22 Parametric equations continued

22.1 Circles and ellipses

We will use the following parametrization of the circle all the time.

\[ x(t) = a \cos(\theta) \]
\[ y(t) = a \sin(\theta) \]

⇒ circle \( x^2 + y^2 = a^2 \).

A small modification gives an ellipse.

\[ x(t) = a \cos(t) \]
\[ y(t) = b \sin(t) \]

⇒ ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

Note that for the circle, the parameter \( \theta \) is the usual angle measured from the positive \( x \)-axis.

22.1.1 Parametric and symmetric forms

\( x(t) = a \cos(\theta), \quad y(t) = a \sin(\theta) \) is called parametric form

\( x^2 + y^2 = a^2 \) is called symmetric form.

Example 22.1. For the circle \( \mathbf{r}(t) = b(\cos(t) \mathbf{i} + \sin(t) \mathbf{j}) \) find the curvature.

**answer:** \( \mathbf{v} = \mathbf{r}'(t) = b(-\sin(t) \mathbf{i} + \cos(t) \mathbf{j}) \), \( |\mathbf{v}| = \frac{ds}{dt} = |\mathbf{r}'(t)| = b \).

Using formula 4 (in topic 21): \( \mathbf{a} = -b(\cos(t) \mathbf{i} + \sin(t) \mathbf{j}) \) ⇒ \( \kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = b^2/b^3 = 1/b \), i.e. the curvature of a circle = 1/radius (bigger circle = smaller curvature).

Alternate method: The unit tangent vector is

\[ \mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}|} = \langle -\sin(t), \cos(t) \rangle. \]

As usual, we can write \( \mathbf{T}(t) = \langle \cos(\phi), \sin(\phi) \rangle \). In this case, \( \phi(t) = t + \pi/2 \). Thus, \( \kappa \frac{d\phi}{ds} = \frac{d\phi/dt}{ds/dt} = \frac{1}{b} \).
Example 22.2. Here’s a circle with different parameterization: \( r(t) = b(\cos(t^2), \sin(t^2)) \).

Since curvature is geometric it should be independent of parameterization. Use formula 4 from topic 21 to verify this for this example.

**answer:** First we compute velocity and acceleration.

\[
\mathbf{v} = \langle -2bt \sin(t^2), 2bt \cos(t^2) \rangle, \quad \mathbf{a} = \langle -2b \sin(t^2) - 4bt^2 \cos(t^2), 2b \cos(t^2) - 4bt^2 \sin(t^2) \rangle.
\]

A little algebra gives: \( \mathbf{a} \times \mathbf{v} = -8b^2 t^3 \mathbf{k} \) and \( |\mathbf{v}| = 2bt \). Formula 4 now gives \( \kappa = \frac{8b^2 t^3}{8b^3 t^3} = \frac{1}{b} \).

(same as before!)

Example 22.3. For the circle in Example 22.1, find \( x(t), y(t), T, N, R \) (radius of curvature) and \( C \) (center of curvature).

**answer:** \( x(t) = b \cos(t), \quad y(t) = b \sin(t) \).

\[
\mathbf{T} = \mathbf{v}/|\mathbf{v}| = -\sin(t) \mathbf{i} + \cos(t) \mathbf{j}
\]

\[
\kappa \mathbf{N} = \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} = \frac{-\cos(t) \mathbf{i} - \sin(t) \mathbf{j}}{b} \Rightarrow \kappa = 1/b, \quad \text{and} \quad \mathbf{N} = -(\cos(t) \mathbf{i} + \sin(t) \mathbf{j}).
\]

\( R = 1/\kappa = b \) = the radius of the circle.

\( C = \mathbf{r}(t) + R\mathbf{N} = b(\cos(t) \mathbf{i}, \sin(t)) + b(-\cos(t), \sin(t)) = 0 \) = center of circle.

This example shows that the terms 'radius of curvature' for \( 1/\kappa \) and 'center of curvature' for \( C \) are well chosen!

### 22.2 The cycloid

The cycloid is defined by the following description. Roll a wheel (circle of radius \( a \)) along the \( x \)-axis and follow the trajectory of a point on the wheel. This trajectory is called a cycloid.

The cycloid plays a roll in many important problems: For example, the brachistochrone -Bernouilli, Newton and the tautochrone –Huygens. Wikipedia will tell you all about these things.

There is a nice mathlet that gives a dynamic view of the cycloid. See [http://mathlets.org/mathlets/wheel/](http://mathlets.org/mathlets/wheel/)
22.2.1 Parametric equations for the cycloid

Our next goal is to find a parametrization of the cycloid. To do this we use vectors. In the figure below on the left we see that \( \vec{r}(\theta) = \overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QC} + \overrightarrow{CP} \)

Each of the individual vectors are easy to compute:
\( \overrightarrow{OQ} = (a\theta, 0) = \) amount rolled \((a=\text{radius})\)
\( \overrightarrow{QC} = (0, a) \)
\( \overrightarrow{CP} = (-a\sin(\theta), -a\cos(\theta)) \) (See the right hand figure just below.)

So,
\[
\vec{r}(\theta) = \langle a\theta - a\sin(\theta), a - a\cos(\theta) \rangle \\
= a\langle \theta - \sin(\theta), 1 - \cos(\theta) \rangle \\
= a((\theta - \sin(\theta)) i + (1 - \cos(\theta)) j).
\]

We can also write this as
\[
x(\theta) = a(\theta - \sin(\theta)), \quad y(\theta) = a(1 - \cos(\theta)).
\]

Notes. (1) The symmetric form of the cycloid equations is hard to write down
(2) Physical units like speed and acceleration will depend on \( \theta(t) \).

Example 22.4. For the cycloid in the previous example find the curvature \( \kappa \) and the arclength of one arch.

answer: \( \frac{dr}{d\theta} = a\langle 1 - \cos(\theta), \sin(\theta) \rangle = 2a\langle \sin^2(\theta/2), \sin(\theta/2)\cos(\theta/2) \rangle \).
\[ \left| \frac{dr}{d\theta} \right| = \frac{ds}{d\theta} = 2a \sqrt{\sin^2 \theta/2 = 2a|\sin(\theta/2)|}. \]

Note: \( T = \frac{2a(\sin^2(\theta/2), \sin(\theta/2) \cos(\theta/2))}{2a|\sin(\theta/2)|} = \pm \langle \sin(\theta/2), \cos(\theta/2) \rangle \) (this is a unit vector).

At the cusp \( ds/d\theta = 0 \), i.e., physically, you must stop to make a sudden 180 degree turn.

For one arch, \( 0 < \theta < 2\pi \), \( \frac{ds}{d\theta} = 2a \sin(\theta/2). \) (We can drop the absolute value because \( \sin(\theta/2) > 0 \) for this range of \( \theta \).)

The arclength of one arch = \( \int_0^{2\pi} ds/d\theta \, d\theta = \int_0^{2\pi} 2a \sin(\theta/2) \, d\theta = -4a \cos(\theta/2) \bigg|_0^{2\pi} = 4a. \)

This is called Wren’s theorem: the arclength of one arch of the cycloid = 4a.

To find the curvature \( \kappa \) we compute velocity and acceleration:

\[ \mathbf{r}(\theta) = a(\theta - \sin(\theta)) \mathbf{i} + a(1 - \cos(\theta)) \mathbf{j} \]

\[ \mathbf{v} = a(1 - \cos(\theta)) \mathbf{i} + a\sin(\theta) \mathbf{j} \quad \text{and} \quad \mathbf{a} = a\sin(\theta) \mathbf{i} + a\cos(\theta) \mathbf{j}. \]

Now we compute the formula \( \kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \):

\[ \mathbf{a} \times \mathbf{v} = a^2((1 - \cos(\theta)) \cos(\theta) - \sin(\theta) \sin(\theta)) \mathbf{k} = a^2(\cos(\theta) - 1) \mathbf{k} = -a^2 \sin^2(\theta/2) \mathbf{k} \]

We know that \( |\mathbf{v}| = 2a|\sin(\theta/2)| \). So,

\[ \kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = \frac{a^2(1 - \cos(\theta))}{a^3 2^{3/2}(1 - \cos(\theta))^{3/2}} = \frac{1}{a 2^{3/2} \sqrt{(1 - \cos(\theta))}} = \frac{1}{4a|\sin(\theta/2)|}. \]

Note, \( \kappa \) is undefined when \( \theta = 2n\pi \), i.e. at the cusps of the cycloid.

**22.3 The symmetric form can lose information**

**Example 22.5.** Where symmetric form loses information.

Find the symmetric form for \( x = \cos^2(t), \ y = \sin^2(t) \).

**answer:** Easily we see that \( x + y = 1 \), with \( x, y \) non-negative.

The symmetric form shows a line, while the parametric equations show which part of the line is actually traced out.

\[ \begin{array}{c}
\text{y} \\
1 \\
\hline
1 \\
\text{x}
\end{array} \]

**22.4 Intersection of two planes**

Suppose we have two planes with respective normals \( \vec{N}_1 \) and \( \vec{N}_2 \). The two planes intersect in a line. This line is perpendicular to both normals, i.e. it is in the same direction as \( \vec{N}_1 \times \vec{N}_2 \).
Example 22.6. Find the intersection of the planes \(x + y + z = 1\) and \(y = 2\).

answer: Normals: \(\vec{N}_1 = (1, 1, 1)\) and \(\vec{N}_2 = (0, 1, 0)\).

Direction vector: \(\vec{v} = \vec{N}_1 \times \vec{N}_2 = -\vec{i} + \vec{k} = (-1, 0, 1)\).

One point of intersection: (by elimination) \(y = 2 \Rightarrow x + 2 + z = 3 \Rightarrow P = (1, 2, 0)\).

answer: The intersection is the line \(\langle 1, 2, 0 \rangle + t\langle -1, 0, 1 \rangle\).

22.5 Three dimensional example

Example 22.7. For the helix \(\mathbf{r}(t) = \cos(t) \mathbf{i} + \sin(t) \mathbf{j} + at \mathbf{k}\) find the radius of curvature and center of curvature for arbitrary \(t\).

answer: We will use the Formulas (2), (3) and (4) from topic 21,

\[
\mathbf{v} = -\sin(t) \mathbf{i} + \cos(t) \mathbf{j} + a \mathbf{k}; \quad \mathbf{a} = -\cos(t) \mathbf{i} - \sin(t) \mathbf{j}.
\]

\[|\mathbf{v}| = \sqrt{1 + a^2}; \quad \mathbf{a} \times \mathbf{v} = -a \sin(t) \mathbf{i} + a \cos(t) \mathbf{j} - \mathbf{k}.
\]

Formula (4) \(\Rightarrow \kappa = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3} = \frac{\sqrt{1 + a^2}}{(1 + a^2)^{3/2}} = \frac{1}{1 + a^2}.
\]

radius of convergence \(= R = 1 + a^2\).

The center of curvature \(C = \mathbf{r}(t) + R \mathbf{N} \Rightarrow \) we have to find \(R \mathbf{N}\).

Since we already have \(\mathbf{a} \times \mathbf{v}\) we use Formula (5) from topic 21, i.e. \(\mathbf{v} \times (\mathbf{a} \times \mathbf{v}) = \kappa v^4 \mathbf{N}\).

\[
\mathbf{v} \times (\mathbf{a} \times \mathbf{v}) = (-\sin(t), \cos(t), a) \times (-a \sin(t), a \cos(t), -1)
\]

\[= (-\cos(t) - a^2 \cos(t), -\sin(t) - a^2 \sin(t), 0)\]

\[= (1 + a^2)(-\cos(t), -\sin(t), 0)\]

Now, Formula 5 implies \((1 + a^2)(-\cos(t), -\sin(t), 0) = \kappa v^4 \mathbf{N}\)

Since \(v = \sqrt{1 + a^2}\) this gives

\[
\mathbf{N} = \langle -\cos(t), -\sin(t), 0 \rangle \quad \text{and} \quad \kappa = 1/(1 + a^2) \Rightarrow R = 1 + a^2.
\]

Thus, \(C = \langle \cos(t), \sin(t), at \rangle + R \mathbf{N} = \langle -a^2 \cos(t), -a^2 \sin(t), at \rangle\).