31 Double and iterated integrals

Old, compressed version of topic 31 notes.

Double and iterated integrals are really the same thing. The distinction is that a double integral is geometric, i.e. a slicing and summing, and an iterated integral is analytic, i.e. a computational tool.

Iterated integrals

Example: \( \int_0^1 \int_0^2 xy \, dx \, dy \)

Inner integral: \( \int_0^2 xy \, dx = \frac{x^2}{2} \bigg|_0^1 = 2y \)

Outer integral: \( \int_0^1 2y \, dy = y^2 \bigg|_0^1 = 1 \).

Example: \( \int_0^1 \int_{x^2}^x (2x + 2y) \, dy \, dx \)

Inner integral: \( \int_{x^2}^x (2x + 2y) \, dy = 2xy + y^2 \bigg|_{x^2}^x = 2x^2 + x^2 - (2x^3 + x^4) = 3x^2 - 2x^3 - x^4 \)

Outer integral: \( \int_0^1 3x^2 - 2x^3 - x^4 \, dx = x^3 - \frac{x^4}{2} - \frac{x^5}{5} \bigg|_0^1 = 1 - \frac{1}{2} - \frac{1}{5} \).

Double integrals Mass example

1 dimension:
We start with a single (one dimensional) integral for mass:

\[
\begin{align*}
\text{density} & = \delta(x) \text{ units are mass/length} \\
\text{mass element} dm & = \delta(x) \, dx \\
\text{mass} M & = \int dm = \int_a^b \delta(x) \, dx \\
\end{align*}
\]

2 dimensions:
Now we consider a \textit{double} (two dimensional) integral for mass

\[
\begin{align*}
\text{density} & = \delta(x, y) \text{ units are mass/area} \\
\text{mass element} dm & = \delta(x, y) \, dA = \delta(x, y) \, dx \, dy \\
\text{mass} M & = \iint_R \delta(x, y) \, dA = \int_R \int \delta(x, y) \, dx \, dy.
\end{align*}
\]

This is a ‘sum’ over the infinitesimal boxes with area \( dx \, dy \).
To compute this 'sum' we’ll learn to do an iterated integral. Let’s discover this for ourselves in the simple example of the rectangle shown. The box runs between \( x = 1 \) and \( x = 3 \) and \( y = 1 \) and \( y = 4 \). We’ll assume a variable density \( \delta(x,y) = x^2y \).

To find the mass we have to 'sum' over all the little squares.

The mass in a single square at coordinates \((x,y)\) is

\[
dm = \delta(x,y) \, dx \, dy.
\]

To do the 'sum' we first fix \( y \) and then 'sum' the mass of the boxes over the horizontal strip (see the figure). This gives the mass of the strip is

\[
\int_1^3 \delta(x,y) \, dx \, dy = \int_1^3 x^2y \, dx \, dy = \left[ \frac{x^3}{3} y \, dy \right]_1^3 = \frac{26}{3} y \, dy.
\]

Notice that the integral (i.e. the sum along the horizontal strip) depends on \( y \) –different strips will have different mass.

Next, to find the total mass we have to sum up the mass of all the horizontal strips. This is just an integral in \( y \):

\[
\int_1^4 \int_1^3 \delta(x,y) \, dx \, dy = \int_1^4 \int_1^3 x^2y \, dx \, dy.
\]

**Limits of integration**

To compute the iterated integral over more complicated regions we need to be able to find limits of integration. We start with an example.

**Example:** Find the mass of region \( R \) bounded by

\[ y = x + 1, \quad y = x^2, \quad x = 0, \quad x = 1, \quad \text{density} = \delta(x,y) = xy. \]

Inner limits: \( y \) from \( x^2 \) to \( x + 1 \).

Outer limits: \( x \) from 0 to 1.

\[
\Rightarrow M = \int \int_R \delta(x,y) \, dA = \int_0^1 \int_{y=x^2}^{x+1} xy \, dy \, dx
\]

Inner:

\[
\int_{x^2}^{x+1} xy \, dy = \left[ \frac{y^2}{2} \right]_{x^2}^{x+1} = \frac{x(x+1)^2}{2} - \frac{x^5}{2} = \frac{x^3}{2} + x^2 + \frac{x}{2} - \frac{x^5}{2}
\]

Outer:

\[
\int_0^1 \frac{x^3}{2} + x^2 + \frac{x}{2} - \frac{x^5}{2} \, dx = \left[ \frac{x^4}{8} + \frac{x^3}{3} + \frac{x^2}{4} - \frac{x^6}{12} \right]_0^1 = \frac{1}{8} + \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{5}{8}.
\]

**Note:** The syntax \( y = x^2 \) in limits is redundant but useful. We know it must be \( y \) because of the \( dy \) matching the integral sign...

**Volume:** Like area: volume = 'sum' of rectangular boxes.
Example: Volume of tetrahedron (see above pictures)
Surface: \( z = 1 - x - y \).

Limits: inner: \( 0 < y < 1 - x \), outer: \( 0 < x < 1 \). \( \Rightarrow \) \( V = \int_0^1 \int_{y=0}^{1-x} 1 - x - y \, dy \, dx \).

Inner: \( \int_{y=0}^{1-x} 1 - x - y \, dy = y - xy - \frac{y^2}{2} \bigg|_0^{1-x} = 1 - x - x + x^2 - \frac{1}{2} + x - \frac{x^2}{2} \)

Outer: \( \int_0^1 \frac{1}{2} - x + \frac{x^2}{2} \, dx = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6} \).

Note: Finding volumes is nice. But just like single integrals and area, it is only one way to think of integrals. The most important way is still to think of them as a 'sum'.

Changing order of integration

Example: \( \int_0^1 \int_{\sqrt{x}}^1 e^y dy \, dx \). –Inner integral is too hard –so change order:

1) Find limits for region \( R \): inner: \( 0 < y < \sqrt{x} \), outer: \( 0 < x < 1 \).

2) Reverse limits: inner: \( 0 < x < y^2 \), outer: \( 0 < y < 1 \).

3) Compute integral: \( \int_{y=0}^1 \int_{x=0}^{y^2} e^y \, dx \, dy \)

Inner: \( x e^y \bigg|_{x=0}^{y^2} = ye^y \) \( \Rightarrow \) Outer: \( \int_0^1 ye^y = ye^y - e^y \bigg|_0^1 = 1. \)