing with more than one variable we use Liebnitz notation to avoid confusing.

The chain rule gives:

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta} \frac{d\theta}{dt}$$

NOTES: 1. This show the two tangent vectors $\frac{d\mathbf{r}}{dt}$ and $\frac{d\mathbf{r}}{d\theta}$ are parallel –as geometric considerations tell us they must be.

2. Likewise $\frac{ds}{d\theta} = \left| \frac{d\mathbf{r}}{d\theta} \right|

**Curvature:** How sharply curved is the trajectory?

That is, how fast does the tangent vector turn per unit arclength?

This is tricky so pay attention. Curvature is the rate $\mathbf{T}$ is turning per unit arclength.

That is, $\kappa = \frac{|d\mathbf{T}|}{ds}$.

(Smaller circle = faster turning = greater curvature.)

Note well, curvature is a geometric idea– we measure the rate with respect to arclength. The speed the point moves over the trajectory is irrelevant.

$\mathbf{T}$ is a unit vector $\Rightarrow \mathbf{T} = \langle \cos \phi, \sin \phi \rangle$ where $\phi$ is the tangent angle.

$$\Rightarrow \frac{d\mathbf{T}}{ds} = \frac{d}{ds} \langle \cos \phi, \sin \phi \rangle = \frac{d\phi}{ds} \langle -\sin \phi, \cos \phi \rangle.$$

Both magnitude and direction of $\frac{d\mathbf{T}}{ds}$ are useful:

Curvature $= \kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\phi}{ds} \right|$.

Direction of $\frac{d\mathbf{T}}{ds} = \mathbf{N} = \text{unit normal } \perp \mathbf{T}$. 
Note: the book doesn’t use the absolute value in its definition of $\kappa$, but it’s more standard to include it.

**Radius of curvature** $= \frac{1}{\kappa}$.

**The center of curvature and the osculating circle:**

The osculating (kissing) circle is the best fitting circle to the curve.

Radius = radius of curvature.

Center along normal direction.

**Nomenclature summary:**

Here are a list of names and formulas. We will motivate and derive them below.

$r(t) = \text{position}$.

$s = \text{arclength}, \quad \text{speed} = v = \frac{ds}{dt}$.

$v(t) = r'(t) = \frac{ds}{dt} T = \text{tangent vector, velocity}$.

$a(t) = \frac{dv}{dt} = \frac{d^2r}{dt^2} = \text{acceleration}$.

$T = \text{unit tangent vector}, \quad N = \text{unit normal vector}$.

$\kappa = \text{curvature}, \quad R = 1/\kappa = \text{radius of curvature}$.

$\phi = \text{tangent angle}$.

$C = \text{Center of curvature} = \text{center of best fitting circle (has radius = radius of curvature)}$.

**Formulas:** (explained in the following pages)

1. Speed $= \frac{ds}{dt} = |v(t)| = \sqrt{(x')^2 + (y')^2}$.

2. $v = \frac{ds}{dt} T, \quad T = \frac{v}{ds/dt}$

3. $a(t) = \frac{d^2s}{dt^2} T + \kappa \left( \frac{ds}{dt} \right)^2 N = \frac{d^2s}{dt^2} T + \frac{v^2}{R} N$

4. $\kappa = \left| \frac{dT}{ds} \right| = \left| \frac{d\phi}{ds} \right| = \frac{|a \times v|}{|v|^3}$ (check dimensions), $\frac{dT}{ds} = \kappa N$. 
4a. For plane curves \( \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} \): \( \kappa = \frac{|x''y' - x'y''|}{((x')^2 + (y')^2)^{3/2}}. \)

5. \( \mathbf{v} \times (\mathbf{a} \times \mathbf{v}) = \kappa \mathbf{v} \)

6. \( \mathbf{C} = \mathbf{r} + R \mathbf{N} = \mathbf{r} + \frac{1}{\kappa} \mathbf{N}. \)

Example 21.5. For the parabola \( \mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} \) find \( \mathbf{v}, \mathbf{a}, \mathbf{T}, \frac{ds}{dt}, \kappa, R, \mathbf{N} \) and \( \mathbf{C} \) for arbitrary \( t \).

\( \mathbf{v} = \mathbf{i} + 2t \mathbf{j}, \mathbf{a} = 2 \mathbf{j}. \)

\( \Rightarrow \frac{ds}{dt} = |\mathbf{v}| = \sqrt{1 + 4t^2}, \quad \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i}}{\sqrt{1+4t^2}} + \frac{2t}{\sqrt{1+4t^2}} \mathbf{j}. \)

Formula 4: \( \mathbf{a} \times \mathbf{v} = -2 \mathbf{k}. \) \( \Rightarrow \kappa = \frac{2}{(1+4t^2)^{3/2}}. \)

(Maximum curvature at \( t = 0 \) as expected.)

\( R = \frac{1}{\kappa} = \frac{(1+4t^2)^{3/2}}{2}. \)

Formula 5: \( \mathbf{v} \times (\mathbf{a} \times \mathbf{v}) = -4t \mathbf{i} + 2 \mathbf{j}. \) \( \Rightarrow \mathbf{N} = \frac{1}{2\sqrt{1+4t^2}}(-4t \mathbf{i} + 2 \mathbf{j}). \)

Formula 6: \( \mathbf{C} = \mathbf{r} + R \mathbf{N} = (t \mathbf{i} + t^2 \mathbf{j}) + \frac{1+4t^2}{4}(-4t \mathbf{i} + 2 \mathbf{j}). \)

Example 21.6. For the line through \((0, -2, 1)\) and \((1, 0, 2)\) (this is from a previous example) find \( x(t), y(t), z(t), \mathbf{v}, \mathbf{T}, \mathbf{a}, \kappa, \mathbf{N}, R, \mathbf{C}. \)

Answer: \( x(t) = t, \quad y(t) = -2 + 2t, \quad z(t) = 1 + t. \)

\( \mathbf{v}(t) = \langle x', y', z' \rangle = \langle 1, 2, 1 \rangle \) (constant velocity). \( \Rightarrow \mathbf{T}(t) = \mathbf{v}/|\mathbf{v}| = \frac{1}{\sqrt{6}} \langle 1, 2, 1 \rangle. \)

\( \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = 0. \)

\( \kappa = 0 \) (easy using any of the formulas for \( \kappa \)). \( \Rightarrow R = \infty, \mathbf{N}, \mathbf{C} \) are undefined.

Proofs of formulas 3-5:

Note: we will repeatedly use that \( v = \frac{ds}{dt} \).

Formula 3 is an application of the product and chain rules:

Start with \( \mathbf{v} = \frac{ds}{dt} \mathbf{T}. \)

\[ \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt} \frac{ds}{dt} \mathbf{T} + \frac{ds}{dt} \frac{d\mathbf{T}}{dt} = \frac{d}{dt} \mathbf{T} + \frac{ds}{dt} \frac{d\mathbf{T}}{ds} \frac{ds}{dt} \]

\[ = \frac{d^2s}{dt^2} \mathbf{T} + \left( \frac{ds}{dt} \right)^2 \kappa \mathbf{N}. \]

In physics this is the decomposition of acceleration into tangential and radial components.

Formula 4 now follows from formula 3 since \( \mathbf{T} \) and \( \mathbf{N} \) are orthogonal unit vectors:
\[ \mathbf{a} \times \mathbf{v} = \left( \frac{d^2 s}{dt^2} \mathbf{T} + \left( \frac{ds}{dt} \right)^2 \kappa \mathbf{N} \right) \times \frac{ds}{dt} \mathbf{T} = \left( \frac{ds}{dt} \right)^3 \kappa (\mathbf{N} \times \mathbf{T}). \]

Since \( \mathbf{N} \) and \( \mathbf{T} \) are orthogonal unit vectors \( \mathbf{N} \times \mathbf{T} \) is a unit vector
\[ \Rightarrow |\mathbf{a} \times \mathbf{v}| = \left( \frac{ds}{dt} \right)^3 \kappa = v^3 \kappa. \quad \blacksquare \]

The second part of formula 4 is just the first in coordinates:
\[ \mathbf{v} = x' \hat{i} + y' \hat{j} \text{ and } \mathbf{a} = x'' \hat{i} + y'' \hat{j} \]
\[ \Rightarrow \mathbf{a} \times \mathbf{v} = (x'' y' - x' y'') \hat{k} \text{ and } v = \sqrt{(x')^2 + (y')^2} \]
\[ \Rightarrow \text{what we want.} \]

Formula 5 now follows from what we just did. We found
\[ \mathbf{a} \times \mathbf{v} = v^3 \kappa (\mathbf{N} \times \mathbf{T}). \Rightarrow \mathbf{v} \times (\mathbf{a} \times \mathbf{v}) = v^4 \kappa \mathbf{T} \times (\mathbf{N} \times \mathbf{T}) = v^4 \kappa \mathbf{N}. \]

The last equality is easy using your right hand (since \( \mathbf{T} \) and \( \mathbf{N} \) are orthogonal unit vectors).

Next time we’ll do a number of examples of all these concepts.